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Comparisons of Parallel Systems with Components Having Proportional Reversed Hazard Rates and Starting Devices

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Abstract: In this paper, we consider stochastic comparisons of parallel systems with proportional reversed hazard rate (PRHR) distributed components equipped with starting devices. By considering parallel systems with two components that PRHR and starting devices, we prove the hazard rate and reversed hazard rate orders. These results are then generalized for such parallel systems with n components in terms of usual stochastic order. The establish results are illustrated with some examples.

Keywords: parallel systems; hazard rate order; reversed hazard rate order; usual stochastic order; proportional reversed hazard rate distribution; starting devices

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1. Introduction

Comparison of important characteristics of lifetimes of technical systems is of interest in many problems. Let $X_1, \cdots, X_n$ be non-negative independent random variables representing lifetimes of components of a system. Let $I_{p_1}, \cdots, I_{p_n}$ be independent Bernoulli random variables with $I_{p_i} = 1$ if the i-th component survives from random shocks and $I_{p_i} = 0$ if the i-th component fails from the shocks and $P(I_{p_i} = 1) = p_i$, for $i = 1, \cdots, n$. Further, let them also be independent of X_is. For a given time period, we can then use $I_{p_1}X_1, \cdots, I_{p_n}X_n$ to denote the lifetimes of components that are subject to random shocks. Of special interest are $Y_{n:n} = \max(I_{p_1}X_1, \cdots, I_{p_n}X_n)$ and $Y_{1:n} = \min(I_{p_1}X_1, \cdots, I_{p_n}X_n)$ corresponding to lifetimes of parallel and series systems, respectively. Throughout this work, we use the term "heterogeneity" to mean that components have different lifetime distributions. A similar assumption is also made on the survival probabilities. It is then of natural interest to evaluate the influence of heterogeneity among the components and the random shocks on the lifetimes of parallel and series systems, and this reliability problem forms the main basis for the present work.

We can present a different motivation for this problem as follows. Consider a finite system with each of its components equipped with a starter whose performance is modelled by a Bernoulli random variable, and with all component lifetimes being independent. As a starter may fail to initiate the component, the total number of components in operation would thus be random. Such situations arise naturally in a number of applications. Some possible examples are as follows: start-ups of power plants with gas turbines, length of time of a conference online being the maximum online time of those who successfully register for the conference, and the maximum loss of an insured individual who has a policy covering multiple risks being the maxima of those invoked losses. Another interesting scenario discussed by [1,2] in auction theory is when an auctioneer attracts some predetermined potential bidders by advertising a valuable object; in this case, the largest bid of those participants defines the price of the object for sale. One may additionally refer to [3–6] for the role of random extremes in financial economics, reliability theory, actuarial science, hydrology, and so on.

In actuarial set-up, the claims sizes may be represented by variables X_is, and the variables I_{p_i}s represent their occurrences. In this case, $Y_{n:n} = \max(I_{p_1}X_1, \cdots, I_{p_n}X_n)$ and $Y_{1:n} = \min(I_{p_1}X_1, \cdots, I_{p_n}X_n)$ correspond to the largest and smallest claim amounts in a portfolio of risks, respectively.

Considerable attention has been paid in the actuarial literature to different stochastic comparisons of numbers of claims and aggregate claim amounts. In particular, [7] consider a general scale model and discuss orderings of smallest and largest claim amounts, while [8] focus on the comparison of smallest and largest claim amounts from two sets of heterogeneous portfolios. These authors have specifically discussed the ordering results in the presence of heterogeneity among the sample sizes and the probabilities of claims and also in the presence of dependence between claim sizes and probabilities of claims.

The flexible family of distributions offered by the proportional reversed hazard rate (PRHR) model has found key applications in lifetime data analysis. For a system consisting of n components, let X_i and $\tilde{r}_i$ (for $i = 1, \cdots, n$) denote the lifetime and the reversed hazard rate of the i-th component. Then, when

$$\tilde{r}_i(x) = \lambda_i \tilde{r}(x), \qquad for\ i = 1, \cdots, n,$$

the variables X_is are said to have the $PRHR$ model, where $\tilde{r}(x)$ is referred to as the baseline reversed hazard rate function and λ_is (all positive) are the proportionality constants. It is then easy in this case to see that $F_i(x)$, the distribution function of X_i, is given by $(F(x))^{\lambda_i}$, $i = 1, \cdots, n$, where $F(x)$ is the baseline distribution function corresponding to $\tilde{r}(x)$. The PRHR family of distributions include many commonly used lifetime distributions as special cases such as generalized exponential and exponentiated Weibull distributions. In addition, when the proportionality constants λ_is are integers, then X_is are in fact the lifetimes of parallel systems consisting of λ_i components with their lifetimes being independent and identically distributed with distribution function $F(x)$. As parallel systems with more components are less prone to failure, the PRHR model is also referred to as resilience model in the reliability literature; one may refer to [9] for relevant details.

Suppose $Y_i = X_i I_{p_i}$, $i = 1, 2$. Then, the survival function of series systems, $V_{1:2} = \min\{X_1 I_{p_1}, X_2 I_{p_2}\}$, is given by

$$\bar{F}_{V_{1:2}}(x) = \left(\prod_{i=1}^{2} p_i\right) \bar{F}_{X_{1:2}}(x), \qquad x \geq 0.$$

Similarly, the survival function of $W_{1:2} = \min\{X_1^* I_{p_1^*}, X_2^* I_{p_2^*}\}$ is given by

$$\bar{F}_{W_{1:2}}(x) = \left(\prod_{i=1}^{2} p_i^*\right) \bar{F}_{X_{1:2}^*}(x), \qquad x \geq 0.$$

Then, the stochastic comparison between $V_{1:2}$ and $W_{1:2}$ is equivalent to the comparison between $X_{1:2}$ and $X_{1:2}^*$. It should be mentioned that the comparison between $X_{1:2}$ and $X_{1:2}^*$ has been investigated by many authors earlier. For this reason, we have not considered this problem in the present work.

In this paper, we consider only stochastic comparisons of parallel systems with proportional reversed hazard rate (PRHR) distributed components equipped with starting devices. We specifically establish the hazard rate, reversed hazard rate and usual stochastic orders of parallel systems with PRHR distributed components equipped with starting devices.

The rest of this paper is organized as follows. In Section 2, we present some basic definitions and notation pertaining to stochastic orders and majorization orders that are used in the present work. Section 3 discusses stochastic comparisons of parallel systems for different probabilities of starters in terms of hazard rate order. In Section 4, stochastic comparisons of parallel systems are established for different probabilities of starters in terms of reversed hazard rate order. Section 5 discusses stochastic comparisons of parallel systems for different probabilities of the starters in terms of usual stochastic order. Finally, some concluding remarks are made in Section 6.

2. Preliminaries

In this section, we present some basic definitions and lemmas that will be useful for all subsequent developments. For convenience, we use the notation $a \overset{sgn}{=} b$ to denote that both sides of an equality have the same sign.

Definition 1. *Suppose X and Y are two non-negative continuous random variables with distribution functions F_X and F_Y, survival functions $\bar{F}_X$ and $\bar{F}_Y$, hazard rate functions r_X and r_Y, and reversed hazard rate functions $\tilde{r}_X$ and $\tilde{r}_Y$. We assume that all involved expectations exist. Then:*

(i) *X is said to be larger than Y in the usual stochastic order (denoted by $X \geq_{st} Y$) if $\bar{F}_X(t) \geq \bar{F}_Y(t)$ for all $t \in \mathbb{R}_+$. This is equivalent to saying that $\mathbb{E}(\phi(X)) \geq \mathbb{E}(\phi(Y))$ for all increasing functions $\phi : \mathbb{R}_+ \to \mathbb{R}$;*

(ii) *X is said to be larger than Y in the hazard rate order (denoted by $X \geq_{hr} Y$) if and only if $\bar{F}_X(t)/\bar{F}_Y(t)$ increases in $t \in \mathbb{R}_+$. This is equivalent to saying that $r_Y(t) \geq r_X(t)$ for all $t \in \mathbb{R}_+$;*

(iii) *X is said to be larger than Y in the reversed hazard rate order (denoted by $X \geq_{rh} Y$) if and only if $F_X(t)/F_Y(t)$ increases in $t \in \mathbb{R}_+$. This is equivalent to saying that $\tilde{r}_X(t) \geq \tilde{r}_Y(t)$ for all $t \in \mathbb{R}_+$.*

It is known that the usual stochastic order is included in both hazard rate and reversed hazard rate orders. The books by [10,11] provide elaborate details on various stochastic orders and their applications to a wide array of problems.

Definition 2. *Consider two vectors $\boldsymbol{a} = (a_1, \cdots, a_n)$ and $\boldsymbol{b} = (b_1, \cdots, b_n)$ with corresponding increasing arrangements $a_{(1)} \leq \cdots \leq a_{(n)}$ and $b_{(1)} \leq \cdots \leq b_{(n)}$, respectively. Then:*

(i) *$\boldsymbol{a}$ is said to majorize $\boldsymbol{b}$, denoted by $\boldsymbol{a} \overset{m}{\succeq} \boldsymbol{b}$, if $\sum_{j=1}^{i} a_{(j)} \leq \sum_{j=1}^{i} b_{(j)}$ for $i = 1, \cdots, n-1$, and $\sum_{j=1}^{n} a_{(j)} = \sum_{j=1}^{n} b_{(j)}$;*

(ii) *$\boldsymbol{a}$ is said to weakly submajorize $\boldsymbol{b}$, denoted by $\boldsymbol{a} \succeq_w \boldsymbol{b}$, if $\sum_{j=i}^{n} a_{(j)} \geq \sum_{j=i}^{n} b_{(j)}$ for $i = 1, \cdots, n$.*

The concept of majorization is a way of comparing two vectors of the same dimension, in terms of the dispersion of their components, in which the order $\boldsymbol{u} \overset{m}{\succeq} \boldsymbol{v}$ means that u_is are more dispersive than v_is for a fixed sum. For example, we always have $\boldsymbol{u} \overset{m}{\succeq} \bar{\boldsymbol{u}}$, where $\bar{\boldsymbol{u}} = (\bar{u}, \cdots, \bar{u})$ with $\bar{u} = \frac{1}{n} \sum_{i=1}^{n} u_i$. It is evident that the majorization order implies weak submajorization order.

Definition 3. *A real-valued function ϕ, defined on a set $\mathbb{A} \subseteq \mathbb{R}^n$, is said to be Schur-convex on $\mathbb{A}$ if*

$$\boldsymbol{u} \overset{m}{\succeq} \boldsymbol{v} \Rightarrow \phi(\boldsymbol{u}) \geq \phi(\boldsymbol{v}) \qquad \textit{for any } \boldsymbol{u}, \boldsymbol{v} \in \mathbb{A}.$$

Further, ϕ is said to be Schur-concave function on $\mathbb{A}$ if $-\phi$ is Schur-convex on $\mathbb{A}$.

Lemma 1. *([12], p. 84) Suppose $J \subset \mathbb{R}$ is an open interval and $\phi : J^n \to \mathbb{R}$ is continuously differentiable. Then, necessary and sufficient conditions for ϕ to be Schur-convex (Schur-concave) on J^n are*

(i) *ϕ is symmetric on J^n;*

(ii) *for all* $i \neq j$ *and all* $\mathbf{z} \in J^n$,

$$(z_i - z_j)\left(\frac{\partial\phi(\mathbf{z})}{\partial z_i} - \frac{\partial\phi(\mathbf{z})}{\partial z_j}\right) \geq 0\,(\leq 0),$$

where $\partial\phi(\mathbf{z})/\partial z_i$ *denotes the partial derivative of* ϕ *with respect to its i-th argument.*

Lemma 2. *([12], p. 87) Consider the real-valued function* φ*, defined on a set* $\mathbb{A} \subseteq \mathbb{R}^n$*. Then,* $\boldsymbol{u} \succeq_w \boldsymbol{v}$ *implies* $\phi(\boldsymbol{u}) \geq \phi(\boldsymbol{v})$ *if and only if* ϕ *is increasing and Schur-convex on* $\mathbb{A}$*.*

3. Hazard Rate Order

In this section, we discuss stochastic comparisons of parallel systems for different probabilities of starters in terms of hazard rate order.

Theorem 1. *Suppose* X_1 *and* X_2 *are independent non-negative random variables with* $X_i \sim PRHR(\lambda_i)$*. Further, suppose* I_{p_1}*,* I_{p_2}*,* $I_{p_1^*}$*, and* $I_{p_2^*}$ *are independent Bernoulli random variables, independently of* X_i*s, with* $E(I_{p_i}) = p_i$ *and* $E(I_{p_i^*}) = p_i^*$*,* $i = 1, 2$*. Let* $V_{2:2} = \max\{X_1 I_{p_1}, X_2 I_{p_2}\}$ *and* $W_{2:2} = \max\{X_1 I_{p_1^*}, X_2 I_{p_2^*}\}$*. Then, the following statements hold true:*

(i) *If* $p_1 = p_1^*$ *(*$p_2 = p_2^*$*) and* $\lambda_1 \leq \lambda_2$ *(* $\lambda_2 \leq \lambda_1$*), then*

$$p_2 \geq p_2^* \;(p_1 \geq p_1^*) \Longleftrightarrow V_{2:2} \geq_{hr} W_{2:2};$$

(ii) *If* $p_1 \geq p_1^*$*, then*

$$p_1 \geq p_2\;,\; p_1^* \geq p_2^* \;\text{ and }\; p_1 - p_2 \leq p_1^* - p_2^* \implies V_{2:2} \geq_{hr} W_{2:2}.$$

Proof. (i) The survival functions of $V_{2:2}$ and $W_{2:2}$ are given by

$$\bar{F}_{V_{2:2}}(x) = p_1\left(1 - F^{\lambda_1}(x)\right) + p_2\left(1 - F^{\lambda_2}(x)\right) - p_1 p_2\left(1 - F^{\lambda_1}(x)\right)\left(1 - F^{\lambda_2}(x)\right)$$

and

$$\bar{F}_{W_{2:2}}(x) = p_1\left(1 - F^{\lambda_1}(x)\right) + p_2^*\left(1 - F^{\lambda_2}(x)\right) - p_1 p_2^*\left(1 - F^{\lambda_1}(x)\right)\left(1 - F^{\lambda_2}(x)\right).$$

For the necessity part, note that $V_{2:2} \geq_{hr} W_{2:2}$ implies that $V_{2:2} \geq_{st} W_{2:2}$ and so $\bar{F}_{V_{2:2}} \geq \bar{F}_{W_{2:2}}$. We thus have

$$\begin{aligned} &p_1\left(1 - F^{\lambda_1}(x)\right) + p_2\left(1 - F^{\lambda_2}(x)\right) - p_1 p_2\left(1 - F^{\lambda_1}(x)\right)\left(1 - F^{\lambda_2}(x)\right) \\ &\geq p_1\left(1 - F^{\lambda_1}(x)\right) + p_2^*\left(1 - F^{\lambda_2}(x)\right) - p_1 p_2^*\left(1 - F^{\lambda_1}(x)\right)\left(1 - F^{\lambda_2}(x)\right). \end{aligned}$$

Therefore, we get

$$p_2\left(1 - F^{\lambda_2}(x)\right)\left\{1 - p_1\left(1 - F^{\lambda_1}(x)\right)\right\} \geq p_2^*\left(1 - F^{\lambda_2}(x)\right)\left\{1 - p_1\left(1 - F^{\lambda_1}(x)\right)\right\},$$

which implies that $p_2 \geq p_2^*$. Now, for the sufficiency part, let us consider

$$\phi(x) = \frac{p_1\left(1 - F^{\lambda_1}(x)\right) + p_2\left(1 - F^{\lambda_2}(x)\right) - p_1 p_2\left(1 - F^{\lambda_1}(x)\right)\left(1 - F^{\lambda_2}(x)\right)}{p_1\left(1 - F^{\lambda_1}(x)\right) + p_2^*\left(1 - F^{\lambda_2}(x)\right) - p_1 p_2^*\left(1 - F^{\lambda_1}(x)\right)\left(1 - F^{\lambda_2}(x)\right)},$$

and it is then enough to show that ϕ is increasing in x. Upon differentiating ϕ with respect to x, we get

$$
\begin{aligned}
\frac{\partial\phi(x)}{\partial x} \overset{sgn}{=} & \left\{p_1\Big(1-F^{\lambda_1}(x)\Big)+p_2^*\Big(1-F^{\lambda_2}(x)\Big)-p_1p_2^*\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big)\right\} \\
\times & \left\{p_1p_2\lambda_1F^{\lambda_1}(x)\Big(1-F^{\lambda_2}(x)\Big)+p_1p_2\lambda_2F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)-p_1\lambda_1F^{\lambda_1}(x)\right. \\
- & \left. p_2\lambda_2F^{\lambda_2}(x)\right\} \\
- & \left\{p_1\Big(1-F^{\lambda_1}(x)\Big)+p_2\Big(1-F^{\lambda_2}(x)\Big)-p_1p_2\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big)\right\} \\
\times & \left\{p_1p_2^*\lambda_1F^{\lambda_1}(x)\Big(1-F^{\lambda_2}(x)\Big)+p_1p_2^*\lambda_2F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)-p_1\lambda_1F^{\lambda_1}(x)\right. \\
- & \left. p_2^*\lambda_2F^{\lambda_2}(x)\right\} \\
= & \; p_1^2p_2\lambda_1F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big)+p_1^2p_2\lambda_2F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)^2 \\
- & \; p_1^2\lambda_1F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big)-p_1p_2\lambda_2F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big) \\
+ & \; p_1p_2p_2^*\lambda_1F^{\lambda_1}(x)\Big(1-F^{\lambda_2}(x)\Big)^2+p_1p_2p_2^*\lambda_2F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big) \\
- & \; p_1p_2^*\lambda_1F^{\lambda_1}(x)\Big(1-F^{\lambda_2}(x)\Big)-p_2p_2^*\lambda_2F^{\lambda_2}(x)\Big(1-F^{\lambda_2}(x)\Big) \\
- & \; p_1^2p_2p_2^*\lambda_1F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big)^2 \\
- & \; p_1^2p_2p_2^*\lambda_2F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)^2\Big(1-F^{\lambda_2}(x)\Big) \\
+ & \; p_1^2p_2^*\lambda_1F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big) \\
+ & \; p_1p_2p_2^*\lambda_2F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big) \\
- & \; p_1^2p_2^*\lambda_1F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big) \\
- & \; p_1^2p_2^*\lambda_2F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)^2+p_1^2\lambda_1F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big) \\
+ & \; p_1p_2^*\lambda_2F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)-p_1p_2p_2^*\lambda_1F^{\lambda_1}(x)\Big(1-F^{\lambda_2}(x)\Big)^2 \\
- & \; p_1p_2p_2^*\lambda_2F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big)+p_1p_2\lambda_1F^{\lambda_1}(x)\Big(1-F^{\lambda_2}(x)\Big) \\
+ & \; p_2p_2^*\lambda_2F^{\lambda_2}(x)\Big(1-F^{\lambda_2}(x)\Big)+p_1^2p_2p_2^*\lambda_1F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big)^2 \\
+ & \; p_1^2p_2p_2^*\lambda_2F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)^2\Big(1-F^{\lambda_2}(x)\Big) \\
- & \; p_1^2p_2\lambda_1F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&- \quad p_1 p_2 p_2^* \lambda_2 F^{\lambda_2}(x)\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big) \\
&= \quad p_1\Big(p_2 - p_2^*\Big)\bigg\{p_1 \lambda_2 F^{\lambda_2}(x)\Big(1 - F^{\lambda_1}(x)\Big)^2 + \lambda_1 F^{\lambda_1}(x)\Big(1 - F^{\lambda_2}(x)\Big) \\
&- \quad \lambda_2 F^{\lambda_2}(x)\Big(1 - F^{\lambda_1}(x)\Big)\bigg\}.
\end{aligned}
$$

For proving the increasing property of ϕ, it is enough to show that

$$
D(x) = \lambda_1 F^{\lambda_1}(x)\Big(1 - F^{\lambda_2}(x)\Big) - \lambda_2 F^{\lambda_2}(x)\Big(1 - F^{\lambda_1}(x)\Big) \tag{1}
$$

is positive. For this purpose, we find

$$
\begin{aligned}
D(x) &= \lambda_1 F^{\lambda_1}(x) - \lambda_2 F^{\lambda_2}(x) + (\lambda_2 - \lambda_1) F^{\lambda_1 + \lambda_2}(x) \\
&\overset{sgn}{=} \lambda_1 - \lambda_2 F^{\lambda_2 - \lambda_1}(x) + (\lambda_2 - \lambda_1) F^{\lambda_2}(x).
\end{aligned}
$$

Let us now consider $E(x) = \lambda_1 - \lambda_2 F^{\lambda_2 - \lambda_1}(x) + (\lambda_2 - \lambda_1) F^{\lambda_2}(x)$. Then, we have $\lim_{x \to \infty} E(x) = 0$; since $\lambda_1 \leq \lambda_2$, we find

$$
\begin{aligned}
E'(x) &= -\lambda_2(\lambda_2 - \lambda_1) F^{\lambda_2 - \lambda_1 - 1}(x) f(x) + \lambda_2(\lambda_2 - \lambda_1) F^{\lambda_2 - 1}(x) f(x) \\
&\overset{sgn}{=} (\lambda_2 - \lambda_1) F^{\lambda_2 - 1}(x)\Big(- F^{-\lambda_1}(x) + 1\Big) \\
&\overset{sgn}{=} F^{\lambda_1}(x) - 1 \\
&\leq 0,
\end{aligned}
$$

and so $E(x)$ is decreasing. Consequently, $E(x) \geq 0$ for $x \geq 0$, and so $D(x) \geq 0$. Thus, Part (i) of the theorem is proved.

(ii) The survival functions of $V_{2:2}$ and $W_{2:2}$, for $x \geq 0$, are

$$
\bar{F}_{V_{2:2}}(x) = p_1\Big(1 - F^{\lambda_1}(x)\Big) + p_2\Big(1 - F^{\lambda_2}(x)\Big) - p_1 p_2\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big)
$$

and

$$
\bar{F}_{W_{2:2}}(x) = p_1^*\Big(1 - F^{\lambda_1}(x)\Big) + p_2^*\Big(1 - F^{\lambda_2}(x)\Big) - p_1^* p_2^*\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big),
$$

respectively. Then, it is enough to show that

$$
\psi(x) = \frac{p_1\Big(1 - F^{\lambda_1}(x)\Big) + p_2\Big(1 - F^{\lambda_2}(x)\Big) - p_1 p_2\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big)}{p_1^*\Big(1 - F^{\lambda_1}(x)\Big) + p_2^*\Big(1 - F^{\lambda_2}(x)\Big) - p_1^* p_2^*\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big)}
$$

is increasing in x. We can then see easily that

$$
\begin{aligned}
\frac{\partial \psi(x)}{\partial x} &\overset{sgn}{=} \bigg\{ - p_1 \lambda_1 F^{\lambda_1}(x) - p_2 \lambda_2 F^{\lambda_2}(x) + p_1 p_2 \lambda_1 F^{\lambda_1}(x)\Big(1 - F^{\lambda_2}(x)\Big) \\
&+ \quad p_1 p_2 \lambda_2 F^{\lambda_2}(x)\Big(1 - F^{\lambda_1}(x)\Big)\bigg\} \\
&\times \quad \bigg\{p_1^*\Big(1 - F^{\lambda_1}(x)\Big) + p_2^*\Big(1 - F^{\lambda_2}(x)\Big) - p_1^* p_2^*\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big)\bigg\} \\
&+ \quad \bigg\{p_1^* \lambda_1 F^{\lambda_1}(x) + p_2^* \lambda_2 F^{\lambda_2}(x) - p_1^* p_2^* \lambda_1 F^{\lambda_1}(x)
\end{aligned}
$$

$$
\begin{aligned}
\times &\quad \Big(1 - F^{\lambda_2}(x)\Big) - p_1^* p_2^* \lambda_2 F^{\lambda_2}(x)\Big(1 - F^{\lambda_1}(x)\Big)\Bigg\} \\
\times &\quad \Bigg\{p_1\Big(1 - F^{\lambda_1}(x)\Big) + p_2\Big(1 - F^{\lambda_2}(x)\Big) - p_1 p_2\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big)\Bigg\} \\
= &\quad -p_1 p_1^* \lambda_1 F^{\lambda_1}(x)\Big(1 - F^{\lambda_1}(x)\Big) - p_1 p_2^* \lambda_1 F^{\lambda_1}(x)\Big(1 - F^{\lambda_2}(x)\Big) \\
+ &\quad p_1 p_1^* p_2^* \lambda_1 F^{\lambda_1}(x)\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big) \\
- &\quad p_2 p_1^* \lambda_2 F^{\lambda_2}(x)\Big(1 - F^{\lambda_1}(x)\Big) - p_2 p_2^* \lambda_2 F^{\lambda_2}(x)\Big(1 - F^{\lambda_2}(x)\Big) \\
+ &\quad p_2 p_1^* p_2^* \lambda_2 F^{\lambda_2}(x)\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big) \\
+ &\quad p_1 p_2 p_1^* \lambda_1 F^{\lambda_1}(x)\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big) + p_1 p_2 p_2^* \lambda_1 F^{\lambda_1}(x)\Big(1 - F^{\lambda_2}(x)\Big)^2 \\
- &\quad p_1 p_2 p_1^* p_2^* \lambda_1 F^{\lambda_1}(x)\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big)^2 \\
+ &\quad p_1 p_2 p_1^* \lambda_2 F^{\lambda_2}(x)\Big(1 - F^{\lambda_1}(x)\Big)^2 + p_1 p_2 p_2^* \lambda_2 F^{\lambda_2}(x)\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big) \\
- &\quad p_1 p_2 p_1^* p_2^* \lambda_2 F^{\lambda_2}(x)\Big(1 - F^{\lambda_1}(x)\Big)^2\Big(1 - F^{\lambda_2}(x)\Big) \\
+ &\quad p_1 p_1^* \lambda_1 F^{\lambda_1}(x)\Big(1 - F^{\lambda_1}(x)\Big) + p_1^* p_2 \lambda_1 F^{\lambda_1}(x)\Big(1 - F^{\lambda_2}(x)\Big) - \Bigg\{p_1^* p_1 p_2 \lambda_1 F^{\lambda_1}(x) \\
\times &\quad \Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big)\Bigg\} \\
+ &\quad p_2^* p_1 \lambda_2 F^{\lambda_2}(x)\Big(1 - F^{\lambda_1}(x)\Big) + p_2^* p_2^* \lambda_2 F^{\lambda_2}(x)\Big(1 - F^{\lambda_2}(x)\Big) \\
- &\quad p_2^* p_1 p_2 \lambda_2 F^{\lambda_2}(x)\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big) \\
- &\quad p_1^* p_2^* p_1 \lambda_1 F^{\lambda_1}(x)\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big) - p_1^* p_2^* p_2 \lambda_1 F^{\lambda_1}(x)\Big(1 - F^{\lambda_2}(x)\Big)^2 \\
+ &\quad p_1^* p_2^* p_1 p_2 \lambda_1 F^{\lambda_1}(x)\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big)^2 \\
- &\quad p_1^* p_2^* p_1 \lambda_2 F^{\lambda_2}(x)\Big(1 - F^{\lambda_1}(x)\Big)^2 - p_1^* p_2^* p_2 \lambda_2 F^{\lambda_2}(x)\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big) \\
+ &\quad p_1^* p_2^* p_1 p_2 \lambda_2 F^{\lambda_2}(x)\Big(1 - F^{\lambda_1}(x)\Big)^2\Big(1 - F^{\lambda_2}(x)\Big) \\
= &\quad (p_1^* p_2 - p_1 p_2^*)\Big\{\lambda_1 F^{\lambda_1}(x)\Big(1 - F^{\lambda_2}(x)\Big) - \lambda_2 F^{\lambda_2}(x)\Big(1 - F^{\lambda_1}(x)\Big)\Big\} \\
+ &\quad p_2 p_2^* \lambda_1 F^{\lambda_1}(x)\Big(1 - F^{\lambda_2}(x)\Big)^2 (p_1 - p_1^*) + p_1 p_1^* \lambda_2 F^{\lambda_2}(x)\Big(1 - F^{\lambda_1}(x)\Big)^2 (p_2 - p_2^*)
\end{aligned}
$$

Based on Part (i) and Equation (1), for $\lambda_1 \leq \lambda_2$, we have

$$\lambda_1 F^{\lambda_1}(x)\Big(1 - F^{\lambda_2}(x)\Big) - \lambda_2 F^{\lambda_2}(x)\Big(1 - F^{\lambda_1}(x)\Big)$$

being positive. From $p_1 \geq p_2$ and $p_1^* \geq p_2^*$, there exist positive real numbers d and c such that $p_1 = p_2 + d$ and $p_1^* = p_2^* + c$. Now, from $p_1 - p_2 \leq p_1^* - p_2^*$, we have $d \leq c$, and also from $p_1 \geq p_1^*$, we get $p_2 + d \geq p_2^* + c$ and then $p_2 - p_2^* \geq c - d \geq 0$. So, $p_2 \geq p_2^*$ and clearly $cp_2 \geq dp_2^*$. Furthermore, we have

$$p_1^* p_2 - p_1 p_2^* = (p_2^* + c) - (p_2 + d)p_2^* = cp_2 - dp_2^* \geq 0.$$

Therefore, all terms of the last equality in $\frac{\partial \psi(x)}{\partial x}$ are positive, and the desired result is obtained.

$\square$

We now present an example to show that Theorem 1 (under $p_1 = p_1^*$) may not hold when the condition $\lambda_2 \geq \lambda_1$ is not satisfied.

Example 1. *Let us consider Beta*$(1,1)$ *as baseline distribution function. Set* $(p_1, p_2) = (0.3, 0.8)$, $(p_1^*, p_2^*) = (0.3, 0.2)$ *and* $(\lambda_1, \lambda_2) = (20, 10)$*. Then, we find*

$$\phi(x) = \frac{0.3(1 - x^{20}) + 0.8(1 - x^{10}) - 0.24(1 - x^{20})(1 - x^{10})}{0.3(1 - x^{20}) + 0.2(1 - x^{10}) - 0.06(1 - x^{20})(1 - x^{10})}$$

to be not monotone when $x \geq 0.95$ *(as seen in Figure 1), and this negates the results that* $V_{2:2} \leq_{hr} W_{2:2}$ *and* $V_{2:2} \geq_{hr} W_{2:2}$.

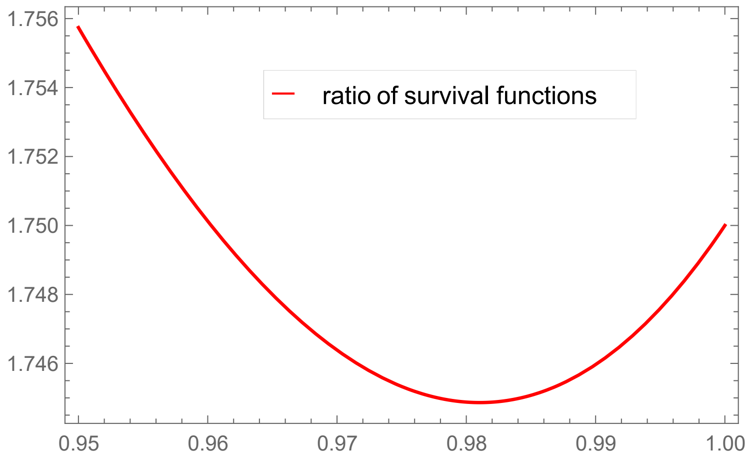

Figure 1. Plot of the ratio of survival functions of $V_{2:2}$ and $W_{2:2}$ for $x \in [0.95, 1]$ in Example 1.

Remark 1. *Under the* $\lambda_1 \leq \lambda_2$*, the result of Theorem 1 also hold under the following conditions:*

1. $p_1 \geq p_2 \geq p_1^* \geq p_2^*$ *and* $p_1 - p_2 \leq p_1^* - p_2^*$;
2. $p_1 \geq p_1^* \geq p_2 \geq p_2^*$ *and* $p_1 - p_2 \leq p_1^* - p_2^*$.

We now present an example to show that Part (ii) of Theorem 1 may not hold when $p_1 - p_2 > p_1^* - p_2^*$.

Example 2. *Let us consider the standard exponential as baseline distribution with* $F(x) = 1 - e^{-x}$, *for* $x > 0$*. Set* $(p_1, p_2) = (0.8, 0.4)$, $(p_1^*, p_2^*) = (0.5, 0.3)$ *and* $(\lambda_1, \lambda_2) = (2, 10)$*. Clearly,* $p_1 \geq p_1^* \geq p_2 \geq p_2^*$ *and* $p_1 - p_2 > p_1^* - p_2^*$*. We then have*

$$\psi(x) = \frac{0.8(1 - (1 - e^{-x})^2) + 0.4(1 - (1 - e^{-x})^{10}) - 0.32(1 - (1 - e^{-x})^2)(1 - (1 - e^{-x})^{10})}{0.5(1 - (1 - e^{-x})^2) + 0.3(1 - (1 - e^{-x})^{10}) - 0.15(1 - (1 - e^{-x})^2)(1 - (1 - e^{-x})^{10})},$$

to be not monotone in $0 \leq x \leq 1.5$ *(as seen in Figure 2), and this negates the results that* $V_{2:2} \leq_{hr} W_{2:2}$ *and* $V_{2:2} \geq_{hr} W_{2:2}$.

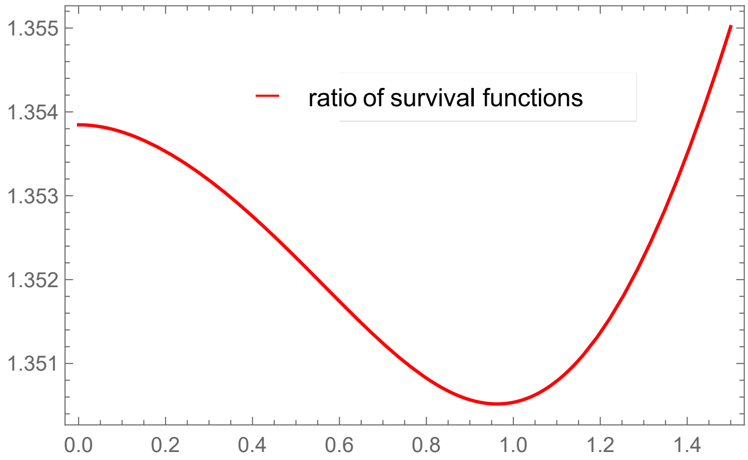

Figure 2. Plot of the ratio of survival functions of $V_{2:2}$ and $W_{2:2}$ for $x \in [0, 1.5]$ in Example 2.

4. Reversed Hazard Rate Order

In this section, we discuss stochastic comparisons of parallel systems for different probabilities of starters in terms of reversed hazard rate order. For this purpose, we first prove the following lemma.

Lemma 3. *Suppose function $g(x; a, b)$ is a differentiable function in x and*

$$\phi(x) = \frac{g(x; p, d)}{g(x; p^*, c)}.$$

If we consider $L(x; a, t)$ as

$$L(x; a, t) = \frac{g'(x; a, t)}{g(x; a, t)},$$

where $g'(x)$ denotes the derivative of function $g(x)$ with respect to x, then:

$$\frac{\partial \phi(x)}{\partial x} = \phi(x)\Big\{L(x; p, d) - L(x; p^*, c)\Big\}.$$

Proof. We can observe that

$$\begin{aligned} \phi(x)\Big\{L(x; p, d) - L(x; p^*, c)\Big\} &= \frac{g(x; p, d)}{g(x; p^*, c)}\left\{\frac{g'(x; p, d)}{g(x; p, d)} - \frac{g'(x; p^*, c)}{g(x; p^*, c)}\right\} \\ &= \frac{g'(x; p, d)g(x; p^*, c) - g'(x; p^*, c)g(x; p, d)}{\Big[g(x; p^*, c)\Big]^2} \\ &= \frac{\partial \phi(x)}{\partial x}. \end{aligned}$$

□

Theorem 2. *Suppose X_1 and X_2 are independent non-negative random variables with $X_i \sim PRHR(\lambda_i)$. Further, suppose I_{p_1}, I_{p_2}, $I_{p_1^*}$, and $I_{p_2^*}$ are independent Bernoulli random variables, independently of X_is, with $E(I_{p_i}) = p_i$ and $E(I_{p_i^*}) = p_i^*$, $i = 1, 2$. Let $V_{2:2} = \max\{X_1 I_{p_1}, X_2 I_{p_2}\}$ and $W_{2:2} = \max\{X_1 I_{p_1^*}, X_2 I_{p_2^*}\}$. Then, the following statements hold true:*

(i) *If $p_1 = p_1^*$ ($p_2 = p_2^*$) and $\lambda_1 \le \lambda_2$ ($\lambda_2 \le \lambda_1$), then*

$$p_2 \ge p_2^* \; (p_1 \ge p_1^*) \iff V_{2:2} \ge_{rh} W_{2:2};$$

(ii) *If $p_2 \geq p_2^*$, then*

$$p_1 \geq p_2 \ , \ p_1^* \geq p_2^* \ \textit{and} \ p_1 - p_2 \geq p_1^* - p_2^* \implies V_{2:2} \geq_{rh} W_{2:2}.$$

Proof. (i) The distribution functions of $V_{2:2}$ and $W_{2:2}$ are given by

$$F_{V_{2:2}}(x) = 1 - p_1\Big(1 - F^{\lambda_1}(x)\Big) - p_2\Big(1 - F^{\lambda_2}(x)\Big) + p_1p_2\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big),$$
$$F_{W_{2:2}}(x) = 1 - p_1\Big(1 - F^{\lambda_1}(x)\Big) - p_2^*\Big(1 - F^{\lambda_2}(x)\Big) + p_1p_2^*\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big).$$

For the necessity part, from $V_{2:2} \geq_{rh} W_{2:2}$, we have $F_{V_{2:2}}(x) \leq F_{W_{2:2}}(x)$, and so

$$\begin{aligned} &1 - p_1\Big(1 - F^{\lambda_1}(x)\Big) - p_2\Big(1 - F^{\lambda_2}(x)\Big) + p_1p_2\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big) \\ &\leq 1 - p_1\Big(1 - F^{\lambda_1}(x)\Big) - p_2^*\Big(1 - F^{\lambda_2}(x)\Big) + p_1p_2^*\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big), \end{aligned}$$

which implies that

$$\begin{aligned} &p_1\Big(1 - F^{\lambda_1}(x)\Big) + p_2\Big(1 - F^{\lambda_2}(x)\Big) - p_1p_2\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big) \\ &\geq p_1\Big(1 - F^{\lambda_1}(x)\Big) + p_2^*\Big(1 - F^{\lambda_2}(x)\Big) - p_1p_2^*\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big). \end{aligned}$$

Then, we have

$$p_2\Big(1 - F^{\lambda_2}(x)\Big)\Big\{1 - p_1\Big(1 - F^{\lambda_1}(x)\Big)\Big\} \geq p_2^*\Big(1 - F^{\lambda_2}(x)\Big)\Big\{1 - p_1\Big(1 - F^{\lambda_1}(x)\Big)\Big\},$$

which implies that $p_2 \geq p_2^*$. Next, for the sufficiency part, let us consider the function

$$\chi(x) = \frac{1 - p_1\Big(1 - F^{\lambda_1}(x)\Big) - p_2\Big(1 - F^{\lambda_2}(x)\Big) + p_1p_2\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big)}{1 - p_1\Big(1 - F^{\lambda_1}(x)\Big) - p_2^*\Big(1 - F^{\lambda_2}(x)\Big) + p_1p_2^*\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big)}.$$

It is then enough to show that χ is increasing in x. Upon differentiating $\chi(x)$ with respect to x, we find

$$\begin{aligned} \frac{\partial \chi(x)}{\partial x} &\overset{sgn}{=} \Big\{1 - p_1\Big(1 - F^{\lambda_1}(x)\Big) - p_2^*\Big(1 - F^{\lambda_2}(x)\Big) + p_1p_2^*\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big)\Big\} \\ &\times \Big\{p_1\lambda_1F^{\lambda_1}(x) + p_2\lambda_2F^{\lambda_2}(x) - p_1p_2\lambda_1F^{\lambda_1}(x)\Big(1 - F^{\lambda_2}(x)\Big) \\ &- p_1p_2\lambda_2F^{\lambda_2}(x)\Big(1 - F^{\lambda_1}(x)\Big)\Big\} \\ &- \Big\{1 - p_1\Big(1 - F^{\lambda_1}(x)\Big) - p_2\Big(1 - F^{\lambda_2}(x)\Big) + p_1p_2\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big)\Big\} \\ &\times \Big\{p_1\lambda_1F^{\lambda_1}(x) + p_2^*\lambda_2F^{\lambda_2}(x) - p_1p_2^*\lambda_1F^{\lambda_1}(x)\Big(1 - F^{\lambda_2}(x)\Big) \\ &- p_1p_2^*\lambda_2F^{\lambda_2}(x)\Big(1 - F^{\lambda_1}(x)\Big)\Big\} \\ &= p_1\lambda_1F^{\lambda_1}(x) + p_2\lambda_2F^{\lambda_2}(x) - p_1p_2\lambda_1F^{\lambda_1}(x)\Big(1 - F^{\lambda_2}(x)\Big) \end{aligned}$$

$$
\begin{aligned}
- &\quad p_1p_2\lambda_2F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big) - p_1^2\lambda_1F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big) \\
- &\quad p_1p_2\lambda_2F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big) + p_1^2p_2\lambda_1F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big) \\
+ &\quad p_1^2p_2\lambda_2F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)^2 - p_2^*p_1\lambda_1F^{\lambda_1}(x)\Big(1-F^{\lambda_2}(x)\Big) \\
- &\quad p_2^*p_2\lambda_2F^{\lambda_2}(x)\Big(1-F^{\lambda_2}(x)\Big) + p_1p_2p_2^*\lambda_1F^{\lambda_1}(x)\Big(1-F^{\lambda_2}(x)\Big)^2 \\
+ &\quad p_1p_2p_2^*\lambda_2F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big) \\
+ &\quad p_1^2p_2^*\lambda_1F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big) \\
+ &\quad p_1p_2p_2^*\lambda_2F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big) \\
- &\quad p_1^2p_2p_2^*\lambda_1F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big)^2 \\
- &\quad p_1^2p_2p_2^*\lambda_2F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)^2\Big(1-F^{\lambda_2}(x)\Big) \\
- &\quad p_1\lambda_1F^{\lambda_1}(x) - p_2^*\lambda_2F^{\lambda_2}(x) + p_1p_2^*\lambda_1F^{\lambda_1}(x)\Big(1-F^{\lambda_2}(x)\Big) \\
+ &\quad p_1p_2^*\lambda_2F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big) + p_1^2\lambda_1F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big) \\
+ &\quad p_1p_2^*\lambda_2F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big) - p_1^2p_2^*\lambda_1F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big) \\
- &\quad p_1^2p_2^*\lambda_2F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)^2 + p_1p_2\lambda_1F^{\lambda_1}(x)\Big(1-F^{\lambda_2}(x)\Big) \\
+ &\quad p_2p_2^*\lambda_2F^{\lambda_2}(x)\Big(1-F^{\lambda_2}(x)\Big) - p_1p_2p_2^*\lambda_1F^{\lambda_1}(x)\Big(1-F^{\lambda_2}(x)\Big)^2 \\
- &\quad p_1p_2p_2^*\lambda_2F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big) \\
- &\quad p_1^2p_2\lambda_1F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big) \\
- &\quad p_1p_2p_2^*\lambda_2F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big) \\
+ &\quad p_1^2p_2p_2^*\lambda_1F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big)^2 + p_1^2p_2p_2^*\lambda_2F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)^2 \\
\times &\quad \Big(1-F^{\lambda_2}(x)\Big) \\
= &\quad (p_2-p_2^*)\lambda_2F^{\lambda_2}(x)\left\{1-2p_1\Big(1-F^{\lambda_1}(x)\Big)+p_1^2\Big(1-F^{\lambda_1}(x)\Big)^2\right\} \\
= &\quad (p_2-p_2^*)\lambda_2F^{\lambda_2}(x)\left\{1-p_1\Big(1-F^{\lambda_1}(x)\Big)\right\}^2 \\
\geq &\quad 0.
\end{aligned}
$$

Hence, χ is increasing in x, which completes the proof of Part (i) of the theorem.

(ii) The distribution functions of $V_{2:2}$ and $W_{2:2}$, for $x \geq 0$, are given by

$$F_{V_{2:2}}(x) = 1 - p_1\Big(1-F^{\lambda_1}(x)\Big) - p_2\Big(1-F^{\lambda_2}(x)\Big) + p_1p_2\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big)$$

and

$$F_{W_{2:2}}(x) = 1 - p_1^*\Big(1-F^{\lambda_1}(x)\Big) - p_2^*\Big(1-F^{\lambda_2}(x)\Big) + p_1^*p_2^*\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big),$$

respectively. Then, it is enough to show that

$$
\begin{aligned}
\Omega(x) &= \frac{F_{V_{2:2}}(x)}{F_{W_{2:2}}(x)} \\
&= \left\{1 - p_1\Big(1 - F^{\lambda_1}(x)\Big) - p_2\Big(1 - F^{\lambda_2}(x)\Big) + p_1 p_2\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big)\right\} \\
/ & \left\{1 - p_1^*\Big(1 - F^{\lambda_1}(x)\Big) - p_2^*\Big(1 - F^{\lambda_2}(x)\Big) + p_1^* p_2^*\Big(1 - F^{\lambda_1}(x)\Big)\Big(1 - F^{\lambda_2}(x)\Big)\right\}
\end{aligned}
$$

is increasing in x. Since $p_1 \geq p_2$ and $p_1^* \geq p_2^*$, there exist positive real numbers d and c such that $p_1 = d + p_2$ and $p_1^* = c + p_2^*$, and then we can rewrite $\Omega(x)$ as follows:

$$
\begin{aligned}
\Omega(x) &= \left\{1 - (d + p_2)\Big(1 - F^{\lambda_1}(x)\Big) - p_2\Big(1 - F^{\lambda_2}(x)\Big) + (d + p_2)p_2\Big(1 - F^{\lambda_1}(x)\Big)\right. \\
&\times \left.\Big(1 - F^{\lambda_2}(x)\Big)\right\} \\
&\times \left\{1 - (c + p_2^*)\Big(1 - F^{\lambda_1}(x)\Big) - p_2^*\Big(1 - F^{\lambda_2}(x)\Big) + (c + p_2^*)p_2^*\Big(1 - F^{\lambda_1}(x)\Big)\right. \\
&\times \left.\Big(1 - F^{\lambda_2}(x)\Big)\right\}^{-1}.
\end{aligned}
$$

Let us consider $L(x; a, b)$ as follows:

$$
\begin{aligned}
L(x; a, b) &= \tilde{r}(x) \times \left\{(b + a)\lambda_1 F^{\lambda_1}(x) + a\lambda_2 F^{\lambda_2}(x) - (b + a)a\lambda_1 F^{\lambda_1}(x)\Big(1 - F^{\lambda_2}(x)\Big)\right. \\
&- \left.(b + a)a\lambda_2 F^{\lambda_2}(x)\Big(1 - F^{\lambda_1}(x)\Big)\right\} \\
&\times \left\{1 - (b + a)\Big(1 - F^{\lambda_1}(x)\Big) - a\Big(1 - F^{\lambda_2}(x)\Big) + (b + a)a\Big(1 - F^{\lambda_1}(x)\Big)\right. \\
&\times \left.\Big(1 - F^{\lambda_2}(x)\Big)\right\}^{-1},
\end{aligned}
$$

where $\tilde{r}(x) = \frac{f(x)}{F(x)}$. We can then see easily that

$$
L(x; a, b) = \frac{g'(x; a, b)}{g(x; a, b)},
$$

where $g'(x; a, b) = \frac{\partial g(x; a, b)}{\partial x}$ and

$$
\begin{aligned}
g(x; a, b) &= 1 - (b + a)\Big(1 - F^{\lambda_1}(x)\Big) - a\Big(1 - F^{\lambda_2}(x)\Big) + (b + a)a\Big(1 - F^{\lambda_1}(x)\Big) \\
&\times \Big(1 - F^{\lambda_2}(x)\Big).
\end{aligned}
$$

Because

$$
\Omega(x) = \frac{g(x; p_2, d)}{g(x; p_2^*, c)},
$$

according to Lemma 3, we have

$$\frac{\partial \Omega(x)}{\partial x} = \Omega(x)\Big\{L(x;p_2,d) - L(x;p_2^*,c)\Big\}, \tag{2}$$

and clearly, for proving increasing property of $\Omega(x)$ with respect to x, it is enough to prove that the function $L(x;a,b)$ is increasing in a and also in b. First, we have

$$\begin{aligned}
\frac{\partial L(x;a,b)}{\partial a} &\overset{sgn}{=} \Big\{\lambda_1 F^{\lambda_1}(x) + \lambda_2 F^{\lambda_2}(x) - (2a+b)\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_2}(x)\Big) \\
&- (2a+b)\lambda_2 F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big\} \\
&\times \Big\{1-(b+a)\Big(1-F^{\lambda_1}(x)\Big) - a\Big(1-F^{\lambda_2}(x)\Big) \\
&+ (b+a)a\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big)\Big\} \\
&- \Big\{-\Big(1-F^{\lambda_1}(x)\Big) - \Big(1-F^{\lambda_2}(x)\Big) + (2a+b)\Big(1-F^{\lambda_1}(x)\Big) \\
&\times \Big(1-F^{\lambda_2}(x)\Big)\Big\} \\
&\times \Big\{(b+a)\lambda_1 F^{\lambda_1}(x) + a\lambda_2 F^{\lambda_2}(x) - (b+a)a\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_2}(x)\Big) \\
&- (b+a)a\lambda_2 F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big\} \\
&= \lambda_1 F^{\lambda_1}(x) - (b+a)\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big) - a\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_2}(x)\Big) \\
&+ (b+a)a\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big) \\
&+ \lambda_2 F^{\lambda_2}(x) - (b+a)\lambda_2 F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big) - a\lambda_2 F^{\lambda_2}(x)\Big(1-F^{\lambda_2}(x)\Big) \\
&+ (b+a)a\lambda_2 F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big) \\
&- (2a+b)\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_2}(x)\Big) + \Big\{(2a+b)(b+a)\lambda_1 F^{\lambda_1}(x) \\
&\times \Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big)\Big\} + (2a+b)a\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_2}(x)\Big)^2 \\
&- (2a+b)(b+a)a\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big)^2 \\
&- (2a+b)\lambda_2 F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big) \\
&+ (2a+b)(b+a)\lambda_2 F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)^2 + (2a+b)a\lambda_2 F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big) \\
&- (2a+b)(b+a)a\lambda_2 F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)^2\Big(1-F^{\lambda_2}(x)\Big) \\
&+ (b+a)\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big) + a\lambda_2 F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big) \\
&- (b+a)a\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big) \\
&- (b+a)a\lambda_2 F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)^2 + (b+a)\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_2}(x)\Big) \\
&+ a\lambda_2 F^{\lambda_2}(x)\Big(1-F^{\lambda_2}(x)\Big) - (b+a)a\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_2}(x)\Big)^2 \\
&- (b+a)a\lambda_2 F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big) \\
&- (2a+b)(b+a)\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big) \\
&- (2a+b)a\lambda_2 F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big)
\end{aligned}$$

$$
\begin{aligned}
&+ \quad (2a+b)(b+a)a\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big)^2 \\
&+ \quad (2a+b)(b+a)a\lambda_2 F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)^2\Big(1-F^{\lambda_2}(x)\Big) \\
&= \quad \lambda_1 F^{\lambda_1}(x) - 2a\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_2}(x)\Big) + a^2\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_2}(x)\Big)^2 \\
&+ \quad \lambda_2 F^{\lambda_2}(x) - 2(b+a)\lambda_2 F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big) + (b+a)^2\lambda_2 F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)^2 \\
&= \quad \lambda_1 F^{\lambda_1}(x)\Big[1-a\Big(1-F^{\lambda_2}(x)\Big)\Big]^2 + \lambda_2 F^{\lambda_2}(x)\Big[1-(b+a)\Big(1-F^{\lambda_1}(x)\Big)\Big]^2 \\
&\geq \quad 0,
\end{aligned}
$$

which shows that $L(x;a,b)$ is increasing in a. Next, we also have

$$
\begin{aligned}
\frac{\partial L(x;a,b)}{\partial b} \quad &\overset{sgn}{=} \quad \Big\{\lambda_1 F^{\lambda_1}(x) - a\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_2}(x)\Big) - a\lambda_2 F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big\} \\
&\times \quad \Big\{1-(b+a)\Big(1-F^{\lambda_1}(x)\Big) - a\Big(1-F^{\lambda_2}(x)\Big) + (b+a)a\Big(1-F^{\lambda_1}(x)\Big) \\
&\times \quad \Big(1-F^{\lambda_2}(x)\Big)\Big\} \\
&- \quad \Big\{-\Big(1-F^{\lambda_1}(x)\Big) + a\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big)\Big\} \\
&\times \quad \Big\{(b+a)\lambda_1 F^{\lambda_1}(x) + a\lambda_2 F^{\lambda_2}(x) - (b+a)a\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_2}(x)\Big) \\
&- \quad (b+a)a\lambda_2 F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big\} \\
&= \quad \lambda_1 F^{\lambda_1}(x) - (b+a)\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big) - a\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_2}(x)\Big) \\
&+ \quad (b+a)a\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big) \\
&- \quad a\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_2}(x)\Big) + (b+a)a\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big) \\
&+ \quad a^2\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_2}(x)\Big)^2 - (b+a)a^2\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big) \\
&\times \quad \Big(1-F^{\lambda_2}(x)\Big)^2 - a\lambda_2 F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big) \\
&+ \quad (b+a)a\lambda_2 F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)^2 \\
&+ \quad a^2\lambda_2 F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big) - \Big\{(b+a)a^2\lambda_2 F^{\lambda_2}(x) \\
&\times \quad \Big(1-F^{\lambda_1}(x)\Big)^2\Big(1-F^{\lambda_2}(x)\Big)\Big\} + (b+a)\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big) \\
&+ \quad a\lambda_2 F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big) - \Big\{(b+a)a\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big) \\
&\times \quad \Big(1-F^{\lambda_2}(x)\Big)\Big\} - (b+a)a\lambda_2 F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)^2 \\
&- \quad (b+a)a\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big) \\
&- \quad a^2\lambda_2 F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big) \\
&+ \quad (b+a)a^2\lambda_1 F^{\lambda_1}(x)\Big(1-F^{\lambda_1}(x)\Big)\Big(1-F^{\lambda_2}(x)\Big)^2 \\
&+ \quad (b+a)a^2\lambda_2 F^{\lambda_2}(x)\Big(1-F^{\lambda_1}(x)\Big)^2\Big(1-F^{\lambda_2}(x)\Big) \\
&= \quad \lambda_1 F^{\lambda_1}(x)\Big[1-2a\Big(1-F^{\lambda_2}(x)\Big) + \Big(1-F^{\lambda_2}(x)\Big)^2\Big]
\end{aligned}
$$

$$
\begin{aligned}
&= \lambda_1 F^{\lambda_1}(x)\left[1 - a\left(1 - F^{\lambda_2}(x)\right)\right]^2 \\
&\geq 0,
\end{aligned}
$$

which shows that $L(x;a,b)$ is increasing in b. Now, since $L(x;a,b)$ is increasing in a and $p_2 \geq p_2^*$, we have

$$L(x; p_2, d) \geq L(x; p_2^*, d). \tag{3}$$

Further, since $L(x;a,b)$ is increasing in b and $d \geq c$ (because $d = p_1 - p_2 \geq p_1^* - p_2^* = c$), we have

$$L(x; p_2^*, d) \geq L(x; p_2^*, c). \tag{4}$$

Upon combining (3) and (4), we have

$$L(x; p_2, d) \geq L(x; p_2^*, c),$$

or

$$L(x; p_2, d) - L(x; p_2^*, c) \geq 0, \tag{5}$$

Next, from (2) and (5), we get $\frac{\partial \Omega(x)}{\partial x} \geq 0$, and then the desired result is obtained. □

Remark 2. *The results of Theorem 2 also hold under the following conditions:*

1. $p_2^* \leq p_1^* \leq p_2 \leq p_1$ *and* $p_1 - p_2 = p_1^* - p_2^*$;
2. $p_2^* \leq p_2 \leq p_1^* \leq p_1$ *and* $p_1 - p_2 = p_1^* - p_2^*$.

5. Usual Stochastic Order

In this section, we present some stochastic comparisons of parallel systems for different components and different probabilities of starters in terms of the usual stochastic order.

Theorem 3. *Suppose* $X_1, \cdots, X_n$ ($X_1^*, \cdots, X_n^*$) *are independent non-negative random variables with* $X_i \sim PRHR(\lambda)$ ($X_i^* \sim PRHR(\lambda^*)$), $i = 1, \cdots, n$. *Further, suppose* $I_{p_1}, \cdots, I_{p_n}$ ($I_{p_1^*}, \cdots, I_{p_n^*}$) *are independent Bernoulli random variables, independently of* X_i*s and* X_i^**s, with* $E(I_{p_i}) = p_i$ *and* $E(I_{p_i^*}) = p_i^*$, $i = 1, \cdots, n$. *Let* $V_{n:n} = \max\{X_1 I_{p_1}, \cdots, X_n I_{p_n}\}$ *and* $W_{n:n} = \max\{X_1^* I_{p_1^*}, \cdots, X_n^* I_{p_n^*}\}$. *If* $\lambda \geq \lambda^*$ *and* $\boldsymbol{p} \overset{m}{\succeq} \boldsymbol{p}^*$, *then* $V_{n:n} \geq_{st} W_{n:n}$.

Proof. Let us denote $s(\boldsymbol{p}, \lambda; x) = 1 - \prod_{i=1}^{n}\left[1 - p_i\left(1 - F^{\lambda}(x)\right)\right]$. For $\lambda \geq \lambda^*$, we can observe that $s(\boldsymbol{p}, \lambda; x) \geq s(\boldsymbol{p}, \lambda^*; x)$. For obtaining the desired result, it is sufficient to observe that $s(\boldsymbol{p}, \lambda^*; x) \geq s(\boldsymbol{p}^*, \lambda^*; x)$. Therefore, we have to check Conditions (i) and (ii) of Lemma 1. It is then evident that $s(\boldsymbol{p}, \lambda^*; x)$ is symmetric with respect to $\boldsymbol{p}$, for any x. Additionally, for any $i \neq j$, we have

$$\frac{\partial s(\boldsymbol{p}, \lambda^*; x)}{\partial p_k} = \left(1 - F^{\lambda^*}(x)\right) \prod_{i=1, i\neq k}^{n} \left[1 - p_i\left(1 - F^{\lambda^*}(x)\right)\right].$$

Thus, we get

$$
\begin{aligned}
(p_i - p_j)\left\{\frac{\partial s(\boldsymbol{p}, \lambda^*; x)}{\partial p_i} - \frac{\partial s(\boldsymbol{p}, \lambda^*; x)}{\partial p_j}\right\} &= (p_i - p_j)\left(1 - F^{\lambda^*}(x)\right) \prod_{k=1, k\neq i, k\neq j}^{n} \\
&\times \left[1 - p_k\left(1 - F^{\lambda^*}(x)\right)\right] \\
&\times \left\{\left[1 - p_j\left(1 - F^{\lambda^*}(x)\right)\right] - \left[1 - p_i\left(1 - F^{\lambda^*}(x)\right)\right]\right\} \\
&= (p_i - p_j)^2\left(1 - F^{\lambda^*}(x)\right)^2 \\
&\times \prod_{k=1, k\neq i, k\neq j}^{n} \left[1 - p_k\left(1 - F^{\lambda^*}(x)\right)\right] \\
&\geq 0.
\end{aligned}
$$

Hence, $s(\boldsymbol{p}, \lambda^*; x)$ is Schur-convex with respect to $\boldsymbol{p}$, for any x. This implies that $V_{n:n} \geq_{st} W_{n:n}$, as required. □

6. Concluding Remarks

A parallel system is one of the most commonly used coherent systems in practice. For this reason, a careful study of its performance characteristics, such as reliability function, hazard function and reversed hazard function, based on the characteristics of the component lifetime distribution, is of great interest to reliability engineers. In this work, we have focused our attention primarily on a parallel system with two components as a parallel system with more components can be decomposed into many subsystems with two components in parallel. One of the prominent examples of a two-component parallel system is a twin-engine jet system, which, in addition to being safer than a single-engine jet system, is more efficient in terms of fuel consumption than a jet system with more than two engines.

Specifically, we have proved the hazard rate and reversed hazard rate orders of parallel systems with two components having proportional hazard rates and starting devices.

It will be of interest to consider the problems discussed here by allowing dependence between components using some general copulas for the joint distribution of lifetime components. We are working in this direction at the present time and will present the corresponding results in the future.

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References

1. Fang, R.; Li, X. Advertising a second-price auction. *J. Math. Econ.* **2015**, *61*, 246–252. [CrossRef]
2. Li, C.; Li, X. Stochastic comparisons of parallel and series systems of dependent components equipped with starting devices. *Commun. Stat. Theory Methods* **2019**, *48*, 694–708. [CrossRef]
3. Li, X.; Zuo, M.J. Preservation of stochastic orders for random minima and maxima, with applications. *Nav. Res. Logist.* **2004**, *51*, 332–344. [CrossRef]
4. Da, G.; Ding, W.; Li, X. On hazard rate ordering of parallel systems with two independent components. *J. Stat. Plan. Inference* **2010**, *140*, 2148–2154. [CrossRef]
5. Joo, S.; Mi, J. Some properties of hazard rate functions of systems with two components. *J. Stat. Plan. Inference* **2010**, *140*, 444–453. [CrossRef]
6. Di Crescenzo, A.; Di Gironimo, P. Stochastic comparisons and dynamic information of random lifetimes in a replacement model. *Mathematics* **2018**, *6*, 204. [CrossRef]
7. Barmalzan, G.; Payandeh Najafabadi, A.T.; Balakrishnan, N. Ordering properties of the smallest and largest claim amounts in a general scale model. *Scand. Actuar. J.* **2017**, *2017*, 105–124. [CrossRef]
8. Zhang, Y.; Cai, X.; Zhao, P. Ordering properties of extreme claim amounts from heterogeneous portfolios. *ASTIN Bull.* **2019**, *49*, 525–554. [CrossRef]
9. Marshall, A.W.; Olkin, I. *Life Distributions*; Springer: New York, NY, USA, 2007.
10. Müller, A.; Stoyan, D. *Comparison Methods for Stochastic Models and Risks*; John Wiley & Sons: New York, NY, USA, 2002.
11. Shaked, M.; Shanthikumar, J.G. *Stochastic Orders*; Springer: New York, NY, USA, 2007.
12. Marshall, A.W.; Olkin, I.; Arnold, B.C. *Inequalities: Theory of Majorization and Its Applications*, 2nd ed.; Springer: New York, NY, USA, 2011.