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Synchronization of Fractional-Order Chaotic Systems with Model Uncertainty and External Disturbance

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Abstract: This paper is concerned with complete synchronization of fractional-order chaotic systems with both model uncertainty and external disturbance. Firstly, we propose a new dynamic feedback control method for complete synchronization of fractional-order nominal systems (without both uncertainty and disturbance). Then, a new uncertainty and disturbance estimator (UDE)-based dynamic feedback control method for the fractional-order systems with both uncertainty and disturbance is presented, by which the synchronization problem of such fractional-order chaotic systems is realized. Finally, the fractional-order Lorenz system is used to demonstrate the practicability of the proposed results.

Keywords: complete synchronization; fractional-order systems; uncertainty; disturbance; dynamic feedback control method

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1. Introduction

As a branch of mathematical analysis, fractional calculus was proposed around the same time as Newton Leibniz's integral calculus, and it can be traced back to 1695. After that, the fractional differential equations have been applied to describe many practical systems, such as circuits [1], viscoelastic beams [2,3], diffusion models [4,5], nonholonomic systems [6,7], chaotic systems [8–13] and so on. Additionally, many systems in reality have fractional-order dynamic behavior. Hence, fractional calculus has developed rapidly during the past few decades. As an important branch of fractional calculus, fractional-order systems are also widely used in many practical applications, such as signal processing, image processing, automatic control, robotics, etc.

Chaos is a kind of special nonlinear dynamic system which is highly sensitive to the change of parameters and initial conditions. Many integer-order chaotic systems exhibit fractional-order dynamic behavior, such as Lorenz [9], Chen [10,11], Rössler [12], Lü [13], etc., which are called fractional-order chaotic systems. The concept of chaos synchronization was first proposed by Pecora and Carroll in their 1990's paper [14]. Since then, chaos synchronization is one of the important branches of chaos control and has great application potential in fields such as secure communication, signal encryption and fault diagnosis. In recent years, synchronization of fractional-order chaotic systems has attracted a lot of attention and various control methods have been proposed, such as adaptive control [15–20], passive control [21], active control [22], fuzzy control [23], sliding mode control [24], feedback control [25], and so forth. Simultaneously, many synchronization types of the fractional-order chaotic systems have been studied including complete synchronization [26,27], projective synchronization [28], lag synchronization [29], etc. Although scholars have made great achievements in the control of fractional-order chaotic systems, there are still many challenges and problems to be solved. For instance, there are too many combinations of controllers and control channels, and the control technology designed in [30–34] does not take the uncertainty of the system into account.

It is well known that chaotic systems are very sensitive to parameter disturbances and external disturbances. Therefore, it is difficult to realize synchronization of chaotic systems with parameter disturbances and external disturbances. Fortunately, there are some works in studying the synchronization problem of integer-order chaotic systems with parameter disturbances and external disturbances. But, there are some limitations in the research results of synchronization of chaotic systems with model uncertainties and external disturbances. For example, both model uncertainty and external disturbance are supposed to be bounded, and the bounds are usually small. Such as, $d(t) \in L_2^n[0, +\infty)$. And, the method used is based on linear matrix inequalities (LMIs), thus the obtained result is too conservative in some cases. Recently, the UDE-based method has shown some advantages over the existing results, see [35–38]. Therefore, we shall investigate the synchronization problem of the fractional-order chaotic systems by extending the existing UDE-based control method.

Inspired by the above discussions, we focus on complete synchronization of the fractional-order chaotic systems with model uncertainty and external perturbation. A new UDE-based control method is proposed to realize the synchronization problem of such fractional-order chaotic systems. The rest of this paper is arranged as follows. Section 2 introduces the preliminary of fractional-order chaotic systems and problem formulation. In Section 3, main results of this paper are presented. In Section 4, an illustrative example is studied to show the correctness and effectiveness of the main results. The last section gives the conclusion.

2. Preliminaries and Problem Formation

2.1. Preliminaries

In this section, we give the definition and some preliminaries of fractional-order system which can be used in the next.

The derivative of α-order Caputo is defined as

$$D_t^\alpha f(x) = \frac{D^\alpha f(x)}{dt^\alpha} = \begin{cases} \frac{1}{\Gamma(\alpha-n)} \int_\alpha^t \frac{f^{(n)}(\varsigma)}{(t-\varsigma)^{\alpha+1-n}} d\varsigma, & n-1 < \alpha < n \\ \frac{d^n f(t)}{dt^n}, & \alpha = n \end{cases}$$

where $n = [\alpha]$, $\Gamma(.)$ is a function which is defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt.$$

Consider the following fractional-order nonlinear system

$$D_t^\alpha x = f(x) \tag{1}$$

here $x \in R^n$ is a state vector, $f(x) \in R^n$ is a continuous vector function.

Many properties of fractional-order calculus are introduced as follows.

Property 1 ([39])**.** *The fractional-order calculus defined by Caputo is a linear operator and satisfies*

$$D_t^\alpha(\lambda f(t) + \mu g(t)) = \lambda D_t^\alpha f(t) + \mu D_t^\alpha g(t)$$

where λ, μ are real constants.

Property 2 ([40])**.** *For fractional-order nonlinear system (1), $f(x)$ meets the following Lipschitz condition:*

$$\|f(y) - f(x)\| \le L\|y - x\|$$

where $\|\cdot\|$ is an ∞-norm, L is a positive real number.

Property 3 ([41])**.** *Suppose that $x(t) \in R$ is a continuous and derivable function. Then, for $\forall \alpha \in (0,1)$ and $\forall t \geq t_0$,*

$$\frac{1}{2} D_t^\alpha x^2 \leq x D_t^\alpha x.$$

For the stability problem of fractional-order nonlinear system, there are many theories to judge the stability of the system, but the Mittag–Leffler stability theory is most used.

Lemma 1 ([42,43])**.** *Assume that $x = 0$ is the equilibrium point of system (1), and $D \subset R^n$ is the region containing the origin. If there exists a Lyapunov function $V(x) : [0, \infty) \times D \longrightarrow R$ satisfying the local Lipschitz condition with respect to x:*

$$\beta_1(\|x\|) \leq V(x) \leq \beta_2(\|x\|)$$

$$D_t^\alpha V(x) \leq -\beta_3(\|x\|)$$

where $t \geq 0$, $x \in D$, $\alpha \in (0,1)$, β_1, β_2, β_3 are class-κ functions, then the equilibrium point $x = 0$ of system (1) is asymptotically stable.

2.2. Problem Formation

The fractional-order chaotic system with model uncertainty and external disturbance can be expressed as

$$D_t^\alpha x = f(x) + u_d + Bu \tag{2}$$

where $x \in R^n$, $f(x) \in R^n$ is a concinnous nonlinear vector function, $u_d = \triangle f(x) + d(t)$, $\triangle f(x)$ is the model uncertainty, $d(t)$ is the external disturbance, $B \in R^{n \times r}$ is the constant matrix, $r \geq 1$, and $u \in R^r$ is the controller to be designed.

Suppose that system (2) is chosen as the drive system. Then the response system can be described as

$$D_t^\alpha y = f(y) \tag{3}$$

where $y \in R^n$, $f(y) \in R^n$ is a concinnous nonlinear vector function.

The error system ($e = x - y$) is presented as

$$D_t^\alpha e = f(x) - f(y) + u_d + Bu \tag{4}$$

where u_d, B are given in Equation (2).

In the following, our goal is to design a controller u satisfying the performance in the form of

$$\lim_{t \to +\infty} \|e(t)\| = 0. \tag{5}$$

3. Main Results

In this section, we investigate how to design a simple and physical controller u to satisfy (5), that is to say, the designed controller u can stabilize the error system (4).

Firstly, we study the stabilization of the error system (4) with $u_d = 0$ and get the controller u_s. A conclusion is obtained as follows.

Theorem 1. *Consider the error system (4) with $u_d = 0$. If $(f(x) - f(y), B)$ can be stabilized, then the dynamic feedback controller is designed as*

$$u_s = Ke \tag{6}$$

where $K = k(t)B^T$, and the feedback gain $k(t)$ is updated

$$D_t^\alpha k(t) \leq -\gamma \sum_{i=1}^{n} e_i^2 \tag{7}$$

and

$$\dot{k} = -\gamma \|e(t)\|^2, \ \gamma > 0. \tag{8}$$

Proof. Define the following non-negative function:

$$V = \frac{1}{2}\sum_{i=1}^{n} e_i^2 + \frac{1}{2\gamma}(k(t) + M)^2 \tag{9}$$

where M is a sufficiently large constant and satisfied $nL \leq M$.

Calculating the Caputo derivative of V along the system in Equation (9), and using Property 1 and Property 3, we obtain that

$$D_t^\alpha V \leq \sum_{i=1}^{n} e_i D_t^\alpha e_i + \frac{k(t) + M}{\gamma} D_t^\alpha (k(t) + M) = \sum_{i=1}^{n} e_i D_t^\alpha e_i + \frac{k(t) + M}{\gamma} D_t^\alpha k(t). \tag{10}$$

In view of the error system (4) with $u_d = 0$, we conclude that

$$\begin{aligned} D_t^\alpha V &\leq \sum_{i=1}^{n} e_i (f_i(x) - f_i(y) + k(t) e_i) + \frac{k(t) + M}{\gamma} D_t^\alpha k(t) \\ &= \sum_{i=1}^{n} e_i (f_i(x) - f_i(y)) + \sum_{i=1}^{n} k(t) e_i^2 + \frac{k(t) + M}{\gamma} D_t^\alpha k(t). \end{aligned}$$

Combining Property 2 and the condition given in Equation (7), we get

$$D_t^\alpha V \leq nL \sum_{i=1}^{n} e_i^2 + \sum_{i=1}^{n} k(t) e_i^2 - (k(t) + M) \sum_{i=1}^{n} e_i^2 = (nL - M) \sum_{i=1}^{n} e_i^2.$$

Applying the condition of $nL \leq M$ and using the following notation $V_1(t) = \sum\limits_{i=1}^{n} e_i^2 = \|e\|^2$ yield that

$$D_t^\alpha V \leq (nL - M) V_1(t) \leq 0. \tag{11}$$

In the sequel, there are two cases for $\lim\limits_{t \to \infty} \int_{t_0}^{t} V_1(s) ds$. As in the proof of Theorem 1 in [44], we get the desired conclusion that $\lim\limits_{t \to +\infty} V_1(t) = \lim\limits_{t \to +\infty} \|e(t)\| = 0$.

Therefore, the drive system (2) with $u_d = 0$ and the response system (3) achieve complete synchronization. □

In the next, we investigate the stabilization of the error system (4) and present the following result.

Theorem 2. *Consider the error system (4). If $(f(x) - f(y), B)$ can be stabilized and there exists a suitable filter $g_f(t)$ such that*

$$\tilde{u}_d = u_d - \hat{u}_d \to 0, \ t \to \infty \tag{12}$$

where

$$\hat{u}_d = u_d * g_f(t) = (D_t^\alpha e - F(x, e) - B u_{ude}) * g_f(t) \tag{13}$$

and u_d satisfies the following structural constraints

$$[I_n - BB^+] u_d \equiv 0 \tag{14}$$

I_n is the identity matrix of order n, then the UDE-based controller u is designed as

$$u = u_s + u_{ude} \tag{15}$$

where

$$u_s = K(t)e(t) = k(t)B^T e(t) \tag{16}$$

$$u_{ude} = B^+ \left\{ \ell^{-1}\left[\frac{G_f(s)}{1-G_f(s)}\right] * F(x,e) - \ell^{-1}\left[\frac{s^\alpha G_f(s)}{1-G_f(s)}\right] * e(t) \right\},\ 0<\alpha\le 1 \tag{17}$$

$F(x,e) = f(x) - f(y) + Bu_s$, $B^+ = (B^TB)^{-1}B^T$, $G_f(s) = \ell[g_f(t)]$, ℓ *represents Laplace transform,* ℓ^{-1} *represents Laplace inverse transform,* $*$ *represents convolution, and the feedback* $k(t)$ *is updated by*

$$D_t^\alpha k(t) \le -\gamma \sum_{i=1}^{n} e_i^2$$

and

$$\dot{k} = -\gamma \|e(t)\|^2,\ \gamma > 0. \tag{18}$$

Proof. Taking into account the controller u in Equation (16) and the error system (4), one has

$$D_t^\alpha e(t) = F(x,e) + u_d + Bu_{ude}.$$

It is noted that

$$u_d = D_t^\alpha e(t) - F(x,e) - Bu_{ude}$$

and the system $D_t^\alpha e(t) = F(x,e)$ is asymptotically stable according to Theorem 1.

According to condition given in Equation (12), if the controller u_{ude} meets the following equation

$$Bu_{ude} = -\hat{u}_d = -u_d * g_f(t) = -(D_t^\alpha e(t) - F(x,e) - Bu_{ude}) * g_f(t) \tag{19}$$

then this controller is proposed.

Taking the Laplace transform of both sides of Equation (19), it yields that

$$Bu_{ude}(s) = -s^\alpha e(s)G_f(s) + F(s)G_f(s) + Bu_{ude}(s)G_f(s)$$

i.e.,

$$Bu_{ude}(s) - Bu_{ude}(s)G_f(s) = -s^\alpha e(s)G_f(s) + F(s)G_f(s).$$

Furthermore, we obtain

$$u_{ude}(s) = B^+ \left\{ \left[\frac{G_f(s)}{1-G_f(s)}\right] F(s) - \left[\frac{s^\alpha G_f(s)}{1-G_f(s)}\right] e(s) \right\} \tag{20}$$

that is

$$u_{ude}(t) = B^+ \left\{ \ell^{-1}\left[\frac{G_f(s)}{1-G_f(s)}\right] * F(x,e) \right\} - B^+ \left\{ \ell^{-1}\left[\frac{s^\alpha G_f(s)}{1-G_f(s)}\right] * e(t) \right\},\ 0<\alpha\le 1 \tag{21}$$

which completes the proof. □

4. Numerical Simulations

Next, we take the famous fractional-order Lorenz system as an illustrative example to show our proposed results.

Example 1. *Consider the controlled fractional-order Lorenz system with both uncertainty and disturbance*

$$D_t^\alpha x = f(x) + u_d + Bu \tag{22}$$

where $x \in R^3$ *is the state, and* $u_d = \Delta f(x) + d(t)$*, i.e.,*

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} = \begin{pmatrix} 10(x_2 - x_1) \\ 28x_1 - x_2 - x_1x_3 \\ -\frac{8}{3}x_3 + x_1x_2 \end{pmatrix}, \Delta f(x) = \begin{pmatrix} 0 \\ 0.3x_1x_2 \\ 0 \end{pmatrix}, \ d(t) = \begin{pmatrix} 0 \\ 10 \\ 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \tag{23}$$

System (22) is chosen as the drive system, the corresponding response system is noted as

$$D_t^\alpha y = f(y) \tag{24}$$

where $y \in R^3$ is a state vector, and

$$f(y) = \begin{pmatrix} f_1(y) \\ f_2(y) \\ f_3(y) \end{pmatrix} = \begin{pmatrix} 10(y_2 - y_1) \\ 28y_1 - y_2 - y_1y_3 \\ -\frac{8}{3}y_3 + y_1y_2 \end{pmatrix}. \tag{25}$$

The error dynamical system ($e = x - y$) is presented as

$$D_t^\alpha e = F(x, e) + u_d + Bu \tag{26}$$

where $e \in R^3$ is the state, and

$$F(x, e) = \begin{pmatrix} 10(e_2 - e_1) \\ 28e_1 - e_2 - x_1e_3 - x_3e_1 + e_1e_3 \\ -\frac{8}{3}e_3 - e_1e_2 + x_1e_2 + x_2e_1 \end{pmatrix}, \ u_d = \Delta f(x) + d(t). \tag{27}$$

In order to design the controller u, we firstly investigate complete synchronization of the following fractional-order nominal Lorenz system

$$D_t^\alpha x = f(x) + Bu_s \tag{28}$$

where $x \in R^3$ is the state, and $f(x), B$ are given in Equation (23), u_s is the controller to be designed.

Suppose that system (28) is the drive system, the corresponding response system is in the form of (24). Let $e = x - y$, then the nominal error system is

$$D_t^\alpha e = F(x, e) + Bu_s \tag{29}$$

where $e \in R^3$ is the state, B is given in Equation (23), $F(x, e)$ is given in Equation (27), u_s is given in Equation (28).

Note that if $e_2 = 0$, then the following two-dimensional system

$$\begin{aligned} D_t^\alpha e_1 &= -10e_1 \\ D_t^\alpha e_3 &= -\tfrac{8}{3}e_3 \end{aligned}$$

is asymptotically stable.

Therefore, $(F(x, e),\ B)$ can be stabilized. Based on Theorem 1, the controller u_s can be designed as

$$u_s = k(t)B^T e = k(t)\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} e = k(t)e_2 \tag{30}$$

where $k(t)$ is defined in Equation (18).

Substituting the controller u_s given in Equation (30) into the error system (29), the controlled error system is presented as follows

$$\begin{aligned} D_t^\alpha e_1 &= 10(e_2 - e_1) \\ D_t^\alpha e_2 &= 28e_1 - e_2 - x_1e_3 - x_3e_1 + e_1e_3 + k(t)e_2 \\ D_t^\alpha e_3 &= -\tfrac{8}{3}e_3 - e_1e_2 + x_1e_2 + x_2e_1. \end{aligned} \tag{31}$$

Numerical simulation is carried out by choosing the initial conditions of the drive system (28): $x(0) = [5, -4, 3]^T$, the initial conditions of the response system (24): $y(0) = [-1, -1, -1]^T$, and $k(0) = -1$, $\gamma = 1$, $\alpha = 0.95$. Figure 1 shows that the error system (29) is asymptotically stable, that is to say, the drive system (28) and the response system (24) realize complete synchronization. The states of the drive system (28) and the response system (24) are displayed in Figure 2, respectively. Figure 3 demonstrates $k(t)$ converges to a constant.

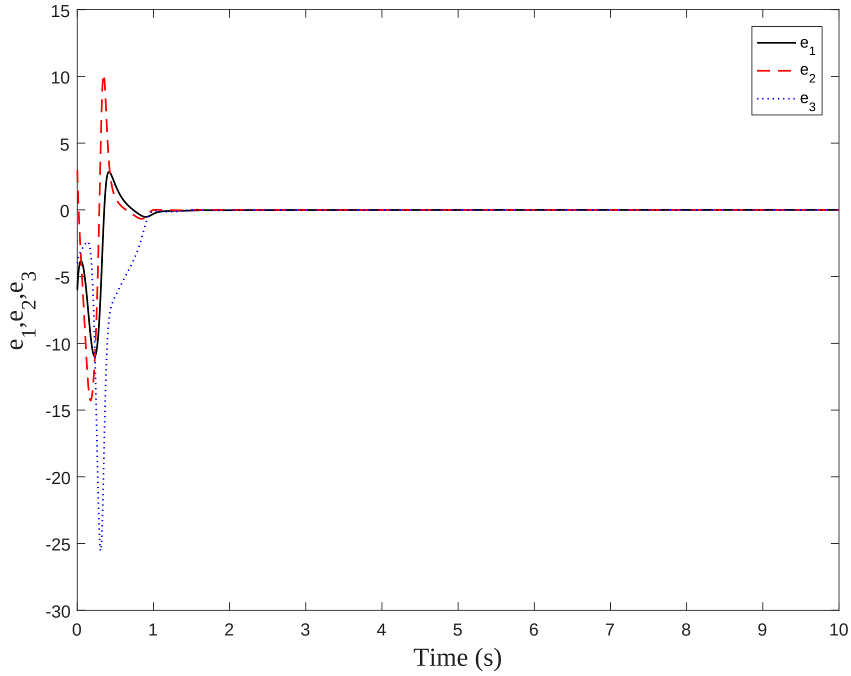

Figure 1. The error system (29) is asymptotically stable.

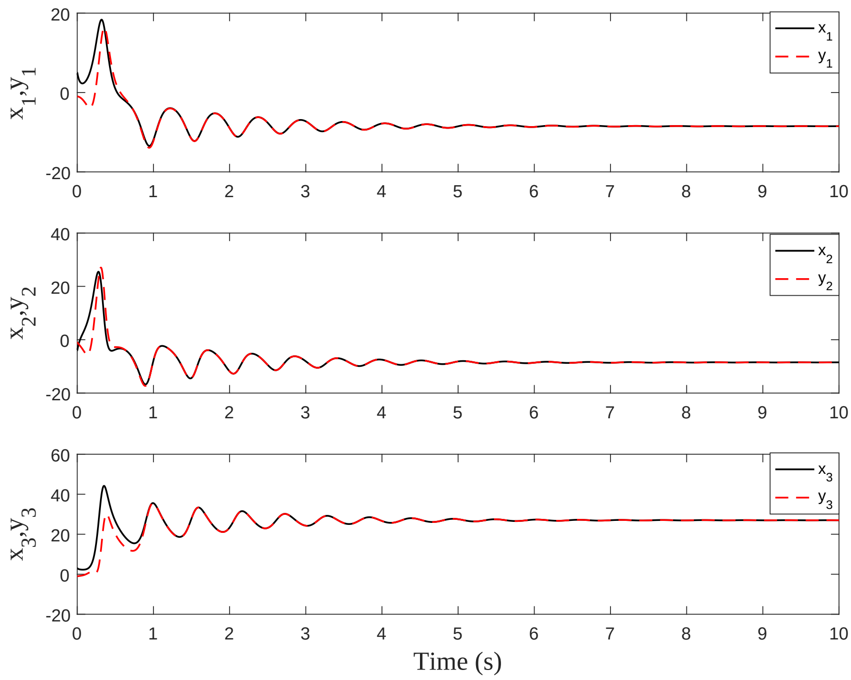

Figure 2. The states of the drive system (28) and the response system (24).

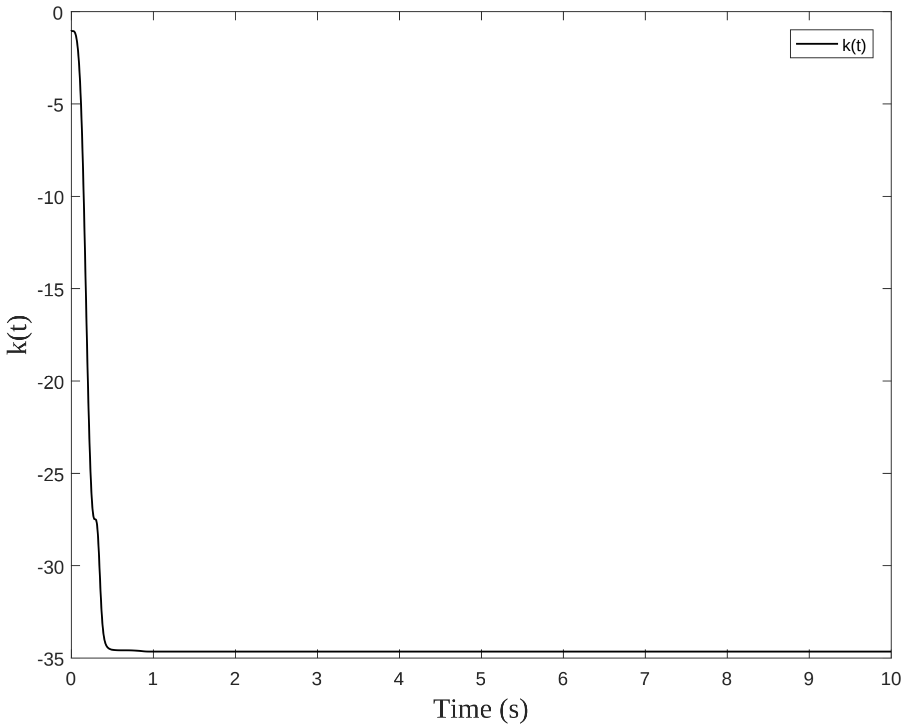

Figure 3. $k(t)$ tends to a constant.

Noticing that the structural constraint condition (14) is satisfied, according to Theorem 2, the controller u_{ude} is designed as

$$u_{ude}(t) = B^{+}\left\{\ell^{-1}\left[\frac{G_f(s)}{1-G_f(s)}\right] * F(x,e)\right\} - B^{+}\left\{\ell^{-1}\left[\frac{s^{\alpha}G_f(s)}{1-G_f(s)}\right] * e(t)\right\},\ 0<\alpha\leq 1 \tag{32}$$

where the filter $G_f(s)$ is proposed as

$$G_f = \frac{1}{\tau s+1},\quad \tau = 0.001.$$

Therefore, the controlled Lorenz system is

$$\begin{aligned} D_t^{\alpha}e_1 &= 10(e_2-e_1) \\ D_t^{\alpha}e_2 &= 28e_1 - e_2 - x_1e_3 - x_3e_1 + e_1e_3 + 0.3x_1x_2 + 10 + k(t)e_2 + u_{ude} \\ D_t^{\alpha}e_3 &= -\tfrac{8}{3}e_3 - e_1e_2 + x_1e_2 + x_2e_1. \end{aligned}$$

Numerical simulation is done with the initial conditions of the drive system (22) and the response system (24): $x(0) = [5,-4,3]^T$, $y(0) = [-1,-1,-1]^T$, respectively, and $k(0) = -1$, $\gamma = 1$, $\alpha = 0.95$. Figure 4 displays the synchronization error $e(t)$ tends to zero, which implies that the drive system (22) and the response system (24) reach complete synchronization. Figure 5 demonstrates the states of the drive system (22) and the response system (24), respectively. Figure 6 represents that the feedback gain $k(t)$ tends to a constant. From Figure 7, it is clear that $\hat{u}_d$ converges asymptotically to u_d.

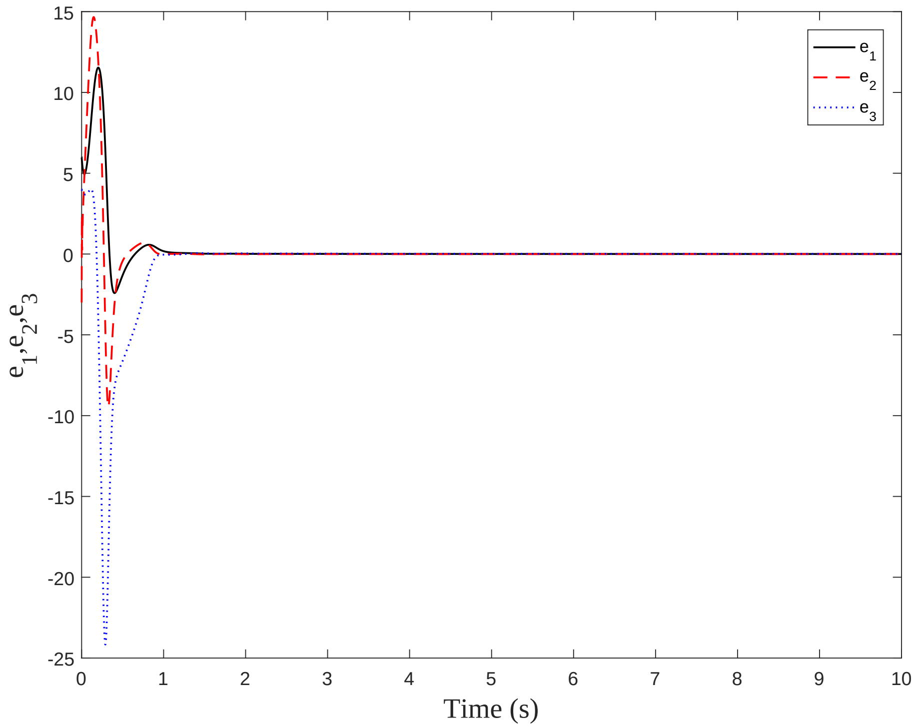

Figure 4. The error system (26) is asymptotically stable.

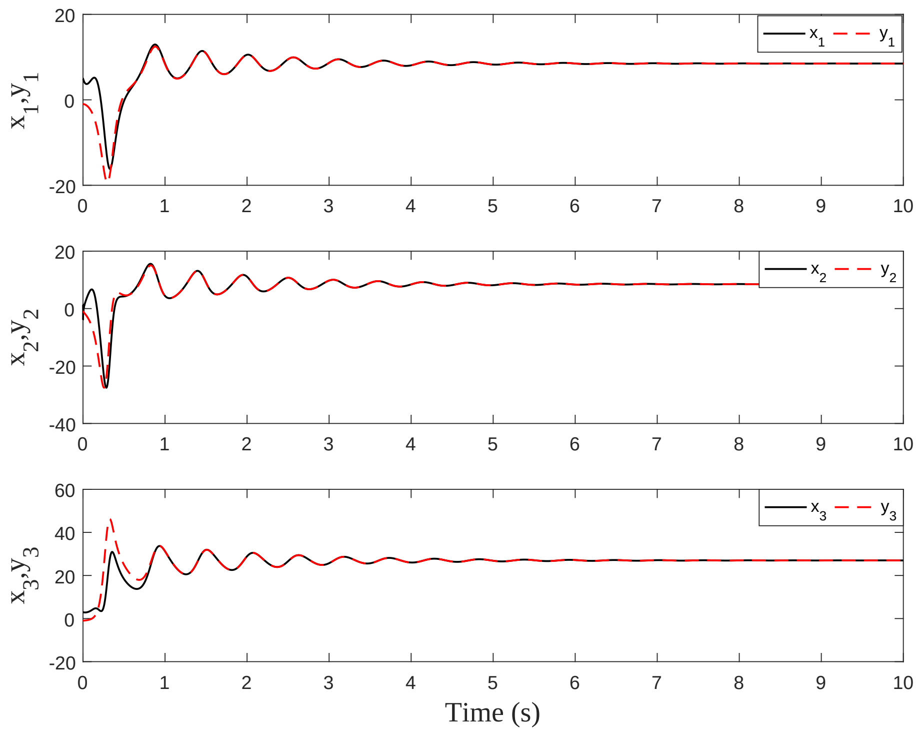

Figure 5. The states of the drive system (22) and the response system (24).

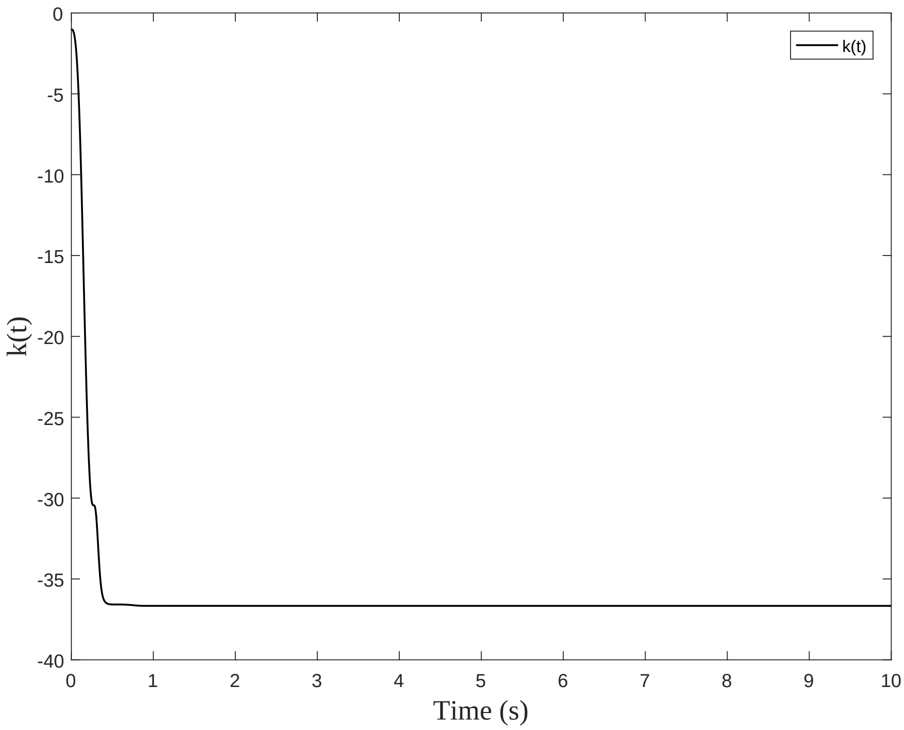

Figure 6. $k(t)$ tends to a constant.

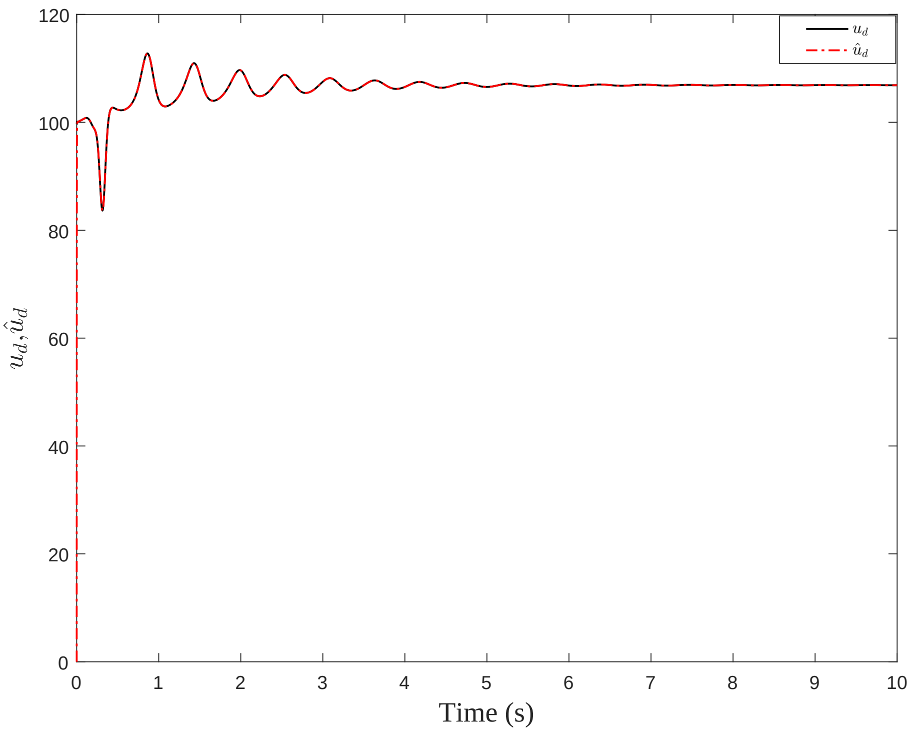

Figure 7. $\hat{u}_d$ converges asymptotically to u_d.

5. Conclusions

This paper has investigated complete synchronization of the fractional-order systems with both model uncertainty and external disturbance. Based on the fractional-order nominal systems, we propose a new dynamic feedback control method. Then, a new UDE-based control method for the fractional-order system has been obtained by extending the existing UDE-based control method. Finally, complete synchronization of the fractional-order chaotic systems has been realized by the above method, and an illustrative example has been used to show the practicability of the obtained results. The simulation results have shown that the UDE-based dynamic feedback control method has good performance.

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