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Optimal Software Feature-Limited Freemium Model Design: A New Consumer Learning Theoretical Framework

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Abstract: The software industry is increasingly adopting a feature-limited freemium business model that combines "free" and "premium" contents in one product, to sell its products. How to determine the optimal product quality differences between the free and premium versions of software is a central business problem facing many software vendors. In this paper, we study the optimal feature-limited freemium software strategy design, as well as the associated pricing strategies based on consumer learning and network externality effects. We propose a new consumer learning framework induced by cross-module synergies that contains both direct and indirect learning processes. By employing a two-stage mathematical theoretical model and a numerical analysis method, we gained some insights regarding the feature-limited free trial strategy design and associated pricing strategies while considering the associated trade-off between the benefits and costs of the free trial strategy. In our modeling and numerical results, consumers' prior beliefs about the quality of premium content before the free trial, network effect intensity, and indirect learning intensity were found to be three conditions that need to be studied to examine software vendors' management decisions. For the software industry, the quality difference between free and premium functionality or the service and price strategy for a feature-limited free trial model can be designed while considering these factors, which will provide some useful guidelines for the industry.

Keywords: freemium strategy; feature-limited freemium model; mathematical modeling; optimization model; software versioning; pricing strategies; consumer learning; cross-module synergies; network effects

1. Introduction

The software industry has grown rapidly over the past few decades. With such a development of information technology, software products are also becoming more diverse and the number of users is increasing. The valuation of the global software industry went from \$265.4 billion in 2010 to \$356.7 billion in 2015 [1]. The development of the PC and mobile phone market has further promoted the development of the software industry. Mobile phone apps and computer software have become indispensable for all aspects of our lives. How to develop appropriate pricing strategies for different types of software products under different market conditions is a problem faced by software vendors.

Software can be regarded as a kind of experience good. Consumers cannot make an accurate assessment of the quality of the product before use, and their understanding of the quality of products is mainly due to their own experience of using products. In order to help consumers gain a better understanding of product quality, the freemium business model proposed by Anderson is becoming popular [2]. This model splits the functionality of the software into two modules—"free" and "premium"—in association with a product. The free module is the basic version that each consumer can experience for free, and the software firms earn their profit only by selling the premium content. Freemium models are spreading quickly in the software industry, such as in the software membership business model. Many software firms (e.g., mobile music apps, mobile video apps, and e-commerce

apps) have launched a membership pricing strategy. Consumers who wish to have a higher level of user experience than free users can purchase premium contents to join the membership of the software.

The freemium model is a new business model that has been developed for the software industry over recent years. The birth of the freemium business strategy has the following economic motivation: First, as a category of information goods, software products are unlike traditional physical goods in that consumers are often unable to assess the quality of goods before their consumption of the software. This consumer uncertainty regarding the functionality or contents of software can reduce their potential willingness to purchase a premium product. Consumer uncertainty is often considered to be an important factor that affects software market failure. Second, the marginal production cost of information goods is negligible compared with traditional goods, which provides a greater advantage for the free trial of software products for the business purpose of reducing consumer uncertainty. Third, according to the theory of the business model proposed by Ostenwalder, customer value for software products is mainly derived from their actual experience using such software. For software firms, user experience is a commercial way to create customer-perceived value, and the freemium free trial model is a useful business strategy to construct customer value that is supported by Barringer's economic theory. Software companies naturally hope to adopt effective strategies, such as the freemium model, so that more consumers will be converted from free users to paying users—the source of the software company's profits.

Determining the optimal quality design of both the free and premium versions of software is a significant issue facing software firms. For software vendors, providing premium content with more advanced features in products can attract more people to become software members, resulting in more profit. There are two opposing effects regarding the quality of the premium version. On the one hand, higher-quality premium content attracts more free users to become paying users, thus allowing vendors enjoy greater returns. On the other hand, if the manufacturer design the quality of the premium content too strong, then the quality of the free version of the software could become too weak to help consumers to learn about the quality or functionality of the software product. Therefore, taking the trade-off between the benefits and costs of product premium functionality or contents into consideration, the optimal freemium strategy design is a central research question that software firms should consider.

Software products are information goods based, to a large extent, on user experience. This experience can be regarded as a consumer's learning process of software functionality and quality, i.e., consumers will update their understanding of the quality of software in line with their free trial using some or all of the functional content of the software. According to existing research, this consumer learning process can be divided into the direct learning process and the indirect learning process. Direct learning refers to consumers updating their perception of the quality of the parts they actually experience through the free trial, while the indirect learning process involves consumers updating their beliefs on the never-before-experienced software features through their free consumption of the free content. The premise is that the features of the two parts (consumers' experienced contents and consumers' non-experienced contents) have some similarities. To the best of our knowledge, most existing research related to software free trial strategies has studied software as a whole. Many studies have only focused on the process of direct learning in feature-limited free trial issues, and very few researchers have considered both direct learning and indirect learning in an analytical model.

In this study, we propose a consumer valuation experience-based learning framework that is induced by cross-module synergies. In this framework, Customers do not know the true quality of the basic and premium versions in the beginning, but they do have their own prior beliefs concerning the quality of both the free and premium modules. They update their beliefs on the quality of the software after they use the basic functionality of the product. In our consumer learning framework, we consider cross-module synergies,

which means that consumers can experience both direct and indirect learning processes. Once consumers have access to the basic module of the software, their belief towards the quality of the free module changes according to their direct learning process and they learn about the quality of the premium module, therefore also updating their beliefs about the premium version (which can be seen as an indirect learning process).

Our research was based on a new consumer learning framework to explore the problem of optimal feature-limited free trial strategy design. Mathematical derivation and numerical analysis approaches were applied in our modeling research work. This paper focuses on analyzing the feature-limited freemium strategy and addressing the issues of how to determine the quality of premium features and the selling price of the software to achieve optimal results. More specifically, we propose the following critical research questions. First, should a software firm offer a feature-limited freemium model? Is this strategy more profitable than the alternative, which is not offering a freemium strategy? If not, under what conditions should a firm employ the freemium model? Second, if it is more profitable to offer a freemium strategy, what is the optimal quality of the premium version and the associated optimal selling price of the software? Finally, what are the factors that affect the optimal quality of the premium content or service and selling price of a product? We sought to examine the optimal selling prices and the quality of the premium version according to different market factors. Our findings can provide a reference for software companies seeking to adopt a feature-limited freemium model design. They can learn how to adopt different free trials by adjusting the quality of free contents and the pricing strategies based on different market conditions or different categories of software.

The remainder of this paper is organized as follows: A review of the relevant literature to the feature-limited software free trial problem is given in Section 2. We propose a new consumer learning theoretical framework to study software firms' optimal feature-limited freemium software model design problem, which involves cross-module synergies based on several basic model assumptions in Section 3. In this section, we give three different optimization problems for the software firm under different market conditions and the corresponding mathematical solutions. In Section 4, we conduct a numerical analysis as a supplement to the mathematically-derived analytical results in order to gain more useful managerial insights for feature-limited freemium model design issues facing the software industry. We conclude the paper and discuss possible future research directions in Section 5.

2. Literature Review

The research topics in IS economics and marketing related to our study include the following aspects: (i) product sampling and free trials, (ii) software versioning, and (iii) the freemium business strategy.

2.1. Product Sampling and Free Trial

The first stream of literature relevant to our study is product sampling and free trials, as the feature-limited freemium model can be seen as a special form of product trial and sampling. Early articles in this research direction mainly focused on the study of product sampling and free trials of physical goods. Most prior studies generally assumed that consumers' beliefs towards products would change according to their free usage experience and examined the impact of the free sampling strategy on changes in consumers' beliefs and attitudes [3,4]. Jain et al. adopted a simulation method to study how to determine the optimal level of product sampling and found that product sampling is significant in the initial stages of a product's life [5]. The results showed several insights on the optimal initial level of product sampling and how the dynamics of product diffusion are affected by sampling strategy. Similarly, Heiman et al. used an analytical model to identify a firm's optimal sampling effort over time [6].

However, software products as information goods have some unique attributes that traditional products do not have. The literature on free trials of software has some differ-

ences from previous literature on physical goods. The biggest difference between software and physical goods is that software has network effects, which most prior research on product trials and sampling has not taken into consideration. Software products are experience goods, and their value is mainly learned by customers when they try them [7]. Based on this hypothesis, most of the literature on free trial strategies for information goods has explores how firms can affect customers' adoption by educating them about the true value of a product. Lee and Tan built a multilevel model to test the effects of sampling strategy on sampling performance among software categories [8]. The results showed that sampling intensity differs among software categories. Faugere and Tayi developed a vertical differentiation game-theoretic model to study the issue of designing free software samples (shareware) [9]. They showed that a software monopolist's optimal free trial strategy largely depends on the category of software. Teo et al. established a theoretical model of a user's coping reactions toward software trial restrictions and conducted a field experiment to explore free trial strategies [10].

In these studies, the update of customers' beliefs of products was mainly based on their actual free trial experience with the products. In our model, the magnitude of belief updates based on consumer learning depends not only on the quality of the free version but also on the strength of cross-module synergistic learning. This is an innovation of our research compared to previous research on product sampling and free trial issues.

2.2. Software Versioning

The second stream of literature relevant to our study is related to software versioning. In feature-limited freemium business models, the vertical differentiation of versioning is achieved by offering a free version with a lower quality and offering a premium version with a higher quality to customers. Many studies have explored optimal software versioning strategies under various utility structures and market assumptions. Raghunathan explored software introduction problems using segmentation theory [11] and found that if cannibalization is low, a vendor should introduce the full software as one edition. When cannibalization is high, introducing multiple editions at the same time is optimal under a variety of conditions. Bhargava and Choudhary explored under what conditions versioning is the optimal strategy for information goods by deriving a single product maximization problem [12]. They found that software versioning is optimal when the optimal market share of the lower quality version offered alone is greater than the optimal market share of the higher-quality version offered alone. Wu and Chen considered how to use software versioning as a potential instrument to fight digital piracy [13]. They showed that the presence of piracy may lead firms to offer more than one quality product and versioning can be an effective way to curb piracy for information goods. Wei and Nault extended this line of work to scenarios of imperfect information on the consumer side [14]. They considered a two-period model in which low-quality version users might decide in the second stage to upgrade to the high-quality version. A common assumption made by these models is that when consumers underestimate a product, it is usually better for the firm to offer both a basic version of the product and a full feature version. However, Lahiri and Dey proposed a new perspective and showed that a manufacturer should adopt versioning even when consumers do not underestimate the product [15]. Their explanation is that the presence of an informed segment of consumers fundamentally changes the information structure in the market, increasing the relative value of information dissemination. Feature-limited freemium business models can be considered a special form of commercial versioning. We extended these studies by combining software versioning with the freemium business model to explore optimal software versioning and optimal freemium strategy design.

2.3. Freemium Business Strategy

The third stream of existing literature relevant to our research is regarding the freemium business strategy, which combines free and premium consumption in association with a product. Freemium models are commonly seen in practice in the following

two forms: feature-limited freemiums and time-limited freemiums. Many studies on freemium models compared the two freemium strategies and determined which strategy (i.e., feature-limited freemiums or time-limited freemiums) merchants should adopt to maximize the benefits under some market conditions. Cheng and Liu examined the trade-off between the effects of reduced uncertainty and demand cannibalization [16], and they described the conditions under which software firms should introduce time-limited or feature-limited free trials. Their research built a unified framework to provide useful guidelines for which freemium model is optimal—time-limited or feature-limited when considering network externalities and consumer uncertainty. As an extension of their research, Cheng and Li proposed a hybrid free trial strategy and developed an analytical model to compare and contrast these three software freemium models [17]. The results showed that the hybrid strategy weakly dominates the other two free trial strategies and that the intensity of the network effects is a key factor in deciding which freemium strategy is optimal. Niculescu and Wu focused on two software business models that involved a free component—feature-limited freemium and uniform seeding—and developed a unified two-period consumer learning framework based on consumer experience-based learning to compare and contrast these business models [18]. In addition to these theoretical studies, there have been some empirical studies on freemium business models. Wagner et al. empirically studied whether a free service's limitations impacted the evaluation of free and premium versions based on the dual mediation hypothesis and the elaboration likelihood model [19]. The results indicated that a strong functional difference between their free and premium services could cause more free users to convert to paying users. Shi et al. investigated how social dynamics in combination with users' past performance can affect purchase behavior in freemium social games online [20]. Koch and Benlian examined the effect of two common free trial strategies on consumer conversion: "Freefirst," where consumers start in the free and then move to the premium version, and "Premiumfirst," where they experienced the versions in reverse order [21]. The analysis revealed that the Premiumfirst experience increases the conversion propensity and that this effect is greater when the premium version and free version are more similar.

Unlike these above-mentioned strategy comparison studies, some studies focused on a single freemium strategy to explore the optimal free trial level. Cheng and Tang studied the optimal strategy for the limited version free trial software by explicitly considering the trade-off between network effects and the cannibalization effect [22]. They showed that when network intensity is strong, a firm will introduce a limited version free trial software, and it is more profitable for the firm to offer a limited free trial than segmenting the market with two versions of different qualities. Dey et al. examined the problem of designing software for time-limited free trials under a general learning function [23]. They found that a time-locked free trial is optimal only when the rate of learning is sufficiently large, and positive network effects have a minimal impact on this optimality. Dou et al. explored optimal seeding strategies and how to engineer optimal network effects by adjusting the investment of social media features [24]. By employing a two-period framework with network effects, experience-based learning, and cross-module synergies to inspect how consumer valuation learning affect the freemium strategy, this study offers a significant contribution to the existing literature on freemium business models.

3. The Model

In this section, we analyze a freemium business design problem facing a software firm. We first give some basic assumptions of the mode in Section 3.1. Then, we describe the theoretical framework of consumer valuation learning based on experience learning and cross-module synergies in Section 3.2. In Section 3.3, the optimization problem for software vendors is proposed and solved mathematically.

3.1. Model Assumptions

We assumed that a software vendor implements featured limited freemium strategy to sell one certain software product. The firm has developed a software product which has two modules: A (basic version) and B (premium version). The features and functionality of module A are free for all consumers in the market, while the functionality of module B requires consumers to pay for a premium version of the software in order to use it. The selling price of the commercial software is P, meaning that the consumer needs to pay the price P if they are willing to obtain the license for the premium contents of the software. The free version of the software helps consumers to better understand the product's quality, and the vendor's profit is only earned from premium content B. Assuming that the quality of the whole software (A and B) is 1, the quality of the free version (module A) is $q_b = q$ $(0 \leq q \leq 1)$, while the quality of the premium features (module B) is $q_p = 1 - q$.

Let $K(K \geq 1)$ be the size of the total population in the market, Q_T be the number of potential customers for the software under consideration, and Q_p be the number of premium users. Without loss of generality, let Q_T be normalized to 1. Let θ denote consumer type, which represents their valuations of the product. Then, θ captures heterogeneity in the consumer's willingness to pay for quality and functionality. For simplicity, θ is uniformly distributed among consumers, which is normalized between $-(K-1)$ and 1. In other words, there is a unit mass of consumers with their type $\theta \sim u[-(K-1), 1]$, where the intervals $[-(K-1), 0]$ and $[0, 1]$ represent the people who are not interested in the product at all and the potential consumers in the market, respectively. This assumption was followed by the model setup of Conner [25]. Without loss of generality, assume that $K - 1 = 0$. Then, θ is uniformly distributed between 0 and 1, $\theta \sim u[0, 1]$. This assumption is for simplicity of mathematical derivation, because it does not affect the results of the mathematical derivation and subsequent analysis. Moreover, each consumer's expected demand quality of the software is q_D, and q_D can be assumed to be uniformly distributed between 0 and 1, $q_D \sim u[0, 1]$. The aggregate cost of using the software, including time and effort spent by consumers to become familiar with the functionality of the software, is c.

This software has the characteristics of network effects, which means a resulting increase in the value of a product because more people use it. The existence of network externality can increase each consumer's valuation of the software. Let the network effects intensity, which reflects the increase in the consumer's willingness to pay when an additional consumer joins the network, be γ $(\gamma > 0)$. Here, the value range of γ can be $0 < \gamma < 1$ or $\gamma > 1$. $\gamma < 1$ means that the network effect of the software is relatively weak, while $\gamma > 1$ means that the network effect is relatively strong.

We considered an experience-based learning framework induced by cross-module synergies, including direct and indirect learning processes from consumers' experiences. This setup of consumer learning was consistent with the work of Niculescu and Wu [18]. Under the direct learning process, we assumed that customers update their valuation for the product according to the direct experience of any functionality or module that they have access to, whether they purchased that license or received it for free. Under indirect learning, consumers update their beliefs about the module without having directly tried the functionality; they learn from their experience with only part of the software. In other words, customers will adjust their valuation for premium functionality in module B after trying the free module A based on the cross-module synergies between A and B but without having tried the premium features in module B. Customers update their beliefs regarding the quality of module A (direct learning) and module B (indirect learning) according to their experience of the free version of the software. Each consumer has their own prior belief about the quality of modules A and B before using the software—a_0 and b_0, respectively $(0 \leq a_0, b_0 \leq 1)$. After a free trial of the module A, a consumer's belief about the quality of module A will update from a_0 to $a_1(a_1 = q_b = q)$, and their belief about the quality of module B will update from b_0 to b_1.

Our model is a two-stage game. The first stage is a learning stage in which each consumer updates their belief about the quality of modules A and B according to their use of the free version A. The second stage is the decision stage in which each consumer decides whether or not to pay for premium module B according to the functionality. A summary of the notation in our paper is shown in Table 1.

Table 1. Notation.

q :	The quality of free functionality (module A)
Q_T :	Total number of potential software users
Q_p :	Total number of buyers that purchase for the premium functionality
θ :	Consumer type, i.e., consumer's valuation of the software functionality
K :	The size of the total population in the market
q_D :	Consumer's expected demand quality of the software
c :	Aggregate cost of using the software
a_0 :	Consumer's prior belief about the quality of module A before using the software
b_0 :	Consumer's prior belief about the quality of module B before using the software
a_1 :	Consumer's belief about quality of module A after a free trial with module A
b_1 :	Consumer's belief about quality of module B after a free trial with module A
γ :	The network effect intensity, measuring how much each addition in the number of users increases the software's perceived value
P :	Price of the premium version of the software

3.2. Consumer Learning Framework

Software products are experience goods, and we assumed that customers update their beliefs regarding the quality of the product according to their practical experience with a software's functionality. The belief-updating progress is based on a consumer's own experience. We considered cross-module synergies in the consumer learning framework, which can be divided into direct and indirect learning mechanisms. According to our previous assumptions, we divided the software product into two modules, free module A and premium module B. Consumers do not understand the true quality of modules A and module B, but they do have an initial belief about the quality of each module. We made an assumption that every potential software user would consume the free trial of free module A. Under the mechanism of direct learning, customers immediately learn the true quality of the functionality of the module that they have a chance to experience. Thus, the consumer belief update function under direct learning is $a_1 = q_b = q$.

Meanwhile, we also considered the possibility of indirect learning via their experience with part of the software. In this way, each consumer updates their belief on the quality of premium contents in module B via their use of free trial of module A, although they have not actually experienced the premium functionality of module B. This is because there is a certain correlation between different versions of the software. Every consumer can obtain a new belief on the quality about the functionality of the unused part through their true experience of the free trial. We used the following formula to describe the process of consumer indirect learning process: $b_1 = b_0 + \Delta(q)$. Here, the variable $\Delta(q)$ is used to measure the increase in consumers' beliefs regarding the quality of premium module B through the free trial of module A. The magnitude of $\Delta(q)$ can depend on many factors. On the one hand, $\Delta(q)$ is affected by the quality of free version A, q. If the quality of free module q is larger, consumers will have a greater probability of believing that the quality of the premium module B is larger after the free trial, i.e., $\Delta(q)$ will be larger too. On the other hand, the magnitude of $\Delta(q)$ is also dependent on the differences of the functionality between software versions. If the functionality or service between the free and premium versions of software is closer, the consumer is more likely to gain awareness of the quality of the premium contents through the use of free features. For example, consider the case of a mobile game, "Angry Birds," which is offered under a featured-limited freemium model. The game's users can play the first few levels for free and then have to pay some money to gain a license for the remaining levels. However, in terms of the game

contents, there is a great similarity between the free levels and premium levels. Consumers can easily understand the features of the premium levels through the content of the free part. Thus, $\Delta(q)$ is relatively large. Let us consider another mathematical programming software, MATLAB, which was developed by MathWorks. This software has some basic programming functionality, as well as some professional toolkits for specialized problems such as a machine learning toolbox and a deep learning toolbox. There is a large difference between the basic contents and premium toolkits of this software. It is quite difficult to learn the true quality of functionality in the add-on professional toolbox just based on the basic features of MATLAB. In this case, $\Delta(q)$ is very small. For simplicity, we assumed that

$$\Delta(q) = \delta \cdot q, \tag{1}$$

where q is the quality of free module A and δ represents the possibility of gaining awareness of the functionality of the premium module through the free consumption of module A ($\delta > 0$); δ is a property of the software itself and is related to the difference in functionality between the free and paid versions of the software. We call this parameter the "indirect learning intensity".

Therefore, the mathematical formulae for the consumer learning framework that reflects the consumer belief update process are as follows:

Direct learning:

$$a_1 = q_b = q, \tag{2}$$

Indirect learning:

$$b_1 = b_0 + \Delta(q) = b_0 + \delta q. \tag{3}$$

In the first stage, all consumers conduct a free trial of module A and update their beliefs on the quality on both free module A and premium module B. Then, in the second stage, every consumer decides whether to purchase premium features of the software or not. The utility function of a consumer with type θ when purchasing the premium module B after the learning stage is given as follows:

$$U(\theta) = (\theta + \gamma Q_T) \cdot (b_0 + \delta q) - P - c. \tag{4}$$

The form of this utility function reflects both the increase in the consumer's valuation due to the network effect and the effect of the consumer's reduced uncertainty about the product quality through the free trial. A consumer with type θ will pay for premium content if they perceive $U(\theta) \geq 0$, which is consistent with the work of Bourreau and Lethiais [26].

3.3. Distribution of Potential Consumers

We show the distribution of potential consumers in the market in Figure 1. Let θ_0 denote the marginal type of consumer who is indifferent to buying the premium functionality or services and doing without under the quality of free version q. Let θ_1 denote the marginal type of consumer who has the same demand between purchasing the commercial software product and doing without when there is no feature-limited freemium strategy offered by the firm. In other words, the quality of free version q is equal to 0 under this circumstance.
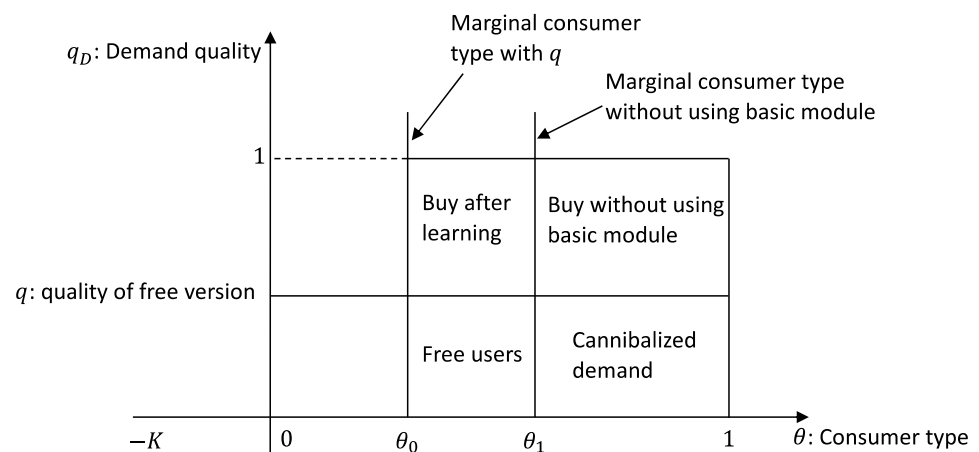
Figure 1. The distribution of potential consumers.

As shown in Figure 1, the consumers whose type θ is higher than θ_0 but lower than θ_1 and whose demand quality q_D is higher than q, that is, $\theta \in [\theta_0, \theta_1]$ and $q_D \in [q, 1]$ (i.e., the upper-left quadrant in Figure 1), will purchase the premium contents of the software after the consumer learning process of a feature-limited free trial. These consumers will become paying users due to the free trial strategy adopted by the merchants. The consumers in the lower-left quadrant in Figure 1 (i.e., $\theta \in [\theta_0, \theta_1]$ and $q_D \in [0, q]$) become free riders because their demand quality q_D is lower than the quality of the free version of software q. For these free riders, the free module of the software product is enough to meet their usage needs for this software product.

We then analyzed the distribution of consumers in the two regions in the right half of Figure 1 (i.e., consumers with type $\theta \in [\theta_1, 1]$). For these consumers, since their consumer type is higher than the marginal consumer type θ_1, even if the software firm does not offer the free trial strategy to potential consumers in the market, they will choose to buy the commercial software. Among these consumers, people whose demand quality for the software is higher than q refers to $q_D \in [q, 1]$ (i.e., the upper-right quadrant in Figure 1) will choose to purchase the premium version of the product without a free trial of the free features of the software. Consumers with a demand quality for the software $q_D < q$ (i.e., the lower-right area in Figure 1) will not buy the software premium version because the quality of the free version provided by the software firm meets their needs. On the contrary, if the company had not offered a free trial strategy, they would have chosen to buy the commercial software. Therefore, we call this group of consumers "cannibalized demand" because of feature-limited freemium strategy. The existence of these consumers causes a loss to the software firm's profit.

3.4. Optimal Freemium Strategies Design Problem

In this section, we analyze a software firm's optimization problems regarding the design of a feature-limited free trial business model and provide a mathematical analytical solution by means of mathematical derivation. Mathematically, we considered the following three optimization problems: (i) the optimal selling price when considering the quality of the free version as an exogenously given parameter, (ii) the optimal quality design of the free version while the selling price if exogenously fixed, and (iii) both the optimal selling price and optimal quality design of free version at the same time when they are regarded as optimized parameters of the model.

3.4.1. Optimization Problems for the Software Firm

We first analyzed and solved marginal consumer type θ_0. We have already seen that θ_0 denotes the marginal-type consumer who is indifferent between adopting the commercial software and doing without and using the free version q. A consumer with type θ will buy the premium version if they perceive $U(\theta|.) \geq 0$ (i.e., the utility function in Equation (4)).

Thus, we let the net utility function in Equation (4) $U(\theta|.) = 0$. Then, we can obtain the marginal consumer type through mathematical derivation:

$$\theta_0 = \frac{P + c}{b_0 + \delta q} - \gamma. \tag{5}$$

From the value range of θ_0 $0 \le \theta_0 \le 1$ and noting that $P \ge 0$, we can obtain a constraint condition of variable P:

$$max\{(b_0 + \delta q)\gamma - c, 0\} \le P \le (b_0 + \delta q)(1 + \gamma) - c. \tag{6}$$

According to our analysis of the distribution of potential consumers in the market shown in Figure 1, the two-dimensional product area $[\theta_0, 1] \times [q, 1]$ represents the market demand of the premium version of the software. Consumer type θ and the quality of free functionality or contents q are uniformly distributed on $[0, 1]$. We can easily give the mathematical formula of market demand D, which is shown as follows:

$$D = (1 - \theta_0)(1 - q). \tag{7}$$

After substituting the expression of θ_0 from Equation (5) into Equation (7), the market demand for the premium version of the software (i.e., total number of buyers of the commercial software Q_p) becomes

$$D = Q_p = \frac{(1 - q) \cdot [(1 + \gamma)(b_0 + \delta q) - (P + c)]}{b_0 + \delta q}. \tag{8}$$

In order to explore the optimal freemium business strategy design problem, the mathematical form of the utility function of the software firm is given as follows. When the consumer's prior belief about the quality of premium module B before the free trial b_0, the possibility or tendency of learning and understanding the contents of the premium module through the free consumption of module A δ, the intensity of network effect γ, and the aggregate cost of using the software c are exogenously provided, the software's business problem is to decide the selling price of the premium module P and the quality of free module q under the constraints of variables to maximize its profit:

$$\max_{P, q} \pi(P, q|b_0, \delta, \gamma, c) = PQ_p = P \cdot \frac{(1-q)[(1+\gamma)(b_0+\delta q)-(P+c)]}{b_0+\delta q},$$
$$\text{s.t. } max\{(b_0 + \delta q)\gamma - c, 0\} \le P \le (b_0 + \delta q)(1 + \gamma) - c, \text{ and } 0 \le q \le 1. \tag{9}$$

3.4.2. Optimal Pricing Strategy When q Is Exogenously Given

In this subsection, we discuss the impact of a change in the price of the premium module P on business profit π. When the quality of the free module q is regarded as an exogenously given constant, we have the partial derivative of π with respect to P :

$$\frac{\partial \pi}{\partial P} = (1 - q)\left(1 + \gamma - \frac{2P + c}{b_0 + \delta q}\right). \tag{10}$$

It is obvious that $1 - q \ge 0$; therefore, in order to determine the sign of the partial derivative $\frac{\partial \pi}{\partial P}$, we only need to judge the sign of $\left(1 + \gamma - \frac{2P+c}{b_0+\delta q}\right)$. Let $\frac{\partial \pi}{\partial P} \ge 0$; then, we have a range of P as follows:

$$P \le \frac{(1 + \gamma)(b_0 + \delta q) - c}{2}. \tag{11}$$

Otherwise, $\frac{\partial \pi}{\partial P} < 0$ implies the following:

$$P > \frac{(1+\gamma)(b_0 + \delta q) - c}{2}. \tag{12}$$

Considering the constraint conditions of P that we obtained in Equation (6) at the same time, we can draw the conclusion that the partial derivative $\frac{\partial \pi}{\partial P} \geq 0$ when $max\{(b_0 + \delta q)\gamma - c, 0\} \leq P \leq \frac{(1+\gamma)(b_0 + \delta q) - c}{2}$, and $\frac{\partial \pi}{\partial P} < 0$ when $\frac{(1+\gamma)(b_0 + \delta q) - c}{2} < P \leq (b_0 + \delta q)(1 + \gamma) - c$. We therefore make the following proposition to summarize the above discussion.

Proposition 1. *Suppose that the quality of free module q is treated as an exogenous variable that is arbitrarily given in advance. The software firm's profit π changes with the independent variable P as follows:*

(i) *When $P \in \left[max\{(b_0 + \delta q)\gamma - c, 0\}, \frac{(1+\gamma)(b_0 + \delta q) - c}{2}\right]$, $\frac{\partial \pi}{\partial P} \geq 0$, the firm's profit $\pi(P|q)$ increases with an increase in the selling price P.*

(ii) *When $P \in \left(\frac{(1+\gamma)(b_0 + \delta q) - c}{2}, (b_0 + \delta q)(1 + \gamma) - c\right]$, $\frac{\partial \pi}{\partial P} < 0$, the firm's profit $\pi(P|q)$ increases with the decreases of the selling price P.*

(iii) *The software firm's optimal pricing strategy for the freemium model and the firm's maximum profit are given by*

$$P^* = \frac{(1+\gamma)(b_0 + \delta q) - c}{2} \tag{13}$$

$$\pi^* = \frac{[(1+\gamma)(b_o + \delta q - c)]^2 (1-q)}{4(b_o + \delta q)}. \tag{14}$$

Proposition 1 mainly provides the change of π with P, while q is regarded as an exogenous variable and the optimal solution of P in this case. We can further explore the effects of b_0, δ, and γ on the optimal price, P^*, through the mathematical formula, Equation (13), of the optimal solution. If the intensity of network effects γ is larger, the software firm can charge a higher price for its premium version. This is because a higher network effect means better interactions between software users, so consumers will have better experiences using the software and the firm can naturally earn more profit from it by raising the selling price. When consumers' prior beliefs about the quality of premium contents b_0 before the free trial is relatively high, the company can also increase the selling price to earn more profit. This means that the consumer group in the market has a better prior understanding of the functionality and quality of the premium part of the software, and they may have a stronger willingness to pay for its premium contents. Therefore, the firm can set a higher price to achieve more income. The greater the possibility of gaining awareness of the quality of the premium contents through the free trial δ, the higher the price of the commercial software can be. Software with a higher δ indicates that consumers can more easily understand the quality of the premium module through free trial consumption, meaning the commercial effect elicited by the free trial strategy is even stronger.

3.4.3. Optimal Quality Design for a Free Version of Software When P Is Given

We also treat the selling price of the premium version P as an exogenously given constant and determine the impact of a change in the quality of free version q on a firm's profit π. We have the partial derivative of π with respect to q:

$$\frac{\partial \pi}{\partial q} = -P\left[(1+\gamma) - \frac{(P+c)(b_0 + \delta)}{(b_0 + \delta q)^2}\right]. \tag{15}$$

Let $\frac{\partial \pi}{\partial q} = 0$, and then one has the solution of q: $q = \frac{\sqrt{(P+c)(b_0+q)} - b_0\sqrt{1+\gamma}}{\delta\sqrt{1+\gamma}}$. We find that $\frac{\partial \pi}{\partial q} \geq 0$ if

$$0 \leq q \leq \frac{\sqrt{(P+c)(b_0+q)} - b_0\sqrt{1+\gamma}}{\delta\sqrt{1+\gamma}}, \tag{16}$$

otherwise, $\frac{\partial \pi}{\partial q} < 0$ if

$$\frac{\sqrt{(P+c)(b_0+q)} - b_0\sqrt{1+\gamma}}{\delta\sqrt{1+\gamma}} < q \leq 1. \tag{17}$$

We then have the following proposition.

Proposition 2. *Based on the assumption that the price of the premium module of the software P is an exogenous variable that is arbitrarily given in advance, the software firm's profit π changes with the independent variable q as follows:*

(i) $\frac{\partial \pi}{\partial q} \geq 0$ *while* $q \in \left[0, \frac{\sqrt{(P+c)(b_0+q)} - b_0\sqrt{1+\gamma}}{\delta\sqrt{1+\gamma}}\right]$*; the firm's profit $\pi(P|q)$ is a non-decreasing function of the price P on* $\left[0, \frac{\sqrt{(P+c)(b_0+q)} - b_0\sqrt{1+\gamma}}{\delta\sqrt{1+\gamma}}\right]$*.*

(ii) $\frac{\partial \pi}{\partial q} < 0$ *while* $q \in \left[\frac{\sqrt{(P+c)(b_0+q)} - b_0\sqrt{1+\gamma}}{\delta\sqrt{1+\gamma}}, 1\right]$*; the firm's profit $\pi(P|q)$ is a non-increasing function of the price P on* $\left[\frac{\sqrt{(P+c)(b_0+q)} - b_0\sqrt{1+\gamma}}{\delta\sqrt{1+\gamma}}, 1\right]$*.*

(iii) *The software firm's optimal freemium model design and the firm's maximum profit are given by*

$$q^* = \frac{\sqrt{(P+c)(b_0+q)} - b_0\sqrt{1+\gamma}}{\delta\sqrt{1+\gamma}} \tag{18}$$

$$\pi^* = \frac{P(1+\gamma)(b_0+\delta) - 2P\sqrt{(P+c)(1+\gamma)(b_0+\delta)} + P(P+c)}{\delta}. \tag{19}$$

Proposition 2 provides the optimal design of the feature-limited freemium strategy of the software firm when P is arbitrarily given in advance. We can also gain some other conclusions and insights from it. If the network effect is stronger, the optimal quality of the free module of the software q^* should be lower. Under the condition that the network effect is relatively strong, consumers are more likely to obtain a higher utility from the network and it is easier for them to gain an understanding of software quality from the word-of-mouth effect. This utility even exceeds the impact of the free trial strategy on consumers. In this case, the functionality and quality of the free module should be lower in order to curb the loss of the firm's revenue from the free riders. For software with a higher δ, the optimal quality of the free module should be lower. A higher δ indicates that the difference between the free module and the premium module of the software is smaller, thus making it easier for consumers to learn the quality of the premium contents of the software through their free trial of the free module. Therefore, the quality of the free module q^* can be designed to be lower; otherwise, more consumers will to choose speculation behavior (i.e., only enjoy the consumption of the free part without purchasing the premium contents of the software).

3.4.4. Solving Both Optimal P^* and Optimal q^* Simultaneously

By relaxing the restriction that the quality of the free version or the price of the premium version is exogenously given in advance, we aimed to find both the optimal price of premium version P^* and the optimal quality of the free version q^* in the feature-limited freemium model. At this point, π should be regarded as a binary function of both P and q ($P, q | b_0, \delta, \gamma, c$) under the condition that b_0, δ, γ, and c are exogenous variables. The optimization problem can be described as follows: The software firm hope to determine the

optimal price of the premium module and the optimal quality of the free module in order to realize the maximum profit. The corresponding mathematical problem is as follows:

$$
\max_{P,\,q} \pi(P, q|b_0, \delta, \gamma, c) = PQ_p = P \cdot \frac{(1-q)[(1+\gamma)(b_0+\delta q)-(P+c)]}{b_0+\delta q},
$$
$$
\text{s.t. } max\{(b_0 + \delta q)\gamma - c, 0\} \leq P \leq (b_0 + \delta q)(1+\gamma) - c, \text{and } 0 \leq q \leq 1. \tag{20}
$$

The detailed mathematical derivation process can be found in Appendix A. We obtained the optimal solution of price P and q by solving the optimization problem of Equation (20). Using this, we make the following proposition.

Proposition 3. *Under the assumption that* b_0*,* δ*,* γ *, and* c *are exogenous variables, the optimal solution to the software firm's feature-limited free trial optimization problem of Equation (20) can be described by:*

$$
P^* = \frac{-4c + (b_0 + \delta)(1+\gamma) + \sqrt{(b_0 + \delta)^2(1+\gamma)^2 + 8c(b_0 + \delta)(1+\gamma)}}{8} \tag{21}
$$

$$
q^* = \frac{(\delta - 3b_0)(1+\gamma) + \sqrt{(b_0 + \delta)^2(1+\gamma)^2 + 8c(b_0 + \delta)(1+\gamma)}}{4\delta(1+\gamma)}. \tag{22}
$$

Proposition 3 shows the optimal solution for the feature-limited freemium strategy problem and the pricing problem. Similar to the previous analysis process, we aimed to explore the effects of parameters b_0, δ, and γ on the free trial and pricing strategies. Note that the mathematical form of the optimal solution of P and q is very complicated, and the form of mathematical analysis results is more complicated. Meanwhile, in order to provide more intuitive results, we performed a numerical analysis and generated a graphical illustration, which is shown in Section 4.

4. Numerical Analysis

As a supplement to the theoretical model, we conducted a numerical analysis and used graphical methods to gain further insight into the software firm's feature-limited free trial problem. First, we discuss the impact of different free trial strategy designs on the company's profit by plotting the approach under the premise that the selling price of the premium contents or services is a fixed parameter. Second, we study the impact of different consumer's prior beliefs regarding the quality of premium contents or services before the free trial, depending on network effect intensities and possibilities of gaining awareness of the quality of the premium module by their free trial on optimal pricing strategies. Third, we explore how the feature-limited free trial strategy design is affected by these three exogenous parameters.

4.1. The Effect of Freemium Model Design on a Firm's Profit

From the perspective of numerical methods, we explored the impact of different free trial strategy designs on the revenue of a software firm; that is, we plotted the image of the functional relationship between the vendor's profit π and the quality of free version q. To control the variables, we made an assumption that the price P was fixed (let $P = 0.2$). We present the graph of this numerical result in Figure 2. In the analysis of Figure 2, we assigned a specific value to each parameter. Let the consumer's prior belief about the premium version $b_0 = 0.1$, the parameter measuring the possibility of gaining awareness of the quality of the premium module by free trial $\delta = 0.6$, and the aggregate cost of using the software $c = 0.1$. We selected three different network effect intensity values: $\gamma = 0.6$, 1.2, and 1.8. It was then easy to verify that our parameter settings met the constraints of the optimization problem of Equation (20).
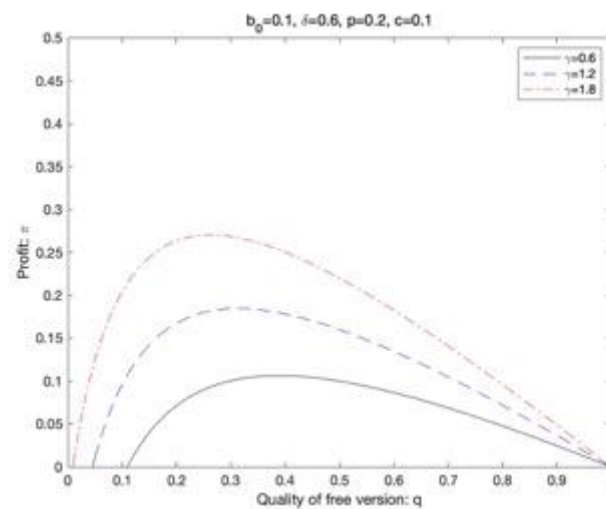
Figure 2. The impact of q on π for different network effect intensities.

Some business insights can be gained from Figure 2. Corresponding to the magnitude of different network effect intensities, the firm's utility π first increases but then decreases with an increase in the quality of the free module q, so there must be a certain value of q in the curves such that π reaches the maximum. When q is relatively low, the overall profit increases with q, which reflects the commercial benefits brought by the free trial strategy. A higher quality of the free version of the software enables consumers to reduce their uncertainty about the quality of the premium version and have a better understanding of the functionality of the premium content. As a result, customers will have a higher willingness to purchase commercial software, thus bringing more profit to the firm. Therefore, when q is below a certain threshold (the threshold given in Proposition 2, denoted by $\bar{q}$, $\bar{q} = \frac{\sqrt{(P+c)(b_0+q)} - b_0\sqrt{1+\gamma}}{\delta\sqrt{1+\gamma}}$), the firm's profit will be higher as q increases ($0 \leq q \leq \bar{q}$).

When q is over a certain threshold $\bar{q}$ ($\bar{q} < q \leq 1$), continuing to increase q will not increase the company's utility, and the profit will actually decrease. This result shows that when q is higher than $\bar{q}$, a larger number of consumers' usage needs for the software will be met and they will become free users without purchasing the premium version. This group of users cannibalizes demand, as mentioned in the previous theoretical analysis. Though the commercial effects that the free trial strategy bring to the software firm still exist, it would be inferior to the loss of the firm due to the cannibalized demand. In this case, increasing q will reduce the firm's profit.

The different network effect intensities in Figure 2 correspond to the curves of different shapes. Another finding from Figure 2 is that for a higher network effect intensity, the corresponding profit of the firm π for a certain q can also be larger. The result shows that the stronger the network effect intensity is, the greater the commercial effect for the firm by adopting feature-limited free trial strategy. The reason for this could be that the network effect expands the spread of the software among consumers. Furthermore, the utility that each user acquires from the network also increases. In this way, the software vendor will benefit more from the free trial strategy due to the network externality.

4.2. Pricing Strategy

Here, we discuss, in depth, how the vendor should adjust its pricing strategy according to the change in consumer's prior beliefs regarding the quality of the premium module before the free trial b_0, and in relation to the network effect intensity γ and possibilities of gaining awareness of the quality of the premium module after a free trial δ. We aim to study the impact of these three parameters on the optimal price by plotting the behaviors of the optimal price P^* with respect to b_0, γ, and δ.

4.2.1. The Impact of b_0

Figure 3 shows the function graph of the optimal price P^* with respect to consumers' prior beliefs b_0 for three different γ, where $\delta = 0.5$, $c = 0.1$, and the intensities of the network effects are given as 0.5, 1, and 1.5. According to this result, the optimal price P^* increases with b_0. A higher b_0 indicates that the consumers in the market have a better initial perception of the quality of the premium module of the software before the free trial. This prior belief of quality allows them to have a better understanding of the software quality at the beginning. The perceived functionalities and lower uncertainty about quality leads to a higher probability of buying the premium version of the software. More consumers are willing to convert from free users to paying users, and software vendors can attain a higher price to make more profit. Notably, parameter b_0 represents an average level of the initial perception of the software quality by potential consumers in the market, rather than one single consumer's initial perception of the software. From the perspective of business management, the information of b_0 for certain software might be understood and evaluated through the method of market research to formulate more reasonable pricing strategies for vendors.
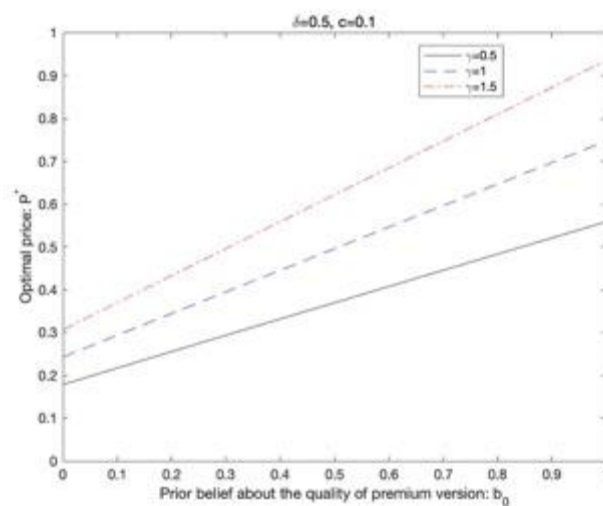

Figure 3. The impact of prior beliefs b_0 on pricing strategies.

4.2.2. The Impact of γ

To plot the impact of network effect intensities on the optimal price, we used $b_0 = 0.1$, $c = 0.1$, and $\delta = 0.3$, 0.55, and 0.8. Figure 4 shows that the optimal price of the premium module increases as the network effect intensity γ increases. For consumers, when γ is higher, the utilities brought by the network that software users can experience are also greater. For the software firm, a stronger network effect will benefit it and facilitate a higher optimal selling price. This relationship does not change with different ranges of consumers' prior beliefs. Furthermore, for a given network effect intensity, the optimal selling price becomes higher when consumers have a higher prior belief about the quality of the premium version.
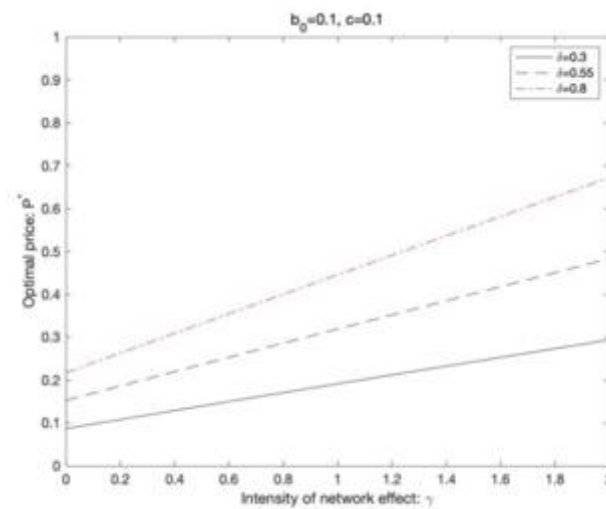
Figure 4. The impact of network effect intensity γ on pricing strategies.

4.2.3. The Impact of δ

Figure 5 shows the change in optimal price with different indirect learning intensities. In Figure 5, $b_0 = 0.1$, $c = 0.1$, and $\gamma = 0.3$, 0.6, and 0.9. Our findings indicated that a larger indirect learning intensity δ corresponds to a higher P^*. If δ is relatively large, consumers in the market will be more likely to learn about the functionality of the premium module through the trial of the free module. Therefore, if the effect of the free trial strategy reducing uncertainty of consumers can be greater, and the firm can benefit by increases its selling price. Parameter δ is a unique attribute of the software itself, and the strengths of δ differ for different categories of software products. For example, some video game software usually has a high degree of similarity between the contents of free and paid levels, which means that consumers can more easily learn the quality and information of the premium module through the free trial. However, with programming software such as MATLAB, the premium module is generally a toolkit suitable for dealing with certain professional mathematical problems. It differs from the free programming functionality, and the strength of δ will therefore be smaller.
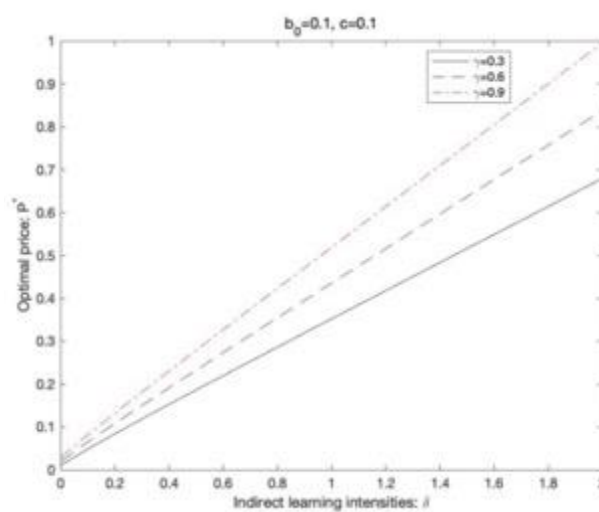

Figure 5. The impact of indirect learning intensity δ on pricing strategies.

4.3. Feature-Limited Freemium Model Design

In order to gain further insights into a software firm's feature-limited freemium strategy design problem, we performed a numerical analysis to study how software firms

should design the optimal free trial strategy based on the parameters of prior belief (b_0), network effect intensity (γ), and indirect learning intensity (δ).

4.3.1. The Impact of b_0

Figure 6 shows the relationship of optimal quality of the free module with consumers' prior belief about the quality of the premium module ($\delta = 0.5$, $c = 0.1$, and $\gamma = 0.4$, 0.8, and 1.6). We found that the optimal quality of the free module q^* decreases as consumers' prior belief about the quality of premium module b_0 increases within a certain range. When b_0 is above a certain threshold (denoted by $\overline{b_0}$, we can set $q^* = 0$ in Equation (22) to get the solution $\overline{b_0} = \frac{\sqrt{(b_0+\delta)^2(1+\gamma)+8c(b_0+\delta)}+\delta\sqrt{1+\gamma}}{3\sqrt{1+\gamma}}$), where q^* will always remain at 0. When $0 \leq b_0 \leq \overline{b_0}$, for a smaller b_0, the software firm should design a higher-quality free module to help consumers realize the functionality of the software and reduce their initial uncertainty about its quality. When $b_0 > \overline{b_0}$, the optimal quality q^* equals 0. Under this condition, the optimal strategy is that no free trial strategy should be offered to consumers because the consumers have a relatively high b_0, so even if there is no free trial strategy adopted by the firm, consumers will have a high willingness to purchase the commercial software. The existence of a free trial strategy will cause more software users to opt for speculative behavior instead of paying for the premium version, which might lead to a loss of profit.
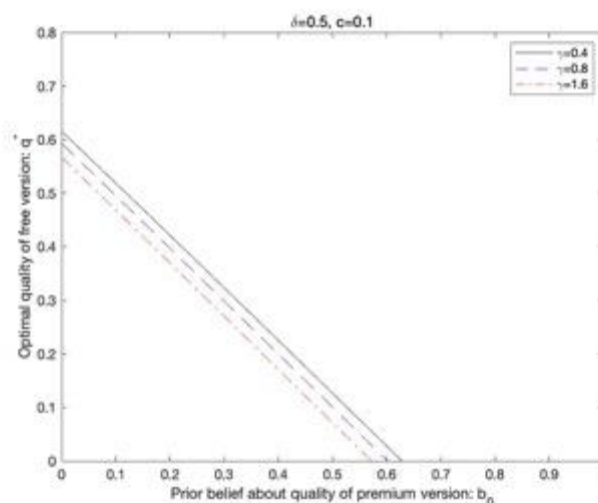

Figure 6. The impact of prior beliefs b_0 on free trial strategy design.

4.3.2. The Impact of γ

Figure 7 shows the influence of the network effect intensity γ on the optimal free trial strategy under different indirect learning intensity δ. The optimal design quality of the free version q^* decreases as γ increases. If the network effect is stronger, there will be more interaction and communication between software users, which allows consumers to receive more information about software features and functionality. To a certain extent, network externalities could be a good supplement to the free trial business strategy, contributing to a benefit for the software firm. In this case, the optimal design quality of the free version should be lower.
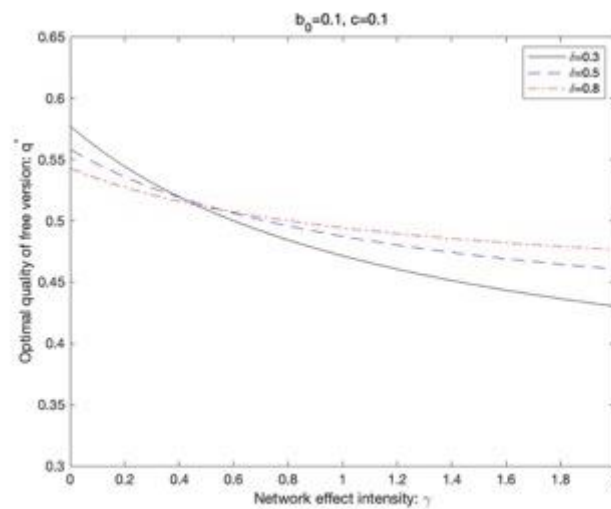
Figure 7. The impact of network effect intensity γ on free trial strategy design.

There are some other interesting findings regarding Figure 7. When the network effect is weak enough, a smaller indirect learning intensity δ leads to a higher q^*; however, when the network effect is stronger than a certain threshold, a larger δ leads to a higher q^*. In the former case, a relatively small γ means that the influence of the network effect is weakened. When the indirect learning intensity is relatively weak, it is not easy for consumers to learn about the quality of the premium module from the free trial. A higher quality of the free version is desirable to make consumers believe that it is worth paying for the premium contents of the software after the free trial process. In the latter case, if the network intensity is over a certain threshold, the utility generated by network effects will increase and may even play a leading role in the software firm's decision-making considerations. In this situation, if the indirect learning effect is stronger, the commercial utility of the free trial strategy will be more significant. The free trial strategy and network externalities will jointly produce a strong interaction effect. A higher-quality free version should be provided so that the vendor can better capture the power of the network.

4.3.3. The Impact of δ

Figure 8 shows the relationship between the optimal free trial strategy design and indirect learning intensity δ. It can be seen that the optimal quality of free contents q^* increases first and then decreases with the parameter of indirect learning intensity δ. Within a certain range, q^* increases as δ increases; however, when δ is greater than a certain threshold, q^* slowly decreases as δ increases. Below a certain threshold, when δ increases, the optimal quality of the free version should be made higher to allow consumers to better understand the functionality and services of the premium version through the free trial. In this way, software firms can make better use of the commercial benefits of the free trial business strategy. When the indirect learning intensity is relatively strong, consumers have a greater probability of gaining awareness of the functionality or services of the premium module through the free trial. For a higher δ, the optimal quality of the free module should be lower to reduce the potential negative effects of a free trial (i.e., cannibalized demand). Furthermore, given the same indirect learning intensity, the optimal quality of the free module becomes lower when there is a higher network effect intensity.
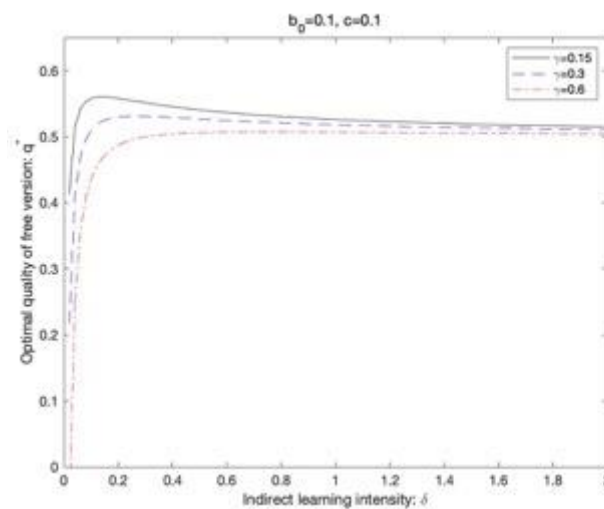
Figure 8. The impact of indirect learning intensity δ on the free trial strategy design.

5. Conclusions and Future Work

In this study, we proposed a consumer valuation experience-based learning framework based on cross-module synergies (i.e., both direct learning and indirect learning process) to explore the issue of feature-limited free trial model design and related price strategies. Under our consumer learning framework, consumers can update not only their cognition about the quality of the free version but also their beliefs about the premium version's quality through the free trial. In the model, we introduced the parameter of indirect learning intensity, which is a new approach compared with previous research. We considered a two-stage theoretical model. In the first stage, all potential consumers in the market take a free consumption of the functionality or services of the free version. In the second stage, each consumer decides whether to purchase the premium version of the software based on their updated belief about the quality of premium contents after the free trial.

Through the comprehensive use of two-stage mathematical theoretical models and numerical analysis, we made the following findings. First, we determined under what conditions a software firm should commercially employ a feature-limited freemium model. Second, we obtained some managerial insights for the feature-limited model and pricing strategies considering the associated trade-off between benefits and costs of free trials and other market conditions. Third, by means of numerical analysis, we acquired some commercial insights concerning the effect of several parameters on the feature-limited freemium model design and pricing strategies, i.e., consumers' prior beliefs about the quality of premium contents before free consumption, the network effect intensity, and indirect learning intensity.

In the mathematical analysis, we outlined three optimization problems and found corresponding analytical solutions. The three mathematical problems were as follows: (1) to solve the optimal price P^* when the quality of free version q is exogenously given, (2) to solve the optimal quality design of the free version q^* when the price P is exogenously fixed, and (3) to simultaneously treat the price P and the quality of the free version q as decision variables, to determine the optimal pricing P^* and optimal quality q^*. In optimization models of (1) and (2), the effects of the consumers' prior belief about the quality of the premium version before free trial b_0, the intensity of the network effect γ, and the intensity of indirect learning δ on the optimal pricing P^* and the optimal quality design of free version q^* were also discussed using mathematical analytical expression in the results. Since the impact of b_0, γ, and δ on P^* and q^* is not intuitive in optimization problem (3), we showed these results with numerical analysis. We used graphs to show the influence of b_0, γ, and δ on P^* and q^*. The graphs of the numerical analysis show the corresponding results from the perspective of pricing strategies and feature-limited freemium model design strategies. From the perspective of the pricing strategies, the relationship between b_0, γ,

and δ on P^* monotonically increased, i.e., the optimal pricing of the premium contents should be raised as b_0, γ, and δ increase. From the perspective of the results of the optimal feature-limited freemium model design, the optimal quality of the free version q^* decreased with the change of b_0 and γ. Interestingly, the effect of the intensity of indirect learning δ's trend on the optimal quality of free version q^* increased first and then decreased. That is, when δ was smaller, q^* increased with δ, but when δ was higher than a certain threshold, q^* decreased with an increase in δ.

Our research results have economic and commercial significance for feature-limited free trial strategies for software firms. We consider the actual commercial implications for the software industry. For software pricing strategies in the free trial business model, our analytical results indicate that the relationship between the optimal selling price of the premium contents of software and consumers' prior beliefs about the quality of premium version before free consumption, network effect intensity, and indirect learning intensity are all positively correlated. In business practice, the corresponding pricing of the premium version of software with stronger network effects also tends to be higher. For example, when we compared electronic book software (e.g., Amazon) and social software (e.g., chat software, LinkedIn, Tencent QQ, and Sina Weibo), it was obvious that the network effect of e-books is weaker than that of social software for the reason that e-books do not have the interactive functionalities of the networks between software users. According to our conclusions, the selling price of e-books should be set lower than that of social media software. This result is the same in commercial practices, and we could see that the prices of electronic books are usually very low. Many books in the Amazon store are less than 1 RMB, and many e-books are free for readers to download. Compared with e-books, the price of some social software with premium functionalities and features is higher. With regard to indirect learning intensity, our results are also consistent with the actual business situation of the software industry. Taking Microsoft Office software as an example, the similarity between the internal features and contents of the software is relatively high, so the indirect learning intensity of consumers is also higher. Therefore, in practical business applications, we observed that Microsoft Office software (e.g., Word, Excel, and PowerPoint) is relatively expensive compared to other categories of software, which also confirms the rationality and commercial implications of our conclusions in relation to pricing strategies.

In addition to the commercial significance of software pricing, our theoretical analysis has important implications for the business design of feature-limited free trial strategies. Our results demonstrated that the trends of optimal quality design for free modules with regard to consumers' prior belief about the quality of the premium version before a free trial and network effect intensity are both negatively correlated, which is reflected in the software industry. Let us first consider the impact of network effect intensity on the optimal quality design of a free software version, again using the example of the Microsoft Office software. The category of the Microsoft Office software is office software, which has relatively weak functionalities for social interaction between different users in the software; that is, the intensity of the network effect is small. According to our analysis, the weaker the network effect, the higher the quality of free content of the software, which coincides with Microsoft's practical business strategies. We have observed that the functionalities or features of the free trial version of Microsoft Office software are very powerful. Almost all of the software contents are available for the free trial to consumers in the market, which is conducive for consumers to better understand its software product's quality and thus leads to increased sales. Then, we discuss the effect of the intensity of indirect learning on the optimal quality design of the free module. Our theoretical results illustrated an interesting conclusion; that is, when the intensity of indirect learning is relatively low, the optimal quality design of the free module increases with the intensity of indirect learning, but when the indirect learning intensity is relatively high, the optimal quality design of the free version decreases with an increase in the indirect learning intensity. In the software industry context, for software with weaker indirect learning intensity, we can

refer to programming software (e.g., MATLAB and Mathematica) and music software (like QQ Music and NetEase CloudMusic). In programming software such as MATLAB, the functionality differences between the basic content and premium functionalities (some professional tool kits such as optimization toolkit or machine learning toolkit) are large. In music software, there are also certain differences between the free and premium songs. Therefore, for these two categories of software, the indirect learning intensity of consumers is relatively weak, and it is not easy for them to learn the quality of the premium contents through the trial of free functionalities. In terms of the comparison between the two, we should point out that music software has a higher indirect learning intensity than programming software, because there are certain correlations between different songs and the premium programming toolkit and basic functionalities of the programming software differ widely in terms of content. We have found in practice that the quality of free content of music software is also higher than that of programming software, which is exactly what our analytical results suggested. For software with relatively strong indirect learning intensity, we consider game software (e.g., Angry Birds) and social software. For games, there is a relatively high degree of similarity between different levels; this is also the case with social software, and different social interaction functionalities have some relevance with other features of the software. As for the comparison between the two, game software has a greater similarity between free levels and paid levels than social software. Therefore, consumers should have a stronger indirect learning intensity for game software than social software. Based on our previous discussion, the optimal quality design of the free contents of game software should be lower, while the optimal quality of the free version of social software is higher. This result is in accordance with the practice of the business strategy of software firms. The experience of free levels of game software can be of relatively lower quality to incentivize consumers to purchase commercial products, while a large proportion of the functionalities of social software are free for users; in other words, the quality of the free version of social software is designed to be of a higher standard.

There are several limitations of this paper that need to be considered. In our model, we considered the free trial strategy of one software monopolist. A possible future research direction is to elicit the competition model of two software firms and discuss the influence of competition on the freemium strategy design and pricing strategies. In addition, we used a two-stage model. It would be interesting to extend our model to the continuous-time setting by introducing a new parameter of time for future work. Finally, a future study could empirically test the results of our game theoretical model.

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Appendix A. Proofs

Proof of Proposition 1. To determine the change of π with P when q is fixed, we need to calculate the first-order partial derivative of π with respect to P:

$$\frac{\partial \pi}{\partial P} = (1-q)\left(1 + \gamma - \frac{2P + c}{b_0 + \delta q}\right).$$

When $0 \leq q \leq 1$, $1 - q \geq 0$. To determine the positive or negative sign of the first-order partial derivative $\frac{\partial \pi}{\partial P}$, we only need to determine the positive or negative sign of $\left(1 + \gamma - \frac{2P+c}{b_0+\delta q}\right)$. We denote this formula to be $f(P)$, $f(P) = \left(1 + \gamma - \frac{2P+c}{b_0+\delta q}\right)$.

Let $f(P) \geq 0$, and then we derive that $P \leq \frac{(1+\gamma)(b_0+\delta q)-c}{2}$; let $f(P) < 0$, and then we have $P > \frac{(1+\gamma)(b_0+\delta q)-c}{2}$. Note that P also has constraints in the optimization problem (6), so we have the following:

(i) $\frac{\partial \pi}{\partial P} \geq 0$ when $max\{(b_0 + \delta q)\gamma - c, 0\} \leq P \leq \frac{(1+\gamma)(b_0+\delta q)-c}{2}$;

(ii) $\frac{\partial \pi}{\partial P} < 0$ when $\frac{(1+\gamma)(b_0+\delta q)-c}{2} < P \leq (b_0 + \delta q)(1 + \gamma) - c$;

(iii) $\pi(P)$ increases in the interval $P \in \left[max\{(b_0 + \delta q)\gamma - c, 0\}, \frac{(1+\gamma)(b_0+\delta q)-c}{2}\right]$ and decreases in the interval $P \in \left(\frac{(1+\gamma)(b_0+\delta q)-c}{2}, (b_0 + \delta q)(1 + \gamma) - c\right]$. Therefore, for the optimization problem of Equation (6), $\pi(P)$ reaches its maximum at $P^* = \frac{(1+\gamma)(b_0+\delta q)-c}{2}$, we can obtain by calculation the corresponding $\pi^* = \frac{[(1+\gamma)(b_o+\delta q-c)]^2(1-q)}{4(b_o+\delta q)}$. $\square$

Proof of Proposition 2. In order to explore the change of π with q when P is exogenously fixed, the first-order partial derivative of π with respect to q should be calculated:

$$\frac{\partial \pi}{\partial q} = -P\left[(1 + \gamma) - \frac{(P+c)(b_0+\delta)}{(b_0+\delta q)^2}\right].$$

Since $P \geq 0$, the signs of the first-order partial derivative $\frac{\partial \pi}{\partial q}$ and $\left[(1 + \gamma) - \frac{(P+c)(b_0+\delta)}{(b_0+\delta q)^2}\right]$ are opposite. Denote $g(q) = (1 + \gamma) - \frac{(P+c)(b_0+\delta)}{(b_0+\delta q)^2}$, and then the problem is transformed into determining the sign of $g(q)$.

$g(q) = 0$ implies that $q = \frac{\sqrt{(P+c)(b_0+q)}-b_0\sqrt{1+\gamma}}{\delta\sqrt{1+\gamma}}$. Let $g(q) > 0$, and we have $q > \frac{\sqrt{(P+c)(b_0+q)}-b_0\sqrt{1+\gamma}}{\delta\sqrt{1+\gamma}}$; let $g(q) \leq 0$, and we can obtain $q \leq \frac{\sqrt{(P+c)(b_0+q)}-b_0\sqrt{1+\gamma}}{\delta\sqrt{1+\gamma}}$. Note that $\frac{\partial \pi}{\partial q}$ and $g(q)$ have opposing signs and $0 \leq q \leq 1$, and then we have the following results:

(i) $\frac{\partial \pi}{\partial q} \geq 0$ when $0 \leq q \leq \frac{\sqrt{(P+c)(b_0+q)}-b_0\sqrt{1+\gamma}}{\delta\sqrt{1+\gamma}}$;

(ii) $\frac{\partial \pi}{\partial q} < 0$ when $\frac{\sqrt{(P+c)(b_0+q)}-b_0\sqrt{1+\gamma}}{\delta\sqrt{1+\gamma}} < q \leq 1$;

(iii) $\pi(q)$ increases while $q \in \left[0, \frac{\sqrt{(P+c)(b_0+q)}-b_0\sqrt{1+\gamma}}{\delta\sqrt{1+\gamma}}\right]$ and decreases while $q \in \left(\frac{\sqrt{(P+c)(b_0+q)}-b_0\sqrt{1+\gamma}}{\delta\sqrt{1+\gamma}}, 1\right]$. The optimal solution for optimization problem (6) is $q^* = \frac{\sqrt{(P+c)(b_0+q)}-b_0\sqrt{1+\gamma}}{\delta\sqrt{1+\gamma}}$, and we can derive this by calculating that $\pi^* = \frac{P(1+\gamma)(b_0+\delta)-2P\sqrt{(P+c)(1+\gamma)(b_0+\delta)}+P(P+c)}{\delta}$. $\square$

Proof of Proposition 3. When both P and q are considered to be optimized variables, the optimization problem we need to solve can be described as follows:

$$\max_{P, q} \pi(P, q | b_0, \delta, \gamma, c) = PQ_p = P \cdot \frac{(1-q)[(1+\gamma)(b_0+\delta q)-(P+c)]}{b_0+\delta q},$$

$$\text{s.t.} \quad -P \leq 0,$$
$$-P + (b_0 + \delta q)\gamma - c \leq 0,$$
$$P - (b_0 + \delta q)(1 + \gamma) + c \leq 0,$$
$$-q \leq 0,$$
$$\text{and } q - 1 \leq 0.$$

According to the convex optimization theory, the Lagrangian function of the above optimization problem is formulated as

$$L(P, q, \lambda) = P \frac{(1-q)[(1+\gamma)(b_0+\delta q)-(P+c)]}{b_0+\delta q} + \lambda_1(-P) + \lambda_2[-P + (b_0 + \delta q)\gamma - c] +$$
$$\lambda_3[P - (b_0 + \delta q)(1 + \gamma) + c] + \lambda_4(-q) + \lambda_5(q - 1)$$

In order to solve the optimization problem, the Karush–Kuhn–Tucker condition must be satisfied:

$$\frac{\partial L(P,q,\lambda)}{\partial P} = (1-q)\left(1 + \gamma - \frac{2P+c}{b_0+\delta q}\right) - \lambda_1 - \lambda_2 + \lambda_3 = 0,$$
$$\frac{\partial L(P,q,\lambda)}{\partial q} = -P\left[1 + \gamma - \frac{(b_0+\delta)(P+c)}{(b_0+\delta q)^2}\right] + \lambda_2\delta\gamma - \lambda_3\delta(1+\gamma) - \lambda_4 + \lambda_5 = 0,$$
$$\lambda_i \geq 0,\ i = 1,\ 2,\ 3,\ 4,\ 5,$$
$$\lambda_1(-P) = 0,\ and - P \leq 0,$$
$$\lambda_2[-P + (b_0 + \delta q)\gamma - c] = 0,\ and - P + (b_0 + \delta q)\gamma - c \leq 0,$$
$$\lambda_3[P - (b_0 + \delta q)(1 + \gamma) + c] = 0,\ and\ P - (b_0 + \delta q)(1 + \gamma) + c \leq 0,$$
$$\lambda_4(-q) = 0,\ and - q \leq 0,$$
$$\lambda_5(q - 1) = 0,\ and\ q - 1 \leq 0.$$

By mathematically solving the above equations, we can obtain that the optimal solution of the original optimization problem should satisfy $\lambda_i = 0$ ($i = 1,\ 2,\ 3,\ 4,\ 5$). The optimal solution is as follows:

$$P^* = \frac{-4c+(b_0+\delta)(1+\gamma)+\sqrt{(b_0+\delta)^2(1+\gamma)^2+8c(b_0+\delta)(1+\gamma)}}{8},\ and$$
$$q^* = \frac{(\delta-3b_0)(1+\gamma)+\sqrt{(b_0+\delta)^2(1+\gamma)^2+8c(b_0+\delta)(1+\gamma)}}{4\delta(1+\gamma)}.$$

□

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