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Conformal Vector Fields and the De-Rham Laplacian on a Riemannian Manifold

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Abstract: We study the effect of a nontrivial conformal vector field on the geometry of compact Riemannian spaces. We find two new characterizations of the m-dimensional sphere $\mathbf{S}^m(c)$ of constant curvature c. The first characterization uses the well known de-Rham Laplace operator, while the second uses a nontrivial solution of the famous Fischer–Marsden differential equation.

Keywords: Riemannian manifold; sphere; conformal vector field; de-Rham Laplace operator; Fischer–Marsden differential equation; Obata's differential equation

MSC: 53C25; 53C42; 58J05; 53A30

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1. Introduction

Conformal vector fields and conformal mappings play important roles in the geometry of (pseudo-)Riemannian manifolds as well as in the general relativity (see, e.g., [1–5]). The characterization of important spaces, such as Euclidean spaces, Euclidean spheres and hyperbolic spaces, represents one of the most fascinating problems in Riemannian geometry. In this respect, the role of conformal vector fields is eminent as these provide one of best tools in obtaining such characterizations (cf. [6–22]).

On a Riemannian manifold (M, g), the Ricci operator S is defined using Ricci tensor Ric, by

$$Ric(X, Y) = g(SX, Y),\ X \in \mathfrak{X}(M),$$

where $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M (see [23]). Similarly, the rough Laplace operator on the Riemannian manifold (M, g), $\Delta : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is defined by [24]

$$\Delta X = \sum_{i=1}^{m} \left(\nabla_{e_i} \nabla_{e_i} X - \nabla_{\nabla_{e_i} e_i} X \right), \quad X \in \mathfrak{X}(M),$$

where ∇ is the Riemannian connection and $\{e_1, ..., e_m\}$ is a local orthonormal frame on M, $m = \dim M$. Rough Laplace operator is used in finding characterizations of spheres as well as of Euclidean spaces (cf. [17,25]). Recall that the de-Rham Laplace operator $\square : \mathfrak{X}(M) \to \mathfrak{X}(M)$ on a Riemannian manifold (M, g) is defined by (cf. [24], p-83)

$$\square = S + \Delta \tag{1}$$

and is used to characterize a Killing vector field on a compact Riemannian manifold. It is known that if ξ is a Killing vector field on a Riemannian manifold (M, g) or a soliton vector field of a Ricci soliton (M, g, ξ, λ), then $\square\xi = 0$ (cf. [11]). In addition, Fischer–Marsden considered the following differential equation (cf. [26]) on a Riemannian manifold (M, g):

$$(\Delta f)g + fRic = Hess(f), \tag{2}$$

where $Hess(f)$ is the Hessian of smooth function f and Δ is the Laplace operator acting on smooth functions of M. It is known that if a complete Riemannian manifold (M, g) has a nontrivial solution f to (2), then the scalar curvature of g is a constant (see [26,27]). We remark that Fischer and Marsden conjectured that if a compact Riemannian manifold admits a nontrivial solution of the differential Equation (2), then it must be an Einstein manifold. Counterexamples to the conjecture were provided by Kobayashi [28] and Lafontaine [29].

If we consider the sphere $\mathbf{S}^m(c)$ of constant curvature c as hypersurface of the Euclidean space $\mathbf{R}^{m+1}$ with unit normal ξ and shape operator $B = -\sqrt{c}I$, where I stands for the identity operator, then it is well known that the Ricci operator S of the sphere $\mathbf{S}^m(c)$ is given by

$$S = (m-1)cI.$$

Now, consider a constant unit vector field ς on the Euclidean space $\mathbf{R}^{m+1}$. Then restricting ς to the sphere $\mathbf{S}^m(c)$ one can express it as

$$\varsigma = \mathbf{u} + f\xi,$$

with $f = \langle \varsigma, \xi \rangle$, where $\mathbf{u}$ is the tangential projection of ς on the sphere and $\langle \cdot, \cdot \rangle$ is the Euclidean metric. Taking covariant derivative of the above equation with respect to a vector field X on the sphere $\mathbf{S}^m(c)$ and using Gauss–Weingarten formulae for hypersurface, we conclude

$$\nabla_X \mathbf{u} = -\sqrt{c} f X, \quad \nabla f = \sqrt{c}\mathbf{u}, \tag{3}$$

where ∇ is the Riemannian connection on the sphere $\mathbf{S}^m(c)$ with respect to the canonical metric g and ∇f is the gradient of the smooth function f on $\mathbf{S}^m(c)$. Then it follows that the rough Laplace operator Δ acting on $\mathbf{u}$ and the Laplace operator acting on the smooth function f are respectively given by

$$\Delta \mathbf{u} = -c\mathbf{u}, \quad \Delta f = -mcf. \tag{4}$$

Now, due to the choice of the constant unit vector field ς on the Euclidean space and the equations in (3), we see that $\mathbf{u}$ is not parallel and that f is a nonconstant function. Further, we observe that the vector field $\mathbf{u}$ on the sphere $\mathbf{S}^m(c)$ satisfies

$$\square \mathbf{u} = (m-2)c\mathbf{u}. \tag{5}$$

In addition, the Hessian of f is given by

$$Hess(f)(X, Y) = g(\nabla_X \nabla f, Y) = -cfg(X, Y), \quad X, Y \in \mathfrak{X}(M)$$

and using Equations (4) and (5), we see that the function f on the sphere $\mathbf{S}^m(c)$ satisfies the Fischer–Marsden Equation (2).

Recall that a smooth vector field $\mathbf{u}$ on a Riemannian manifold (M, g) is said to be a conformal vector field, if

$$\pounds_{\mathbf{u}} g = 2\sigma g, \tag{6}$$

where $\pounds_{\mathbf{u}} g$ is the Lie differentiation of g with respect to the vector field $\mathbf{u}$ and σ is a smooth function on M called the potential function (or the conformal factor) of the conformal vector field $\mathbf{u}$. A conformal vector field is said to be nontrivial if the potential function σ is a nonzero function. We observe that using Equation (4), the vector field $\mathbf{u}$ on the sphere $\mathbf{S}^m(c)$ satisfies

$$\pounds_{\mathbf{u}} g = -2\sqrt{c} f g, \tag{7}$$

that is, $\mathbf{u}$ is a nontrivial conformal vector field with potential function (conformal factor) $-\sqrt{c}f$. Thus, the sphere $\mathbf{S}^m(c)$ admits a nontrivial conformal vector field that is an eigenvector of the de-Rham Laplace operator with eigenvalue $(m-2)c$ (see Equation (5)) and the potential function is solution of the Fischer–Marsden differential Equation (2). These raise two natural questions:

(i) Is a compact Riemannian manifold (M, g) that admits a nontrivial conformal vector field $\mathbf{u}$, which is eigenvector of de-Rham Laplace operator $\square$ corresponding to a positive eigenvalue, necessarily isometric to a sphere?

(ii) Is a compact Riemannian manifold (M, g) that admits a nontrivial conformal vector field $\mathbf{u}$ with potential function a nontrivial solution of the Fischer–Marsden differential equation, necessarily isometric to a sphere?

In this paper, we answer the above two problems, showing that the first question has an affirmative answer (cf. Theorem 1), while an affirmative answer for the second question requires an additional condition on the Ricci curvature (cf. Theorem 2).

2. Preliminaries

Let $\mathbf{u}$ be a nontrivial conformal vector field on an m-dimensional Riemannian manifold (M, g) and $\mathfrak{X}(M)$ be the Lie algebra of smooth vector fields on M. Let γ be the smooth 1-form dual to $\mathbf{u}$, that is

$$\gamma(X) = g(X, \mathbf{u}),\ X \in \mathfrak{X}(M).$$

If we define a skew-symmetric operator G, called the associate operator of $\mathbf{u}$, by

$$\frac{1}{2}d\gamma(X, Y) = g(GX, Y), \quad X, Y \in \mathfrak{X}(M),$$

then using the above equation and Equation (6) in Koszul's formula (see [30] [p. 55, Equation (9)]) we have

$$\nabla_X \mathbf{u} = \sigma X + GX, \quad X \in \mathfrak{X}(M), \tag{8}$$

where ∇ is the Riemannian connection on (M, g). We adopt the following expression for curvature tensor

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad X, Y, Z \in \mathfrak{X}(M),$$

and use Equation (8) to compute

$$R(X, Y)\mathbf{u} = X(\sigma)Y - Y(\sigma)X + (\nabla G)(X, Y) - (\nabla G)(Y, X),$$

where

$$(\nabla G)(X, Y) = \nabla_X GY - G(\nabla_X Y).$$

Using the above equation and the expression for the Ricci tensor

$$Ric(X, Y) = \sum_{i=1}^{m} g(R(e_i, X)Y, e_i),$$

where $\{e_1, ..., e_m\}$ is a local orthonormal frame, we obtain

$$Ric(Y, \mathbf{u}) = -(m-1)Y(\sigma) - \sum_{i=1}^{m} g(Y, (\nabla G)(e_i, e_i)),$$

where we used the skew-symmetry of the operator G. The above equation gives

$$S(\mathbf{u}) = -(m-1)\nabla\sigma - \sum_{i=1}^{m} (\nabla G)(e_i, e_i). \tag{9}$$

Now, using Equation (8), we compute the action of the rough Laplace operator Δ on the vector field $\mathbf{u}$ and find

$$\Delta \mathbf{u} = \nabla\sigma + \sum_{i=1}^{m} (\nabla G)(e_i, e_i). \tag{10}$$

Note that using Equation (8), we get

$$\operatorname{div}\mathbf{u} = m\sigma, \ \operatorname{div}(\sigma\mathbf{u}) = \mathbf{u}(\sigma) + m\sigma^2. \tag{11}$$

Let $\tau = \mathrm{Tr}S$ be the scalar curvature of the Riemannian manifold. Then we have the following expression for the gradient of the scalar curvature

$$\frac{1}{2}\nabla\tau = \sum_{i=1}^{m} (\nabla S)(e_i, e_i),$$

where $\{e_1, ..., e_m\}$ is a local orthonormal frame.

3. Characterizations of Spheres

Let $\mathbf{u}$ be a nontrivial conformal vector field on an m-dimensional Riemannian manifold (M, g) with nonzero potential function σ. In this section, we find two new characterizations of spheres through nontrivial conformal vector fields, using the de-Rham Laplace operator $\Box$ and the Fischer–Marsden differential equation. If $\mathbf{u}$ is a nontrivial conformal vector field with potential function σ on an m-dimensional compact Riemannian manifold (M, g), then using Equation (11), we have

$$\int_M \sigma = 0, \quad \int_M \left(\mathbf{u}(\sigma) + m\sigma^2\right) = 0. \tag{12}$$

Theorem 1. *Let $\mathbf{u}$ be a nontrivial conformal vector field on an m-dimensional compact Riemannian manifold (M, g), $m > 2$. Then $\Box\mathbf{u} = \lambda\mathbf{u}$ for a constant λ, if and only if $\lambda > 0$ and (M, g) is isometric to the sphere $\mathbf{S}^m\left(\frac{\lambda}{m-2}\right)$.*

Proof. Suppose $\mathbf{u}$ is a nontrivial conformal vector field with potential function σ on a compact Riemannian manifold (M, g) that satisfies

$$\Box\mathbf{u} = \lambda\mathbf{u},$$

where λ is a constant. Then using Equations (9) and (10), we conclude

$$\nabla\sigma = -\frac{\lambda}{m-2}\mathbf{u}. \tag{13}$$

If $\lambda = 0$, then the above equation will imply that σ is a constant and then the first Equation in (12) will imply $\sigma = 0$, contrary to our assumption that $\mathbf{u}$ is a nontrivial conformal vector field. Hence, the constant $\lambda \neq 0$. Now, taking covariant derivative in Equation (13) and using Equation (8), we get

$$\nabla_X \nabla\sigma = -\frac{\lambda}{m-2}(\sigma X + GX), \quad X \in \mathfrak{X}(M).$$

Taking the inner product with $X \in \mathfrak{X}(M)$ in the above equation and noticing that G is skew symmetric, we conclude

$$g(\nabla_X \nabla\sigma, X) = -\frac{\lambda\sigma}{m-2} g(X, X).$$

Using polarization in above equation, and noticing that

$$Hess(\sigma)(X,Y) = g(\nabla_X \nabla\sigma, Y)$$

is symmetric, we get

$$Hess(\sigma)(X,Y) = -\frac{\lambda\sigma}{m-2} g(X,Y). \tag{14}$$

Taking trace in above equation, we get

$$\Delta\sigma = -\frac{m\lambda}{m-2}\sigma.$$

Since $\mathbf{u}$ is a nontrivial conformal vector field, it follows that σ is nonconstant due to Equation (12) and consequently, the above equation suggests that σ is an eigenfunction of the Laplace operator with eigenvalue $\frac{m\lambda}{m-2}$. Thus the nonzero constant $\lambda > 0$. Hence, Equation (14) being Obata's differential equation implies that (M,g) is isometric to the sphere $\mathbf{S}^m\left(\frac{\lambda}{m-2}\right)$ (cf. [18,19]).

Conversely, if (M,g) is isometric to the sphere $\mathbf{S}^m\left(\frac{\lambda}{m-2}\right)$, then Equation (5) confirms the existence of nontrivial vector field $\mathbf{u}$ satisfying $\square\mathbf{u} = \lambda\mathbf{u}$ for a constant λ. $\square$

Recall that if an m-dimensional Riemannian manifold (M,g) admits a nontrivial solution of the Fischer–Marsden differential Equation (2), $m > 2$, then the scalar curvature τ is a constant (cf. [26,27]) and the nontrivial solution f satisfies

$$\Delta f = -\frac{\tau}{m-1} f, \tag{15}$$

Now, we consider an m-dimensional Riemannian manifold (M,g) that admits a nontrivial conformal vector field $\mathbf{u}$ with potential function σ that is a nontrivial solution of the Fischer–Marsden differential Equation (2) and define a constant α by $\tau = m(m-1)\alpha$ for this Riemannian manifold. Then we have the following:

Theorem 2. *Let $\mathbf{u}$ be a nontrivial conformal vector field with potential function σ and associated operator G on an m-dimensional compact Riemannian manifold (M,g), $m > 2$. Then σ is a nontrivial solution of the Fischer–Marsden Equation (2) and*

$$Ric(\nabla\sigma + \alpha\mathbf{u}, \nabla\sigma + \alpha\mathbf{u}) \geq \alpha^2 \|G\|^2$$

holds for a constant α, where the constant α is given by $\tau = m(m-1)\alpha$, if and only if $\alpha > 0$ and (M,g) is isometric to the sphere $\mathbf{S}^m(\alpha)$.

Proof. Suppose the potential function σ of a nontrivial conformal vector field $\mathbf{u}$ is a nontrivial solution of the Fischer–Marsden (2) on an m-dimensional compact Riemannian manifold (M,g) and the associated operator G satisfies

$$Ric(\nabla\sigma + \alpha\mathbf{u}, \nabla\sigma + \alpha\mathbf{u}) \geq \alpha^2 \|G\|^2, \tag{16}$$

where the constant α is given by $\tau = m(m-1)\alpha$. Since the potential function is a nontrivial solution of the Equation (2), by Equation (15), we have

$$\Delta\sigma = -m\alpha\sigma. \tag{17}$$

As observed earlier, for a nontrivial conformal vector field $\mathbf{u}$ we have that the potential function σ is nonconstant and by Equation (17) we see that σ is an eigenfunction of the Laplace operator and therefore $\alpha > 0$. Now, using Equation (9), we have

$$Ric(\mathbf{u},\mathbf{u}) = -(m-1)\mathbf{u}(\sigma) - \sum_{i=1}^{m} g(\mathbf{u},(\nabla G)(e_i,e_i)). \tag{18}$$

Using Equation (8) and skew-symmetry of the associated operator G, we find

$$\mathrm{divG}(\mathbf{u}) = -\|\mathrm{G}\|^2 - \sum_{\mathrm{i}=1}^{\mathrm{m}} \mathrm{g}(\mathbf{u},(\nabla \mathrm{G})(\mathrm{e_i},\mathrm{e_i})), \tag{19}$$

where

$$\|G\|^2 = \sum_{i=1}^{m} g(Ge_i,Ge_i)$$

for a local orthonormal frame $\{e_1,...,e_m\}$. Inserting Equation (19) in Equation (18), we get

$$Ric(\mathbf{u},\mathbf{u}) = -(m-1)\mathbf{u}(\sigma) + \mathrm{divG}(\mathbf{u}) + \|\mathrm{G}\|^2$$

and integrating the above equation while using Equation (12), we derive

$$\int_M Ric(\mathbf{u},\mathbf{u}) = \int_M \left(\|G\|^2 + m(m-1)\sigma^2\right). \tag{20}$$

Using the Bochner's formula (cf. [31]) (p. 19, Equation (1.45)), we have

$$\int_M \left(Ric(\nabla\sigma,\nabla\sigma) + |Hess(\sigma)|^2 - (\Delta\sigma)^2\right) = 0. \tag{21}$$

Now, using the symmetry of $Hess(\sigma)$ and skew-symmetry of the operator G in computing $\mathrm{div}(G\nabla\sigma)$, we get

$$\mathrm{div}(G\nabla\sigma) = -\sum_{i=1}^{m} g(\nabla\sigma,(\nabla G)(e_i,e_i))$$

and using Equation (9) in above equation, we have

$$\mathrm{div}(G\nabla\sigma) = Ric(\mathbf{u},\nabla\sigma) + (m-1)\|\nabla\sigma\|^2.$$

Integrating the above equation yields

$$\int_M Ric(\mathbf{u},\nabla\sigma) = -(m-1)\int_M \|\nabla\sigma\|^2. \tag{22}$$

In addition, Equation (17) implies

$$\int_M \|\nabla\sigma\|^2 = m\alpha\int_M \sigma^2. \tag{23}$$

Note that

$$Ric(\nabla\sigma + \alpha\mathbf{u},\nabla\sigma + \alpha\mathbf{u}) = Ric(\nabla\sigma,\nabla\sigma) + 2\alpha Ric(\mathbf{u},\nabla\sigma) + \alpha^2 Ric(\mathbf{u},\mathbf{u})$$

and integrating the above equation while using Equations (20)–(23), we conclude

$$\int_M Ric(\nabla\sigma + \alpha\mathbf{u}, \nabla\sigma + \alpha\mathbf{u}) = \int_M \left((\Delta\sigma)^2 - |Hess(\sigma)|^2 - m(m-1)\alpha^2\sigma^2 + \alpha^2\|G\|^2\right),$$

that is, on using Equation (17), we have

$$\int_M \left(Ric(\nabla\sigma + \alpha\mathbf{u}, \nabla\sigma + \alpha\mathbf{u}) - \alpha^2\|G\|^2\right) = \int_M \left(\frac{1}{m}(\Delta\sigma)^2 - |Hess(\sigma)|^2\right).$$

Now, using inequality (16) and the Schwartz's inequality $|Hess(\sigma)|^2 \geq \frac{1}{m}(\Delta\sigma)^2$ in the above equation, we derive

$$|Hess(\sigma)|^2 = \frac{1}{m}(\Delta\sigma)^2$$

and the above equality holds if and only if

$$Hess(\sigma) = \frac{1}{m}(\Delta\sigma)g.$$

Hence, by Equation (17) we have

$$Hess(\sigma) = -\alpha\sigma g,$$

where α is a positive constant and the potential function α is a nonconstant function due to the fact that $\mathbf{u}$ is a nontrivial conformal vector field and first equation in Equation (12). Hence, by Obata's result, it follows that (M, g) is isometric to the sphere $\mathbf{S}^m(\alpha)$. The converse is trivial as the sphere $\mathbf{S}^m(\alpha)$ admits a nontrivial conformal vector field $\mathbf{u}$ with potential function $\sigma = -\sqrt{\alpha}f$ (see Equation (7)) with $\nabla\sigma = -\alpha\mathbf{u}$ (see Equation (3). Thus, we have

$$\nabla\sigma + \alpha\mathbf{u} = 0$$

and $\mathbf{u}$ being a gradient (see Equation (3)), it follows that $d\gamma = 0$, where γ is the smooth 1-form dual to $\mathbf{u}$, and therefore we derive immediately that $G = 0$. Hence, we conclude that on the sphere $\mathbf{S}^m(\alpha)$ the conditions in the statement of the Theorem 2 hold. □

4. Conclusions

The aim of the present work was to study whether the existence of a nontrivial conformal vector field on an n-dimensional compact Riemannian manifold satisfying some very natural conditions influences the geometry of this space. Investigating this question, we arrived at two characterizations of the standard n-spheres with the help of nontrivial conformal vector fields, using the de-Rham Laplace operator and the Fischer–Marsden differential equation. One of the key ingredients in proving these results was the Obata's celebrated theorem on the characterization of the standard spheres (see [18]). Finally, we would like to mention some possible applications of the results. Obviously, it is unfeasible to obtain results of this type imposing such conditions for a general vector field on a general Riemannian space, but it is expected to be possible to adapt and apply the techniques developed in this article to other remarkable vector fields and famous (partial) differential equations on Riemannian manifolds. It is clear that such characterizations provide us a better insight of the relationship between differential equations and vector fields on Riemannian manifolds. In particular, it is worth mentioning that as immediate applications of the results, we obtain not only characterizations but also obstructions to the existence of certain nontrivial solutions to some (partial) differential equations on spaces of great interest in differential geometry, like Euclidean spheres, complex and quaternion projective spaces (see, e.g, [25]). Applications in physics are also notable, as many complicated physical problems are modeled through differential equations on certain (pseudo)-Riemannian manifolds (see, e.g, the recent books [32,33]). We only mention that the Fisher and Marsden equation investigated in this work in a geometric setting is nothing but the so-called *vacuum static equation* on static spaces introduced by Hawking and Ellis in [34].

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