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Duality Theorems for (ρ, ψ, d)-Quasiinvex Multiobjective Optimization Problems with Interval-Valued Components

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Abstract: The present paper deals with a duality study associated with a new class of multiobjective optimization problems that include the interval-valued components of the ratio vector. More precisely, by using the new notion of (ρ, ψ, d)-quasiinvexity associated with an interval-valued multiple-integral functional, we formulate and prove weak, strong, and converse duality results for the considered class of variational control problems.

Keywords: multiobjective fractional control problem; LU-efficiency conditions; multiple-integral functional; duality; (ρ, ψ, d)-quasiinvexity

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1. Introduction

Due to the effectiveness of duality theory in mathematical programming, it has been extended to more general classes of functions. In this sense, we mention the classical works of Hanson [1], Mond and Hanson [2], Mond and Smart [3], Aggarwal et al. [4], and Mukherjee and Rao [5]. The multiobjective optimization problems with mixed constraints have been investigated by many researchers, and we would be dishonest by specifying only some of them: Zhian and Qingkai [6], Treanţă and Udrişte [7], Zalmai [8], Hachimi and Aghezzaf [9], and Treanţă [10–12]. For various contributions and approaches to multiobjective variational control problems, the reader is directed to Chen [13], Kim and Kim [14], Gulati et al. [15], Nahak and Nanda [16], Antczak and Arana-Jiménez [17], Antczak [18], Khazafi et al. [19], Zhang et al. [20], and Treanţă and Arana-Jiménez [21].

As is well known, the concept of quasiinvexity is a generalization of the notion of quasiconvexity. It (and some modified versions of it) played a fundamental role in formulating and demonstrating sufficient efficiency conditions for certain classes of variational problems (see, for instance, Mititelu [22], Mititelu and Treanţă [23], and Treanţă and Mititelu [24]). In this paper, by considering the new notion of (ρ, ψ, d)-quasiinvexity associated with an interval-valued multiple-integral functional, we establish Mond–Weir weak, strong, and converse duality results for a new class of multiobjective optimization problems with interval-valued components of the ratio vector. The duality model considered in this paper includes a partition associated with a set of indices used for the inequality type constraints. This element is new in multidimensional multiobjective interval-valued optimization problems. In addition, another novel element of this work is represented by the necessary LU-efficiency conditions derived by using a recent research paper of the author (see Treanţă [25]). More precisely, compared with previous research works (see [7,10–12,21,24,25]), the present paper deals with the duality study associated with a new class of multiobjective optimization problems including interval-valued components of the ratio vector. These three highlighted elements, considered together at the same time, are totally new in the related literature. In addition, to illustrate the effectiveness of the results derived in the paper, an example is provided.

The paper is organized as follows: Section 2 includes the notations, preliminary mathematical tools, and formulation of the problem that we are going to study; Section 3 contains the main results of this paper: Mond–Weir weak, strong, and converse duality results are formulated and proved for the new class of multiobjective optimization problems. Finally, Section 4 concludes the paper.

2. Preliminaries and Problem Formulation

In the following, we consider the compact domain Ω in the Euclidean real space $\mathbb{R}^m$, and denote by $t = (t^\alpha)$, $\alpha = \overline{1,m}$, $u = (u^j)$, $j \in \overline{1,k}$, and $a = (a^i)$, $i = \overline{1,n}$, the points in Ω, $\mathbb{R}^k$ and $\mathbb{R}^n$, respectively. Now, we define the following continuously differentiable functions:

$$\mathcal{X} = \left(\mathcal{X}^i_\alpha\right) : \Omega \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^{nm}, \quad i = \overline{1,n},\ \alpha = \overline{1,m}$$

$$\mathcal{Y} = (\mathcal{Y}_1, \dots, \mathcal{Y}_q) = (\mathcal{Y}_\beta) : \Omega \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^q, \quad \beta = \overline{1,q}$$

and we accept that the following Lagrange densities

$$\mathcal{X}_\alpha = \left(\mathcal{X}^i_\alpha\right) : \mathcal{P} \to \mathbb{R}^n, \quad i = \overline{1,n},\ \alpha = \overline{1,m}$$

satisfy the closeness conditions (complete integrability conditions)

$$D_\zeta \mathcal{X}^i_\alpha = D_\alpha \mathcal{X}^i_\zeta, \quad \alpha, \zeta = \overline{1,m},\ \alpha \neq \zeta,\ i = \overline{1,n},$$

where D_ζ is the total derivative operator. We denote by $\mathcal{A}$ the space of all piecewise smooth state functions $a : \Omega \to \mathbb{R}^n$, and by $\mathcal{U}$ the space of all piecewise continuous control functions $u : \Omega \to \mathbb{R}^k$, endowed with the induced norm. By $dt := dt^1 dt^2 \cdots dt^m$, we denote the volume element on $\mathbb{R}^m \supset \Omega$. In addition, in this paper, for any two p-tuples $l = (l_1, \dots, l_p)$, $c = (c_1, \dots, c_p)$ in $\mathbb{R}^p$, we use the following partial ordering:

$$l = c \Leftrightarrow l_r = c_r, \quad l \le c \Leftrightarrow l_r \le c_r,$$

$$l < c \Leftrightarrow l_r < c_r, \quad l \preceq c \Leftrightarrow l \le c,\ l \neq c, \quad r = \overline{1,p}.$$

Consider that K is the set of all closed and bounded real intervals. We denote by $I = [i^L, i^U]$ a closed and bounded real interval, where i^L and i^U indicate the lower and upper bounds of I, respectively. Throughout this paper, the interval operations are performed as follows:

(1) $I = J \Longrightarrow i^L = j^L$ and $i^U = j^U$;
(2) if $i^L = i^U = i$ then $I = [i, i] = i$;
(3) $I + J = [i^L + j^L, i^U + j^U]$;
(4) $-I = -[i^L, i^U] = [-i^U, -i^L]$;
(5) $I - J = [i^L - j^U, i^U - j^L]$;
(6) $k + I = [k + i^L, k + i^U]$, $k \in \mathbb{R}$;
(7) $kA = [ka^L, ka^U]$, $k \in \mathbb{R}$, $k \ge 0$;
(8) $kA = [ka^U, ka^L]$, $k \in \mathbb{R}$, $k < 0$;
(9) $I/J = [i^L/j^L, i^U/j^U]$, $j^L, J^U > 0$.

Definition 1. *Let $I, J \in K$ be two closed and bounded real intervals. We write $I \le J$ if and only if $i^L \le j^L$ and $i^U \le j^U$.*

Definition 2. *Let $I, J \in K$ be two closed and bounded real intervals. We write $I < J$ if and only if $i^L < j^L$ and $i^U < j^U$.*

Definition 3. *A function $f : \Omega \times \mathbb{R}^n \times \mathbb{R}^k \to K$, defined by*

$$f(t, a(t), u(t)) = [f^L(t, a(t), u(t)), f^U(t, a(t), u(t))], \quad t \in \Omega,$$

where $f^L(t,a(t),u(t))$ and $f^U(t,a(t),u(t))$ are real-valued functions and satisfy the condition $f^L(t,a(t),u(t)) \leq f^U(t,a(t),u(t)),\ t \in \Omega$, is said to be an interval-valued function.

In the following (in accordance with Mititelu and Treanţă [23], Treanţă [25]), in order to formulate and prove the main results included in this paper, we introduce the concept of (ρ,ψ,d)-quasiinvexity associated with an interval-valued multiple-integral functional. Consider an interval-valued continuously differentiable function

$$h : \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \times \mathbb{R}^k \to K,$$

$$h = h(t,a(t),a_\alpha(t),u(t)) = [h^L(t,a(t),a_\alpha(t),u(t)), h^U(t,a(t),a_\alpha(t),u(t))],$$

where $a_\alpha(t) := \dfrac{\partial a}{\partial t^\alpha}(t)$, and for $a \in \mathcal{A}$ and $u \in \mathcal{U}$, we introduce the following interval-valued multiple-integral functional:

$$H : \mathcal{A} \times \mathcal{U} \to K, \quad H(a,u) = \int_\Omega h(t,a(t),a_\alpha(t),u(t))dt$$

$$= \left[\int_\Omega h^L(t,a(t),a_\alpha(t),u(t))dt, \int_\Omega h^U(t,a(t),a_\alpha(t),u(t))dt\right].$$

In addition, let ρ be a real number, $\psi : \mathcal{A} \times \mathcal{U} \times \mathcal{A} \times \mathcal{U} \to [0,\infty)$ a positive functional, and $d\Big((a,u),(a^0,u^0)\Big)$ a real-valued function on $(\mathcal{A} \times \mathcal{U})^2$.

Definition 4. *(i) If there exist*

$$\zeta : \Omega \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n,$$

$$\zeta = \zeta\Big(t,a(t),u(t),a^0(t),u^0(t)\Big) = \Big(\zeta_i\Big(t,a(t),u(t),a^0(t),u^0(t)\Big)\Big), \quad i = 1,\ldots,n,$$

of the C^1-class with $\zeta\Big(t,a^0(t),u^0(t),a^0(t),u^0(t)\Big) = 0,\ \forall t \in \Omega,\ \zeta|_{\partial\Omega} = 0$, and

$$\kappa : \Omega \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^k,$$

$$\kappa = \kappa\Big(t,a(t),u(t),a^0(t),u^0(t)\Big) = \Big(\kappa_j\Big(t,a(t),u(t),a^0(t),u^0(t)\Big)\Big), \quad j = 1,\ldots,k,$$

of the C^0-class with $\kappa\Big(t,a^0(t),u^0(t),a^0(t),u^0(t)\Big) = 0,\ \forall t \in \Omega,\ \kappa|_{\partial\Omega} = 0$, such that for every $(a,u) \in \mathcal{A} \times \mathcal{U}$:

$$H(a,u) \leq H\Big(a^0,u^0\Big)$$

$$\Rightarrow \psi\Big(a,u,a^0,u^0\Big) \int_\Omega \Big[h_a^L\Big(t,a^0(t),a_\alpha^0(t),u^0(t)\Big), h_a^U\Big(t,a^0(t),a_\alpha^0(t),u^0(t)\Big)\Big]\zeta dt$$

$$+\psi\Big(a,u,a^0,u^0\Big) \int_\Omega \Big[h_{a_\alpha}^L\Big(t,a^0(t),a_\alpha^0(t),u^0(t)\Big), h_{a_\alpha}^U\Big(t,a^0(t),a_\alpha^0(t),u^0(t)\Big)\Big] D_\alpha\zeta dt$$

$$+\psi\Big(a,u,a^0,u^0\Big) \int_\Omega \Big[h_u^L\Big(t,a^0(t),a_\alpha^0(t),u^0(t)\Big), h_u^U\Big(t,a^0(t),a_\alpha^0(t),u^0(t)\Big)\Big]\kappa dt$$

$$+\rho\psi\Big(a,u,a^0,u^0\Big)d^2\Big((a,u),(a^0,u^0)\Big) \leq [0,0]$$

or, equivalently,

$$\psi\Big(a,u,a^0,u^0\Big) \int_\Omega \Big[h_a^L\Big(t,a^0(t),a_\alpha^0(t),u^0(t)\Big), h_a^U\Big(t,a^0(t),a_\alpha^0(t),u^0(t)\Big)\Big]\zeta dt$$

$$+\psi\Big(a,u,a^0,u^0\Big) \int_\Omega \Big[h_{a_\alpha}^L\Big(t,a^0(t),a_\alpha^0(t),u^0(t)\Big), h_{a_\alpha}^U\Big(t,a^0(t),a_\alpha^0(t),u^0(t)\Big)\Big] D_\alpha\zeta dt$$

$$+\psi\Big(a,u,a^0,u^0\Big)\int_\Omega\Big[h_u^L\Big(t,a^0(t),a^0_\alpha(t),u^0(t)\Big),h_u^U\Big(t,a^0(t),a^0_\alpha(t),u^0(t)\Big)\Big]\kappa dt$$

$$+\rho\psi\Big(a,u,a^0,u^0\Big)d^2\Big((a,u),(a^0,u^0)\Big) > [0,0] \Rightarrow H(a,u) > H\Big(a^0,u^0\Big),$$

then H is said to be (ρ,ψ,d)*-quasiinvex at* $\Big(a^0,u^0\Big)\in\mathcal{A}\times\mathcal{U}$ *with respect to* ζ *and* κ*;*

(ii) If there exist

$$\zeta:\Omega\times\mathbb{R}^n\times\mathbb{R}^k\times\mathbb{R}^n\times\mathbb{R}^k\to\mathbb{R}^n,$$

$$\zeta=\zeta\Big(t,a(t),u(t),a^0(t),u^0(t)\Big)=\Big(\zeta_i\Big(t,a(t),u(t),a^0(t),u^0(t)\Big)\Big),\quad i=1,\dots,n,$$

of the C^1*-class with* $\zeta\Big(t,a^0(t),u^0(t),a^0(t),u^0(t)\Big)=0,\ \forall t\in\Omega,\ \zeta|_{\partial\Omega}=0$*, and*

$$\kappa:\Omega\times\mathbb{R}^n\times\mathbb{R}^k\times\mathbb{R}^n\times\mathbb{R}^k\to\mathbb{R}^k,$$

$$\kappa=\kappa\Big(t,a(t),u(t),a^0(t),u^0(t)\Big)=\Big(\kappa_j\Big(t,a(t),u(t),a^0(t),u^0(t)\Big)\Big),\quad j=1,\dots,k,$$

of the C^0*-class with* $\kappa\Big(t,a^0(t),u^0(t),a^0(t),u^0(t)\Big)=0,\ \forall t\in\Omega,\ \kappa|_{\partial\Omega}=0$*, such that for every* $(a,u)\neq(a^0,u^0)\in\mathcal{A}\times\mathcal{U}$*:*

$$H(a,u)\le H\Big(a^0,u^0\Big)$$

$$\Rightarrow\psi\Big(a,u,a^0,u^0\Big)\int_\Omega\Big[h_a^L\Big(t,a^0(t),a^0_\alpha(t),u^0(t)\Big),h_a^U\Big(t,a^0(t),a^0_\alpha(t),u^0(t)\Big)\Big]\zeta dt$$

$$+\psi\Big(a,u,a^0,u^0\Big)\int_\Omega\Big[h_{a_\alpha}^L\Big(t,a^0(t),a^0_\alpha(t),u^0(t)\Big),h_{a_\alpha}^U\Big(t,a^0(t),a^0_\alpha(t),u^0(t)\Big)\Big]D_\alpha\zeta dt$$

$$+\psi\Big(a,u,a^0,u^0\Big)\int_\Omega\Big[h_u^L\Big(t,a^0(t),a^0_\alpha(t),u^0(t)\Big),h_u^U\Big(t,a^0(t),a^0_\alpha(t),u^0(t)\Big)\Big]\kappa dt$$

$$+\rho\psi\Big(a,u,a^0,u^0\Big)d^2\Big((a,u),(a^0,u^0)\Big) < [0,0]$$

or, equivalently,

$$\psi\Big(a,u,a^0,u^0\Big)\int_\Omega\Big[h_a^L\Big(t,a^0(t),a^0_\alpha(t),u^0(t)\Big),h_a^U\Big(t,a^0(t),a^0_\alpha(t),u^0(t)\Big)\Big]\zeta dt$$

$$+\psi\Big(a,u,a^0,u^0\Big)\int_\Omega\Big[h_{a_\alpha}^L\Big(t,a^0(t),a^0_\alpha(t),u^0(t)\Big),h_{a_\alpha}^U\Big(t,a^0(t),a^0_\alpha(t),u^0(t)\Big)\Big]D_\alpha\zeta dt$$

$$+\psi\Big(a,u,a^0,u^0\Big)\int_\Omega\Big[h_u^L\Big(t,a^0(t),a^0_\alpha(t),u^0(t)\Big),h_u^U\Big(t,a^0(t),a^0_\alpha(t),u^0(t)\Big)\Big]\kappa dt$$

$$+\rho\psi\Big(a,u,a^0,u^0\Big)d^2\Big((a,u),(a^0,u^0)\Big) \ge [0,0] \Rightarrow H(a,u) > H\Big(a^0,u^0\Big),$$

then H is said to be strictly (ρ,ψ,d)*-quasiinvex at* $\Big(a^0,u^0\Big)\in\mathcal{A}\times\mathcal{U}$ *with respect to* ζ *and* κ*.*

Consider the vector continuously differentiable function

$$h:\Omega\times\mathbb{R}^n\times\mathbb{R}^{nm}\times\mathbb{R}^k\to K^p,$$

$$h=\Big(h_r(t,a(t),a_\alpha(t),u(t))\Big)$$

$$=\Big([h_1^L(t,a(t),a_\alpha(t),u(t)),h_1^U(t,a(t),a_\alpha(t),u(t))],$$

$$\cdots,[h_p^L(t,a(t),a_\alpha(t),u(t)),h_p^U(t,a(t),a_\alpha(t),u(t))]\Big).$$

Definition 5. *The vector multiple-integral functional*

$$H : \mathcal{A} \times \mathcal{U} \to K^p, \quad H(a,u) = \int_\Omega h(t, a(t), a_\alpha(t), u(t)) dt$$

$$= \Big([\int_\Omega h_1^L(t, a(t), a_\alpha(t), u(t)) dt, \int_\Omega h_1^U(t, a(t), a_\alpha(t), u(t)) dt],$$

$$\cdots, [\int_\Omega h_p^L(t, a(t), a_\alpha(t), u(t)) dt, \int_\Omega h_p^U(t, a(t), a_\alpha(t), u(t)) dt] \Big)$$

is said to be (ρ, ψ, d)*-quasiinvex (strictly* (ρ, ψ, d)*-quasiinvex) at* $\left(a^0, u^0\right) \in \mathcal{A} \times \mathcal{U}$ *with respect to* ζ *and* κ *if each interval-valued component of the vector is* (ρ, ψ, d)*-quasiinvex (strictly* (ρ, ψ, d)*-quasiinvex) at* $\left(a^0, u^0\right) \in \mathcal{A} \times \mathcal{U}$ *with respect to* ζ *and* κ*.*

Consider the following vector continuously differentiable functions:

$$f = (f_1, \ldots, f_p) = (f_r) : \Omega \times \mathbb{R}^n \times \mathbb{R}^k \to K^p, \quad r = \overline{1,p}$$

$$g = (g_1, \ldots, g_p) = (g_r) : \Omega \times \mathbb{R}^n \times \mathbb{R}^k \to K^p, \quad r = \overline{1,p}.$$

Now, we are in a position to formulate the following new class of multiobjective fractional variational control problems with interval-valued components, which are called a Primal Problem (in short, PP):

$$(PP) \qquad \min_{(a,u)} \left\{ \mathcal{K}(a,u) = \left(\frac{\int_\Omega f_1(t, a(t), u(t)) dt}{\int_\Omega g_1(t, a(t), u(t)) dt}, \ldots, \frac{\int_\Omega f_p(t, a(t), u(t)) dt}{\int_\Omega g_p(t, a(t), u(t)) dt} \right) \right\}$$

subject to

$$\frac{\partial a^i}{\partial t^\alpha}(t) = \mathcal{X}_\alpha^i(t, a(t), u(t)), \quad i = \overline{1,n}, \ \alpha = \overline{1,m}, \ t \in \Omega \tag{1}$$

$$\mathcal{Y}(t, a(t), u(t)) \le 0, \quad t \in \Omega \tag{2}$$

$$a(t)|_{\partial\Omega} = \varphi(t) = \text{ given}, \tag{3}$$

where, for $r = \overline{1,p}$, we have denoted

$$\frac{\int_\Omega f_r(t, a(t), u(t)) dt}{\int_\Omega g_r(t, a(t), u(t)) dt} := \left[\frac{\int_\Omega f_r^L(t, a(t), u(t)) dt}{\int_\Omega g_r^L(t, a(t), u(t)) dt}, \frac{\int_\Omega f_r^U(t, a(t), u(t)) dt}{\int_\Omega g_r^U(t, a(t), u(t)) dt} \right],$$

or, equivalently,

$$\frac{F_r(a,u)}{G_r(a,u)} := \frac{[F_r^L(a,u), F_r^U(a,u)]}{[G_r^L(a,u), G_r^U(a,u)]}$$

and it is assumed that $G_r(a,u) > [0,0]$, $\forall (a,u) \in \mathcal{A} \times \mathcal{U}$.

The set of all feasible solutions in (PP) is defined by

$$\mathcal{D} := \{(a,u) | a \in \mathcal{A}, \ u \in \mathcal{U} \ \text{ satisfying } \ (1), \ (2), \ (3)\}.$$

Definition 6. *A feasible solution* $(a^0, u^0) \in \mathcal{D}$ *in* (PP) *is called an LU-efficient solution if there is no other* $(a,u) \in \mathcal{D}$ *such that* $\mathcal{K}(a,u) \preceq \mathcal{K}(a^0, u^0)$*.*

Taking Treanţă [25] and Mititelu and Treanţă [23] into account, under constraint qualification assumptions, if $(a^0, u^0) \in \mathcal{D}$ is an LU-efficient solution of the variational control problem (PP), then there exist $\theta = (\theta^r_\iota)$, $\mu(t)$ and $\lambda(t)$, with $\mu(t) = (\mu^\beta(t))$, $\lambda(t) = (\lambda^\alpha_i(t))$ piecewise smooth functions satisfying the following conditions (see Einstein summation):

$$\theta^r_\iota\Big[G^\iota_r(a^0,u^0)\frac{\partial f^\iota_r}{\partial a^i}\Big(t,a^0(t),u^0(t)\Big) - F^\iota_r(a^0,u^0)\frac{\partial g^\iota_r}{\partial a^i}\Big(t,a^0(t),u^0(t)\Big)\Big]$$

$$+\lambda^\alpha_i(t)\frac{\partial \mathcal{X}^i_\alpha}{\partial a^i}\Big(t,a^0(t),u^0(t)\Big) + \mu^\beta(t)\frac{\partial \mathcal{Y}_\beta}{\partial a^i}\Big(t,a^0(t),u^0(t)\Big) + \frac{\partial \lambda^\alpha_i}{\partial t^\alpha}(t) = 0, \quad i=\overline{1,n},\ \iota = L,U \tag{4}$$

$$\theta^r_\iota\Big[G^\iota_r(a^0,u^0)\frac{\partial f^\iota_r}{\partial u^j}\Big(t,a^0(t),u^0(t)\Big) - F^\iota_r(a^0,u^0)\frac{\partial g^\iota_r}{\partial u^j}\Big(t,a^0(t),u^0(t)\Big)\Big]$$

$$+\lambda^\alpha_i(t)\frac{\partial \mathcal{X}^i_\alpha}{\partial u^j}\Big(t,a^0(t),u^0(t)\Big) + \mu^\beta(t)\frac{\partial \mathcal{Y}_\beta}{\partial u^j}\Big(t,a^0(t),u^0(t)\Big) = 0, \quad j=\overline{1,k},\ \iota = L,U \tag{5}$$

$$\mu^\beta(t)\mathcal{Y}_\beta\Big(t,a^0(t),u^0(t)\Big) = 0 \quad \text{(no summation)}, \quad (\theta,\mu(t)) \succeq 0, \tag{6}$$

for all $t \in \Omega$, except at discontinuities.

Definition 7. *The feasible solution $(a^0,u^0) \in \mathcal{D}$ is a normal LU-efficient solution for (PP) if the necessary LU-efficiency conditions formulated in $(4)-(6)$ hold for $\theta \succeq 0$ and $e^t\theta_\iota = 1$, $e^t = (1,\dots,1) \in \mathbb{R}^p$, $\iota = L,U$.*

3. Mond–Weir Duality Associated with (PP)

Consider that $\{\mathcal{Q}_1,\mathcal{Q}_2,\cdots,\mathcal{Q}_s\}$ is a partition of the set $\mathcal{Q} = \{1,2,\cdots,q\}$, where $s < q$. For $(b,w) \in \mathcal{A}\times\mathcal{U}$, with the same notations as in Section 2, we associate to (PP) the next multiobjective fractional variational control problem with interval-valued components of the vector, which is called the Dual Problem (in short, DP):

$$(DP) \qquad \max_{(b,w)}\left\{\mathcal{K}(b,w) = \left(\frac{\int_\Omega f_1(t,b(t),w(t))dt}{\int_\Omega g_1(t,b(t),w(t))dt},\dots,\frac{\int_\Omega f_p(t,b(t),w(t))dt}{\int_\Omega g_p(t,b(t),w(t))dt}\right)\right\}$$

subject to

$$\theta^r_\iota\Big[G^\iota_r(b,w)\frac{\partial f^\iota_r}{\partial b^i}(t,b(t),w(t)) - F^\iota_r(b,w)\frac{\partial g^\iota_r}{\partial b^i}(t,b(t),w(t))\Big]$$

$$+\lambda^\alpha_i(t)\frac{\partial \mathcal{X}^i_\alpha}{\partial b^i}(t,b(t),w(t)) + \mu^\beta(t)\frac{\partial \mathcal{Y}_\beta}{\partial b^i}(t,b(t),w(t)) + \frac{\partial \lambda^\alpha_i}{\partial t^\alpha}(t) = 0, \quad i=\overline{1,n},\ \iota = L,U \tag{7}$$

$$\theta^r_\iota\Big[G^\iota_r(b,w)\frac{\partial f^\iota_r}{\partial w^j}(t,b(t),w(t)) - F^\iota_r(b,w)\frac{\partial g^\iota_r}{\partial w^j}(t,b(t),w(t))\Big]$$

$$+\lambda^\alpha_i(t)\frac{\partial \mathcal{X}^i_\alpha}{\partial w^j}(t,b(t),w(t)) + \mu^\beta(t)\frac{\partial \mathcal{Y}_\beta}{\partial w^j}(t,b(t),w(t)) = 0, \quad j=\overline{1,k},\ \iota = L,U \tag{8}$$

$$\lambda^\alpha_i(t)\Big[\mathcal{X}^i_\alpha(t,b(t),w(t)) - \frac{\partial b^i}{\partial t^\alpha}(t)\Big] \geq 0 \tag{9}$$

$$\mu^{\mathcal{Q}_\vartheta}(t)\mathcal{Y}_{\mathcal{Q}_\vartheta}(t,b(t),w(t)) \geq 0, \quad \vartheta = \overline{1,s} \tag{10}$$

$$\theta = (\theta^r_\iota) \succeq 0, \quad \mu(t) = (\mu^\beta(t)) \geq 0, \quad b(t)|_{\partial\Omega} = \varphi(t) = \text{ given},\ \iota = L,U. \tag{11}$$

Remark 1. *In the previous dual problem, the expression $\mu^{\mathcal{Q}_\vartheta}(t)\mathcal{Y}_{\mathcal{Q}_\vartheta}(t,b(t),w(t))$ has the following meaning:*

$$\mu^{\mathcal{Q}_\vartheta}(t)\mathcal{Y}_{\mathcal{Q}_\vartheta}(t,b(t),w(t)) = \sum_{\beta\in\mathcal{Q}_\vartheta}\mu^\beta(t)\mathcal{Y}_\beta(t,b(t),w(t)).$$

In this section, we establish that the multiobjective optimization problems with interval-valued components of the ratio vector, (PP) and (DP), are a Mond–Weir (see [26]) dual pair under (ρ, ψ, d)-quasiinvexity hypotheses. Further, consider that $\triangle$ is the set of all feasible solutions associated with (DP).

Now, we formulate and prove the first duality result.

Theorem 1 (Weak Duality)**.** *Let $(a, u) \in \mathcal{D}$ be a feasible solution of the multiobjective variational control problem with interval-valued components (PP), and let $(b, w, \theta, \lambda, \mu) \in \triangle$ be a feasible solution of the multiobjective variational control problem with interval-valued components (DP). In addition, consider that the following conditions are fulfilled:*

(a) Each functional

$$\mathcal{F}^{b,w}_{r,\iota}(a, u) = \int_\Omega [G^\iota_r(b, w) f^\iota_r(t, a(t), u(t)) - F^\iota_r(b, w) g^\iota_r(t, a(t), u(t))]dt, \quad r = \overline{1, p}, \ \iota = L, U$$

is (ρ^1_r, ψ, d)-quasiinvex at (b, w) with respect to ζ and κ; or, equivalently, each interval-valued multiple-integral functional

$$\mathcal{F}^{b,w}_{r}(a, u) = [\mathcal{F}^{b,w}_{r,L}(a, u), \mathcal{F}^{b,w}_{r,U}(a, u)], \quad r = \overline{1, p}$$

is (ρ^1_r, ψ, d)-quasiinvex at (b, w) with respect to ζ and κ.

(b) The functional $\mathcal{X}(a, u) = \int_\Omega \lambda^\alpha_i(t) \left[\mathcal{X}^i_\alpha(t, a(t), u(t)) - \dfrac{\partial a^i}{\partial t^\alpha}(t)\right] dt$ is (ρ^2, ψ, d)-quasiinvex at (b, w) with respect to ζ and κ.

(c) Each functional

$$\mathcal{Y}_\vartheta(a, u) = \int_\Omega \mu^{\mathcal{Q}_\vartheta}(t) \mathcal{Y}_{\mathcal{Q}_\vartheta}(t, a(t), u(t))dt, \quad \vartheta = \overline{1, s}$$

is $(\rho^3_\vartheta, \psi, d)$-quasiinvex at (b, w) with respect to ζ and κ.

(d) At least one of the functionals given in $a) - c)$ is strictly (ρ, ψ, d)-quasiinvex at (b, w) with respect to ζ and κ, where $\rho = \rho^1_r, \ \rho^2$ or ρ^3_ϑ.

(e) $\theta^r_\iota \rho^1_r + \rho^2 + \sum_{\vartheta=1}^{s} \rho^3_\vartheta \geq 0 \quad (\rho^1_r, \rho^2, \rho^3_\vartheta \in \mathbb{R})$.

Then, the infimum of (PP) is greater than or equal to the supremum of (DP).

Proof. Denote by $\pi(a, u)$ and $\delta(b, w, \theta, \lambda, \mu)$ the value of problem (PP) at $(a, u) \in \mathcal{D}$ and the value of problem (DP) at $(b, w, \theta, \lambda, \mu) \in \triangle$, respectively. Contrary to the result, suppose that $\pi(a, u) \preceq \delta(b, w, \theta, \lambda, \mu)$. Further, for $r = \overline{1, p}, \ \iota = L, U$ and $\vartheta = \overline{1, s}$, consider the following non-empty set:

$$S = \left\{(a, u) \in \mathcal{A} \times \mathcal{U} \mid \mathcal{F}^{b,w}_{r,\iota}(a, u) \leq \mathcal{F}^{b,w}_{r,\iota}(b, w), \ \mathcal{X}(a, u) \leq \mathcal{X}(b, w), \ \mathcal{Y}_\vartheta(a, u) \leq \mathcal{Y}_\vartheta(b, w)\right\}.$$

Using (a), for $(a, u) \in S$ and $r = \overline{1, p}, \ \iota = L, U$, we get

$$\mathcal{F}^{b,w}_{r,\iota}(a, u) \leq \mathcal{F}^{b,w}_{r,\iota}(b, w) \Longrightarrow$$

$$\psi(a, u, b, w) \int_\Omega [G^\iota_r(b, w)(f^\iota_r)_b(t, b(t), w(t)) - F^\iota_r(b, w)(g^\iota_r)_b(t, b(t), w(t))]\zeta dt$$
$$+\psi(a, u, b, w) \int_\Omega [G^\iota_r(b, w)(f^\iota_r)_w(t, b(t), w(t)) - F^\iota_r(b, w)(g^\iota_r)_w(t, b(t), w(t))]\kappa dt$$
$$\leq -\rho^1_r \psi(a, u, b, w) d^2((a, u), (b, w)).$$

Multiplying by $\theta = (\theta_\iota^r) \succeq 0,\ \iota = L, U$ and making summation over $r = \overline{1,p}$, we find

$$\psi(a,u,b,w)\int_\Omega \theta_\iota^r [G_r^\iota(b,w)(f_r^\iota)_b(t,b(t),w(t)) - F_r^\iota(b,w)(g_r^\iota)_b(t,b(t),w(t))]\zeta dt$$
$$+\psi(a,u,b,w)\int_\Omega \theta_\iota^r [G_r^\iota(b,w)(f_r^\iota)_w(t,b(t),w(t)) - F_r^\iota(b,w)(g_r^\iota)_w(t,b(t),w(t))]\kappa dt$$
$$\leq -\theta_\iota^r \rho_r^1 \psi(a,u,b,w) d^2((a,u),(b,w)). \tag{12}$$

For $(a,u) \in S$, the inequality $\mathcal{X}(a,u) \leq \mathcal{X}(b,w)$ holds, and, according to $b)$, it follows that

$$\psi(a,u,b,w)\int_\Omega \Big[\lambda_i^\alpha(t)(\mathcal{X}_\alpha^i)_b(t,b(t),w(t))\zeta - \lambda^\alpha(t) D_\alpha \zeta + \lambda_i^\alpha(t)(\mathcal{X}_\alpha^i)_w(t,b(t),w(t))\kappa\Big] dt$$
$$\leq -\rho^2 \psi(a,u,b,w) d^2((a,u),(b,w)). \tag{13}$$

In addition, for $(a,u) \in S$, the inequality $\mathcal{Y}_\vartheta(a,u) \leq \mathcal{Y}_\vartheta(b,w)$, $\vartheta = \overline{1,s}$, gives (see $c)$)

$$\psi(a,u,b,w)\int_\Omega \Big[\mu^{\mathcal{Q}_\vartheta}(t)(\mathcal{Y}_{\mathcal{Q}_\vartheta})_b(t,b(t),w(t))\zeta + \mu^{\mathcal{Q}_\vartheta}(t)(\mathcal{Y}_{\mathcal{Q}_\vartheta})_w(t,b(t),w(t))\kappa\Big] dt$$
$$\leq -\rho_\vartheta^3 \psi(a,u,b,w) d^2((a,u),(b,w)),$$

and making summation over $\vartheta = \overline{1,s}$ in the previous inequality, it results that

$$\psi(a,u,b,w)\int_\Omega \Big[\mu^\beta(t)(\mathcal{Y}_\beta)_b(t,b(t),w(t))\zeta + \mu^\beta(t)(\mathcal{Y}_\beta)_w(t,b(t),w(t))\kappa\Big] dt$$
$$\leq -\sum_{\vartheta=1}^{s} \rho_\vartheta^3 \psi(a,u,b,w) d^2((a,u),(b,w)). \tag{14}$$

Making the sum (12) + (13) + (14), side by side, and taking $d)$ into account, we have

$$\psi(a,u,b,w)\int_\Omega \theta_\iota^r [G_r^\iota(b,w)(f_r^\iota)_b(t,b(t),w(t)) - F_r^\iota(b,w)(g_r^\iota)_b(t,b(t),w(t))]\zeta dt$$
$$+\psi(a,u,b,w)\int_\Omega \Big[\lambda_i^\alpha(t)(\mathcal{X}_\alpha^i)_b(t,b(t),w(t)) + \mu^\beta(t)(\mathcal{Y}_\beta)_b(t,b(t),w(t))\Big]\zeta dt$$
$$+\psi(a,u,b,w)\int_\Omega \theta^r [G_r^\iota(b,w)(f_r^\iota)_w(t,b(t),w(t)) - F_r^\iota(b,w)(g_r^\iota)_w(t,b(t),w(t))]\kappa dt$$
$$+\psi(a,u,b,w)\int_\Omega \Big[\lambda_i^\alpha(t)(\mathcal{X}_\alpha^i)_w(t,b(t),w(t)) + \mu^\beta(t)(\mathcal{Y}_\beta)_w(t,b(t),w(t))\Big]\kappa dt$$
$$-\psi(a,u,b,w)\int_\Omega [\lambda^\alpha(t) D_\alpha \zeta] dt < -(\theta_\iota^r \rho_r^1 + \rho^2 + \sum_{\vartheta=1}^{s} \rho_\vartheta^3)\psi(a,u,b,w) d^2((a,u),(b,w)),\ \iota = L, U.$$

The previous inequality implies $\psi(a,u,b,w) > 0$ and, as a consequence, we can rewrite it as

$$\int_\Omega \theta_\iota^r [G_r^\iota(b,w)(f_r^\iota)_b(t,b(t),w(t)) - F_r^\iota(b,w)(g_r^\iota)_b(t,b(t),w(t))]\zeta dt$$
$$+\int_\Omega \Big[\lambda_i^\alpha(t)(\mathcal{X}_\alpha^i)_b(t,b(t),w(t)) + \mu^\beta(t)(\mathcal{Y}_\beta)_b(t,b(t),w(t))\Big]\zeta dt$$
$$+\int_\Omega \theta_\iota^r [G_r^\iota(b,w)(f_r^\iota)_w(t,b(t),w(t)) - F_r^\iota(b,w)(g_r^\iota)_w(t,b(t),w(t))]\kappa dt$$
$$+\int_\Omega \Big[\lambda_i^\alpha(t)(\mathcal{X}_\alpha^i)_w(t,b(t),w(t)) + \mu^\beta(t)(\mathcal{Y}_\beta)_w(t,b(t),w(t))\Big]\kappa dt$$
$$-\int_\Omega [\lambda^\alpha(t) D_\alpha \zeta] dt < -(\theta_\iota^r \rho_r^1 + \rho^2 + \sum_{\vartheta=1}^{s} \rho_\vartheta^3) d^2((a,u),(b,w)),\ \iota = L, U.$$

Now, considering the constraints (7), (8) of (DP), we obtain

$$-\int_{\Omega}\zeta D_{\alpha}\lambda^{\alpha}(t)dt-\int_{\Omega}[\lambda^{\alpha}(t)D_{\alpha}\zeta]dt+0<-(\theta_{\iota}^{r}\rho_{r}^{1}+\rho^{2}+\sum_{\vartheta=1}^{s}\rho_{\vartheta}^{3}))d^{2}((a,u),(b,w)),\ \iota=L,U.$$

By direct computation, we get

$$D_{\alpha}[\zeta\lambda^{\alpha}(t)]=\lambda^{\alpha}(t)D_{\alpha}\zeta+\zeta D_{\alpha}\lambda^{\alpha}(t)$$

$$\int_{\Omega}\zeta D_{\alpha}\lambda^{\alpha}(t)dt=\int_{\Omega}D_{\alpha}[\zeta\lambda^{\alpha}(t)]dt-\int_{\Omega}[\lambda^{\alpha}(t)D_{\alpha}\zeta]dt$$

and, applying the condition $\zeta|_{\partial\Omega}=0$ and the flow-divergence formula, we obtain

$$\int_{\Omega}D_{\alpha}[\zeta\lambda^{\alpha}(t)]dt=\int_{\partial\Omega}[\zeta\lambda^{\alpha}(t)]\vec{n}d\sigma=0,$$

where $\vec{n}=(n_{\alpha}),\ \alpha=\overline{1,m}$, is the normal unit vector to the hypersurface $\partial\Omega$. It results that

$$\int_{\Omega}\zeta D_{\alpha}\lambda^{\alpha}(t)dt=-\int_{\Omega}[\lambda^{\alpha}(t)D_{\alpha}\zeta]dt$$

and further,

$$-\int_{\Omega}\zeta D_{\alpha}\lambda^{\alpha}(t)dt-\int_{\Omega}[\lambda^{\alpha}(t)D_{\alpha}\zeta]dt=0.$$

Consequently,

$$0<-(\theta_{\iota}^{r}\rho_{r}^{1}+\rho^{2}+\sum_{\vartheta=1}^{s}\rho_{\vartheta}^{3})d^{2}((a,u),(b,w)),\quad \iota=L,U$$

and applying the hypothesis $e)$ and $d^{2}((a,u),(b,w))\geq 0$, we get a contradiction. Therefore, the infimum of (PP) is greater than or equal to the supremum of (DP). □

The next result establishes a strong duality between the two considered multiobjective optimization problems with interval-valued components.

Theorem 2 (Strong Duality)**.** *Under the same (ρ,ψ,d)-quasiinvexity hypotheses formulated in Theorem 1, if $(a^0,u^0)\in\mathcal{D}$ is a normal LU-efficient solution of the Primal Problem (PP), then there exist $\theta^0,\ \mu^0(t)$ and $\lambda^0(t)$ such that $(a^0,u^0,\theta^0,\lambda^0,\mu^0)\in\triangle$ is an LU-efficient solution of the Dual Problem (DP), and the corresponding objective values are equal.*

Proof. Considering that $(a^0,u^0)\in\mathcal{D}$ is a normal LU-efficient solution in (PP), the necessary LU-efficiency conditions formulated in (4)–(6) involve that there exist $\theta^0,\ \mu^0(t)$, and $\lambda^0(t)$ such that $(a^0,u^0,\theta^0,\lambda^0,\mu^0)$ is feasible solution for (DP). Since

$$\frac{\partial a^{0i}}{\partial t^{\alpha}}(t)=\mathcal{X}_{\alpha}^{i}\Big(t,a^{0}(t),u^{0}(t)\Big),\quad i=\overline{1,n},\ \alpha=\overline{1,m},\ t\in\Omega$$

and (by (6))

$$\mu^{\beta}(t)\mathcal{Y}_{\beta}\Big(t,a^{0}(t),u^{0}(t)\Big)=0,\quad \text{(summation over }\beta\text{)},\quad t\in\Omega,$$

the dual objective has the same value as the primal objective; by Theorem 1, $(a^0,u^0,\theta^0,\lambda^0,\mu^0)\in\triangle$ is an LU-efficient solution of (DP). □

The following theorem formulates a converse duality result associated with the considered multiobjective optimization problems with interval-valued components.

Theorem 3 (Converse Duality). *Consider that $(a^0, u^0, \theta^0, \lambda^0, \mu^0) \in \triangle$ is an LU-efficient solution of (DP). In addition, assume that the following conditions are fulfilled:*
(a) $(\overline{a}, \overline{u}) \in \mathcal{D}$ is a normal LU-efficient solution of (PP).
(b) The hypotheses of Theorem 1 are satisfied for $(a^0, u^0, \theta^0, \lambda^0, \mu^0)$.
Then, $(\overline{a}, \overline{u}) = (a^0, u^0)$ and the corresponding objective values are equal.

Proof. Contrary to the result, let us suppose that (a^0, u^0) is not a normal LU-efficient solution of (PP), that is, $(\overline{a}, \overline{u}) \neq (a^0, u^0)$. As $(\overline{a}, \overline{u}) \in \mathcal{D}$ is a normal LU-efficient solution of (PP), according to Treanţă [25] and Mititelu and Treanţă [23], there exist $\overline{\theta}, \overline{\mu}(t)$ and $\overline{\lambda}(t)$, satisfying (4)–(6) and Definition 7. It follows that

$$\overline{\lambda}_i^{\alpha}(t)\left[\mathcal{X}_{\alpha}^i(t, \overline{a}(t), \overline{u}(t)) - \frac{\partial \overline{a}^i}{\partial t^{\alpha}}(t)\right] \geq 0,$$

$$\overline{\mu}^{\mathcal{Q}_{\vartheta}}(t)\mathcal{Y}_{\mathcal{Q}_{\vartheta}}(t, \overline{a}(t), \overline{u}(t)) \geq 0, \quad \vartheta = \overline{1, s}$$

and, therefore, $(\overline{a}, \overline{u}, \overline{\theta}, \overline{\lambda}, \overline{\mu}) \in \triangle$. Moreover, we have $\pi(\overline{a}, \overline{u}) = \delta(\overline{a}, \overline{u}, \overline{\theta}, \overline{\lambda}, \overline{\mu})$. In accordance with Theorem 1, we have $\pi(\overline{a}, \overline{u}) \npreceq \delta(a^0, u^0, \theta^0, \lambda^0, \mu^0)$, equivalently, $\delta(\overline{a}, \overline{u}, \overline{\theta}, \overline{\lambda}, \overline{\mu}) \npreceq \delta(a^0, u^0, \theta^0, \lambda^0, \mu^0)$. This contradicts the maximal LU-efficiency of $(a^0, u^0, \theta^0, \lambda^0, \mu^0)$. Hence, $(\overline{a}, \overline{u}) = (a^0, u^0)$ and the corresponding objective values are equal. □

Remark 2. *If, for $r = \overline{1, p}$ and $(a, u) \in \mathcal{A} \times \mathcal{U}$, each interval-valued multiple-integral functional $\int_{\Omega} g_r(t, a(t), u(t))dt$ is equal to $[1, 1]$, then we obtain primal and dual multiobjective non-fractional variational control problems with interval-valued components and the corresponding Mond–Weir duality results.*

Illustrative example. To illustrate the derived theoretical results, for $p = 1$ and $\int_{\Omega} g(t, a(t), u(t))dt = [1, 1]$ (see Section 2), we consider the following two-dimensional interval-valued variational control problem:

$$\min_{(a,u)} \int_{\Omega_{0,3}} f(t, a(t)u(t))dt = \left[\int_{\Omega_{0,3}} \left(u^2(t) - 8u(t) + 16\right)dt^1dt^2, \int_{\Omega_{0,3}} u^2(t)dt^1dt^2\right],$$

subject to $\frac{\partial a}{\partial t^1}(t) = \frac{\partial a}{\partial t^2}(t) = 3 - u(t),\ t = \left(t^1, t^2\right) \in \Omega_{0,3}$, $81 - a^2(t) \leq 0,\ t = \left(t^1, t^2\right) \in \Omega_{0,3}$, and $a(0) = a(0,0) = 6,\ a(3) = a(3,3) = 8$, where $a : \Omega_{0,3} \to \mathbb{R},\ u : \Omega_{0,3} \to \left[-\frac{8}{3}, \frac{8}{3}\right]$, and $\Omega_{t_0,t_1} = \Omega_{0,3}$ is a square fixed by the diagonally opposite points $t_0 = \left(t_0^1, t_0^2\right) = (0,0)$ and $t_1 = \left(t_1^1, t_1^2\right) = (3,3)$ in $\mathbb{R}^2$. In addition, we assume in the considered variational control problem that we are only interested in affine state functions. By direct computation, it can be proved that the feasible point

$$a^0(t) = \frac{1}{3}\left(t^1 + t^2\right) + 6, \quad u^0(t) = \frac{8}{3}, \quad t \in \Omega_{0,3}$$

is a normal LU-optimal solution in the optimization problem considered above, with $\lambda = (\lambda^1, \lambda^2) = \left(1, \frac{5}{3}\right),\ \theta = (\theta^L, \theta^U) = (1, 1)$, and $\mu = 0$. Further, it is easy to check the $(\rho, 1, 0)$-quasiinvexity (with $\rho \in \mathbb{R}$) of the involved functionals (see Theorem 1) at (a^0, u^0) with respect to ζ and κ, defined as follows: $\zeta,\ \kappa : \Omega_{0,3} \times (\mathbb{R} \times \mathbb{R})^2 \to \mathbb{R}$, given by

$$\zeta\left(t, a(t), u(t), a^0(t), u^0(t)\right) = \begin{cases} a(t) - a^0(t), & t \in Int(\Omega_{0,3}) \\ 0, & t \in \partial\Omega_{0,3}, \end{cases}$$

$$\kappa\Big(t, a(t), u(t), a^0(t), u^0(t)\Big) = \begin{cases} u(t) - u^0(t), & t \in Int(\Omega_{0,3}) \\ 0, & t \in \partial\Omega_{0,3}. \end{cases}$$

Consequently (see Theorem 2), $\left(\frac{1}{3}\left(t^1 + t^2\right) + 6,\ \frac{8}{3},\ (1,1),\ (1, \frac{5}{3}),\ 0\right)$ is an LU-optimal solution for the following dual problem:

$$\max_{(b,w)} \int_{\Omega_{0,3}} f(t, b(t)w(t))dt = \left[\int_{\Omega_{0,3}} \Big(w^2(t) - 8w(t) + 16\Big) dt^1 dt^2,\ \int_{\Omega_{0,3}} w^2(t) dt^1 dt^2\right],$$

subject to

$$-2\mu(t)b(t) + \frac{\partial\lambda^1}{\partial t^1}(t) + \frac{\partial\lambda^2}{\partial t^2}(t) = 0, \quad t = \left(t^1, t^2\right) \in \Omega_{0,3},$$

$$2\theta^L w(t) - 8\theta^L + 2\theta^U w(t) - \lambda^1(t) - \lambda^2(t) = 0, \quad t = \left(t^1, t^2\right) \in \Omega_{0,3},$$

$$\lambda^1(t)\left(3 - w(t) - \frac{\partial b}{\partial t^1}(t)\right) + \lambda^2(t)\left(3 - w(t) - \frac{\partial b}{\partial t^2}(t)\right) \geq 0, \quad t = \left(t^1, t^2\right) \in \Omega_{0,3},$$

$$\mu(t)\Big(81 - b^2(t)\Big) \geq 0, \quad t = \left(t^1, t^2\right) \in \Omega_{0,3},$$

$$\theta = (\theta^L, \theta^U) \succeq [0,0], \quad \mu(t) \geq 0, \quad b(0) = b(0,0) = 6, \quad b(3) = b(3,3) = 8,$$

and the corresponding objective values are equal.

4. Conclusions

In this paper, based on the totally new concept of (ρ, ψ, d)-quasiinvexity associated with an interval-valued multiple-integral functional, we have formulated and proved Mond–Weir weak, strong, and converse duality results for a new class of multiobjective optimization problems with interval-valued components of the ratio vector. Taking into account the applicability of interval analysis and duality theory in optimization and control, the present paper represents an important outcome for researchers and engineers in applied sciences.

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