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A Comparison of Macaulay Approximations

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Abstract: We discuss several known formulas that use the Macaulay duration and convexity of commonly used cash flow streams to approximate their net present value, and compare them with a new approximation formula that involves hyperbolic functions. Our objective is to assess the reliability of each approximation formula under different scenarios. The results in this note should be of interest to actuarial candidates and educators as well as analysts working in all areas of actuarial practice.

Keywords: Macaulay duration; Macaulay convexity; net present value of cash flows

1. Introduction

Actuaries and actuarial science students at universities all over the world are familiar with approximation formulas for the present value of cash flow streams using some notion of cash flow duration or convexity. For example, the syllabus of Exam FM of the US-based Society of Actuaries includes the topic of approximations using the Macaulay and modified duration and convexity, while the UK-based Institute and Faculty of Actuaries in its material for exam CM1 mentions approximations derived from a Taylor series expansion.

Beside the academic and pedagogical interest in such approximation formulas, one may also consider the practical value in the management of interest rate risk. Although abundant computing power has enabled firms to implement elaborate immunization strategies that incorporate multi-factor stochastic interest rate models, non-parallel yield curve shifts, and complicated asset and liability characteristics, the restrictions posed by a simplistic valuation model are not unreasonable if rates remain historically low, yield curves stay relatively flat, and we can control the potential errors. Indeed, it may be helpful to know which approximation formula proves to be the most reliable, and to use it as a quick validation tool when time constraints preclude the use of a more sophisticated approach.

Alps (2017) describes a realistic scenario involving an investment actuary and her CEO, where the use of an approximation formula would be warranted or even necessitated. This is especially true in today's world of fast-changing rates, when companies have to react almost instantly to benchmark fund rates and quantitative tightening decisions by the Federal Reserve or other central banks.

In this note, we discuss several known formulas that use the Macaulay duration and convexity of commonly used cash flow streams to approximate their net present value, and compare them with a new approximation formula that involves hyperbolic functions. In addition to annuities, dividend stocks, and bonds, we also consider the cases of negative payments and embedded options to perform a deeper assessment. The notions of effective duration and convexity are defined in the next section and used to price the embedded options. Our objective is to measure the reliability of each approximation formula under different scenarios. As alluded to earlier, we only consider parallel interest rate shocks to flat yield curves.

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1.1. Literature Review

The idea of using a bond's duration to approximate changes to its price goes back to Macaulay (1938). Some authors credit Fischer and Weil (1971) with the publication of the first duration–convexity approximation formula. Enhancements of that formula by controlling the missing higher-order terms and incorporating passage of time were announced in Jarjir and Rakotondratsimba (2008, 2012), though the resulting formulas contain parameters that are unintuitive and hard to calibrate. A conceptually simpler formula was given in Barber (1995), and independently in Livingston and Zhou (2005) for the modified duration, which was subsequently generalized to a duration–convexity model in Tchuindjo (2008). Further work in Barber and Dandapani (2017) considered negative-yielding bonds, and Johansson (2012) added passage of time. A separate duration–convexity formula appears in Alps (2017) and is applied to an empirical study of basic immunization strategies in Nie et al. (2021), while a very recent paper by Barber (2022) further generalizes a duration–convexity approximation by introducing an additional 'compounding' parameter. Finally, traditional approximations have been implemented in statistical analysis packages; see Lee (2021) for R code.

1.2. Notation

We denote the net present value of a cash flow stream by P. The interest rate r is annualized and continuously compounded (i.e., force of interest). Δr is the change in interest rates from the initial value r_0 to r. Finally, the annual discount factor v is by definition equal to e^{-r}.

Throughout the remainder of this paper and for convenience, assume $r_0 = 1.6\%$, which is approximately the yield on the 10-year T-bond at the beginning of this year.

2. Materials and Methods

Recall that the *Macaulay duration* of a stream of cash flows $\{CF_{t_j}\}_{j=1}^{n}$ being paid at future times $\{t_j\}_{j=1}^{n}$ is defined by

$$d = \frac{\sum_{j=1}^{n} CF_{t_j}\, v^{t_j}\, t_j}{\sum_{j=1}^{n} CF_{t_j}\, v^{t_j}} = -\frac{dP/dr}{P},$$

while its *Macaulay convexity* is given by

$$c = \frac{\sum_{j=1}^{n} CF_{t_j}\, v^{t_j}\, t_j^2}{\sum_{j=1}^{n} CF_{t_j}\, v^{t_j}} = \frac{d^2P/dr^2}{P}.$$

We do not consider the modified duration here because the Macaulay duration has a more intuitive interpretation (being the 'average' timing of the cash flows) and tends to result in tighter approximations for non-negative rates.

In the case of bonds with embedded options, it will be necessary to price the value of the option using a simple Black model. Recall that the pricing formula for, say, a European call option with expiration at time t and strike K is

$$V = v^t (P\Phi(d_1) - K\Phi(d_2))$$

where Φ represents the standard normal c.d.f. and the quantities $d_{1,2}$ are given by

$$d_{1,2} = \frac{\ln(P/K)}{\sigma\sqrt{t}} \pm \frac{\sigma\sqrt{t}}{2}$$

with σ the bond price volatility. We also need more flexible measures of bond duration and convexity. To that end, define the *effective duration* by means of

$$d_e = -\frac{P(r_0 + \Delta r) - P(r_0 - \Delta r)}{2P_0\,\Delta r}$$

and the *effective convexity* as

$$c_e = \frac{P(r_0 + \Delta r) - 2P_0 + P(r_0 - \Delta r)}{P_0\,(\Delta r)^2}.$$

2.1. Fischer–Weil's Approximation

This follows immediately from Calculus and the definitions above.

$$\frac{\Delta P}{P_0} \approx -d_0\,\Delta r + \frac{c_0}{2}\,(\Delta r)^2. \tag{1}$$

It is assumed that the Macaulay duration and convexity are computed at rate r_0, hence the subscripts.

2.2. Barber' 1995 Approximation

Instead of the second-order Taylor polynomial of P, we consider the first-order Taylor polynomial of $\ln P$, thus obtaining

$$\ln P \approx \ln P_0 - d_0\,\Delta r,$$

thus

$$P \approx P_0\,e^{-d_0\,\Delta r}. \tag{2}$$

Unlike the first-order Taylor polynomial in P that has no convexity, the functional form of Barber's approximation bequeaths it with a certain degree of positive curvature. This leads to good approximation results whenever $c_0 \approx d_0^2$ and poor performance for $c_0 < 0$.

2.3. Tchuindjo' Approximation

Similar to Barber's approximation, but involving the second-order Taylor polynomial of $\ln P$

$$\ln P \approx \ln P_0 - d_0\,\Delta r + \frac{c_0 - d_0^2}{2}\,(\Delta r)^2, \tag{3}$$

from which one solves for P. The added quadratic term gives better results in cases where $c_0 - d_0^2$ is non-trivial, but may still introduce large errors whenever $c_0 < 0$.

2.4. Alps' Approximation

The approximation formula and its derivation can be found in Alps (2017). The central idea in the derivation of this approximation is to compute a Taylor polynomial for the current value of the cash flow stream at time $t = d_0$. This choice results in high accuracy in situations where $d_0 > 0$, and less so for $d_0 < 0$.

We have rewritten it below in terms of continuously compounded interest rates.

$$P \approx P_0\,e^{-d_0\,\Delta r}\left(1 + \frac{c_0 - d_0^2}{2}\left(e^{\Delta r} - 1\right)^2\right). \tag{4}$$

2.5. Hyperbolic Approximation

We have not encountered this approximation formula in the literature and we assume its derivation is presented here for the first time. Consider the homogeneous differential equation $P'' - c_0 P = 0$ that mimics the definition of Macauley convexity given earlier in this note. Its general solution takes the form

$$P = a\,e^{\sqrt{c_0}\,r} + b\,e^{-\sqrt{c_0}\,r}$$

(do not worry for the time being about the case $c_0 < 0$.) Setting $P(r_0) = P_0$ and $P'(r_0) = -d_0\, P_0$, which is a reformulation of the definition of Macauley duration, one obtains the approximation

$$P \approx P_0 \left(\frac{1}{2} \left(1 - \frac{d_0}{\sqrt{c_0}} \right) e^{\sqrt{c_0}\, \Delta r} + \frac{1}{2} \left(1 + \frac{d_0}{\sqrt{c_0}} \right) e^{-\sqrt{c_0}\, \Delta r} \right)$$

which can be rewritten as

$$P \approx P_0 \left(\cosh(\sqrt{c_0}\, \Delta r) - \frac{d_0}{\sqrt{c_0}} \sinh(\sqrt{c_0}\, \Delta r) \right). \tag{5}$$

The well-known trig identities

$$\cosh(i\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta\,, \qquad \sinh(i\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2} = i \sin\theta$$

can be used in the case $c_0 < 0$ to obtain

$$P \approx P_0 \left(\cos\left(\sqrt{|c_0|}\, \Delta r \right) - \frac{d_0}{\sqrt{|c_0|}} \sin\left(\sqrt{|c_0|}\, \Delta r \right) \right),$$

which is useful whenever there is a computational issue with imaginary numbers.

In the next section, we demonstrate that the hyperbolic approximation is less prone to errors than other well-known approximations in situations where the duration and/or convexity are negative. Recall that negative convexity cash flow streams can be easily constructed with the addition of negative cash flows to a stream of positive payments, when considering callable bonds, or with mortgage-backed securities due to the prepayment option in conventional residential mortgages.

3. Results

Approximation formulas such as Equations (1)–(5) should ideally be intuitive and behave well in special cases.

(i) The simplest cash flow is cash, which has trivial duration and convexity and is unaffected by interest rate changes. By substituting $d_0 = c_0 = 0$ or taking the corresponding limit in the case of (5) and using the fact that

$$\lim_{\theta \to 0} \frac{\sinh\theta}{\theta} = \lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1,$$

we obtain $P = P_0$ as expected.

(ii) Next, take a zero-coupon bond, for which $c_0 = d_0^2$: Except for Fischer–Weil's approximation, the rest reduce to Barber's approximation, which is perfectly accurate in this case. On the other hand, the error associated with Fischer-Weil's approximation increases with the bond duration and it can be as high as 0.56% for a 30-year zero-coupon bond after a 100 bp increase in rates.

(iii) For a convexity-hedged ($c_0 \to 0$) portfolio, Fischer–Weil's and the hyperbolic approximations reduce to the first-order approximation $\Delta P \approx -d_0\, P_0\, \Delta r$. The corresponding results for the other approximation formulas are not as intuitive and their accuracy relative to the above approximation cannot be determined without additional details about the cash flow characteristics.

We supplement the theoretical tests above with some concrete examples.

(iv) Consider a 10-year annuity-immediate with annual payments of 10. Recall that our assumption is $r_0 = 1.6\%$ and compute the present value $P_0 = 10 a_{\overline{10|}} = 91.6728$. Another easy calculation gives the Macaulay duration and convexity as $d_0 = 5.3681$ and $c_0 = 37.0554$, respectively.

In Table 1, the exact value of P is computed using the same formula as for P_0 but at the new continuously compounded rate.

Table 1. PV of 10-year annuity with annual payments of 10.

Δr	Exact	Fischer-Weil	Barber	Tchuindjo	Alps	Hyperbolic
−100 bp	96.7682	96.7637	96.7283	96.7682	96.7669	96.7668
−80 bp	95.7207	95.7184	95.6954	95.7207	95.7199	95.7199
−60 bp	94.6876	94.6866	94.6735	94.6876	94.6871	94.6873
−40 bp	93.6687	93.6684	93.6625	93.6687	93.6685	93.6686
−20 bp	92.6639	92.6638	92.6623	92.6639	92.6638	92.6639
0 bp	91.6728	91.6728	91.6728	91.6728	91.6728	91.6728
20 bp	90.6954	90.6954	90.6939	90.6954	90.6953	90.6954
40 bp	89.7313	89.7316	89.7254	89.7313	89.7311	89.7314
60 bp	88.7804	88.7813	88.7672	88.7804	88.7800	88.7807
80 bp	87.8425	87.8447	87.8193	87.8425	87.8417	87.8432
100 bp	86.9173	86.9216	86.8815	86.9173	86.9162	86.9186

We can observe that Tchuindjo's approximation outperforms the rest, while Barber's lags behind for sizable rate changes. This was to be expected, since Barber's approximation lacks a convexity term and will not do well in cases when $c_0 - d_0^2$ is non-trivial. On the other hand, Alps' and the hyperbolic approximations are roughly equally accurate behind Tchuindjo's.

(v) Next, add a negative cash flow at time 20. We have chosen $CF_{20} = -120$ in the example below; the net present value is $P_0 = 4.5349$ and the Macaulay duration and convexity are $d_0 = -275.7817$ and $c_0 = -6,936.8498$, respectively.

Looking at Table 2 below, it may come as a surprise that the approximations by Tchuindjo and Alps blow up completely. However, we can provide a simple mathematical explanation for the bizarre behavior. Whenever $c_0 < 0$, the quadratic term of these two approximations that includes the expression $c_0 - d_0^2$ has the potential to be extremely influential. As Δr increases, said term can overwhelm the baseline value P_0 and the linear term, resulting in large errors. Barber's approximation exhibits the opposite weakness: missing a quadratic term implies that the negative convexity is not accounted for at all. In fact, for suitable CF_{20}, we can obtain $d_0 = 0$, in which case Barber's approximation fails to yield any results.

Table 2. NPV of 10-year annuity with annual payments of 10 and a payment of −120 at time 20.

Δr	Exact	Fischer-Weil	Barber	Tchuindjo	Alps	Hyperbolic
−100 bp	−9.6622	−9.5445	0.2877	0.0045	−0.8684	−8.0590
−80 bp	−6.5366	−6.4769	0.4994	0.0351	−0.7850	−5.7162
−60 bp	−3.5601	−3.5352	0.8669	0.1946	−0.3873	−3.2151
−40 bp	−0.7266	−0.7193	1.5048	0.7747	0.5372	−0.6250
−20 bp	1.9698	1.9707	2.6123	2.2128	2.1924	1.9824
0 bp	4.5349	4.5349	4.5349	4.5349	4.5349	4.5349
20 bp	6.9742	6.9733	7.8725	6.6685	6.6069	6.9619
40 bp	9.2929	9.2859	13.6664	7.0357	4.8785	9.1962
60 bp	11.4960	11.4726	23.7243	5.3262	−10.6004	11.1758
80 bp	13.5885	13.5335	41.1846	2.8930	−64.7471	12.8461
100 bp	15.5748	15.4686	71.4950	1.1275	−215.8388	14.1608

We conclude this example by mentioning that the top-performing approximation is Fischer–Weil's, while the hyperbolic approximation is second-best.

(vi) Let us now consider a dividend stock, whose theoretical price is computed using *Gordon's dividend discount model*

$$P = \frac{D}{r - g}$$

with D representing next year's dividend and g its constant continuously compounded growth rate in perpetuity. A quick calculation gives $d = (r - g)^{-1}$ and $c = 2(r - g)^{-2}$; for $g = 0.6\%$ we obtain $d_0 = 100$ and $c_0 = 20,000$. Assume $D = 1$.

Some of the results in Table 3 may appear counterintuitive at first sight. Gordon's model suggests that P has an inverse relationship to r; however, all approximations except for Alps' and Barber's eventually produce a divergent estimate for P as Δr increases. However, this is explained by the fact that we are attempting to trace a hyperbola using quadratic curves. Moreover, all approximations struggle to keep up with P for large negative values of Δr.

Table 3. Price of a dividend stock with $D = 1$ and $g = 0.6\%$.

Δr	Exact	Fischer-Weil	Barber	Tchuindjo	Alps	Hyperbolic
−100 bp	n/a	300.0000	271.8282	448.1689	403.4619	354.6482
−80 bp	500.0000	244.0000	222.5541	306.4854	291.5285	269.3175
−60 bp	250.0000	196.0000	182.2119	218.1472	213.9771	205.6762
−40 bp	166.6667	156.0000	149.1825	161.6074	160.7412	158.5990
−20 bp	125.0000	124.0000	122.1403	101.8813	101.8813	101.8813
0 bp	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000
20 bp	83.3333	84.0000	81.8731	83.5270	83.4590	83.7590
40 bp	71.4286	76.0000	67.0320	72.6149	72.2257	74.2635
60 bp	62.5000	76.0000	54.8812	65.7047	64.4487	70.7488
80 bp	55.5556	84.0000	44.9329	61.8783	58.8586	72.9319
100 bp	50.0000	100.0000	36.7879	60.6531	54.6026	80.9885

Overall, Alps' approximation proves to be the most dependable for moderate changes in the interest rates.

(vii) Next, consider a 10-year bond with a coupon rate of r_0 and face value of 100. A quick calculation yields $d_0 = 9.3151$ and $c_0 = 90.6932$.

It turns out that the last three approximation formulas clearly outperform the rest, with Tchuindjo's having a slight advantage over Alps' and the hyperbolic approximation, as evidenced from Table 4. The subpar performance of Fischer–Weil on bonds is one of the reasons why this approximation is not widely utilized, despite its robustness in cases such as (v).

It has been shown empirically that although investment-grade bonds fall in price when interest rates rise, that is not necessarily the case with high-yield bonds whose duration can be negative due to default risk; see Melentyev and Yu (2020). For such bonds, care should be exercised when using the approximations by Tchuindjo or Alps.

Table 4. PV of 10-year par bond with coupon rate r_0.

Δr	Exact	Fischer-Weil	Barber	Tchuindjo	Alps	Hyperbolic
−100 bp	109.7839	109.7686	109.7628	109.7843	109.7836	109.7830
−80 bp	107.7501	107.7423	107.7368	107.7503	107.7499	107.7497
−60 bp	105.7556	105.7523	105.7482	105.7557	105.7555	105.7554
−40 bp	103.7996	103.7986	103.7963	103.7996	103.7995	103.7995
−20 bp	101.8813	101.8812	101.8805	101.8813	101.8813	101.8813
0 bp	100.0000	100.0000	100.0000	100.0000	100.0000	100.0000
20 bp	98.1550	98.1551	98.1542	98.1550	98.1550	98.1550
40 bp	96.3456	96.3465	96.3425	96.3455	96.3454	96.3456
60 bp	94.5710	94.5742	94.5642	94.5709	94.5707	94.5712
80 bp	92.8306	92.8381	92.8188	92.8304	92.8301	92.8310
100 bp	91.1238	91.1383	91.1056	91.1234	91.1229	91.1246

(viii) Finally, assume the bond is callable, with the European call strike set at $K = 101.0000$ and bond price volatility $\sigma = 8\%$. The call is exercised a year ahead of the bond's maturity and has price $V = 9.9431$, which is subtracted from the price of a conventional bond to arrive at the callable bond price. Using $\Delta r = 20$ bp in the calculation of the effective duration and convexity, we obtain $d_e = 5.3333$ and $c_e = 50.3993$. The positive convexity may surprise some readers, but note that the convexity turns negative when the interest rate gets closer to 0 and the bond price approaches the strike.

It is important to observe in Table 5 that none of the approximation formulas can consistently outperform the rest, if our objective is to estimate the full range of prices for such a bond. In effect, we are trying to approximate a function with an inflection point using quadratic curves, and thus significant approximation errors are inevitable.

Table 5. NPV of 10-year par bond with coupon rate r_0, callable for 101 at $t = 9$.

Δr	Exact	Fischer-Weil	Barber	Tchuindjo	Alps	Hyperbolic
−100 bp	95.0594	95.0868	94.9903	95.0946	95.0913	95.0910
−80 bp	94.0367	94.0445	93.9824	94.0485	94.0464	94.0466
−60 bp	93.0197	93.0204	92.9853	93.0220	93.0209	93.0213
−40 bp	92.0148	92.0144	91.9987	92.0149	92.0144	92.0146
−20 bp	91.0265	91.0265	91.0226	91.0266	91.0265	91.0266
0 bp	90.0569	90.0569	90.0569	90.0569	90.0569	90.0569
20 bp	89.1053	89.1053	89.1014	89.1053	89.1052	89.1053
40 bp	88.1692	88.1720	88.1560	88.1715	88.1710	88.1717
60 bp	87.2437	87.2568	87.2207	87.2552	87.2541	87.2559
80 bp	86.3227	86.3597	86.2953	86.3559	86.3540	86.3577
100 bp	85.3991	85.4808	85.3797	85.4735	85.4705	85.4768

The only useful conclusion is that the hyperbolic approximation is never the worst one, since it tends to be "sandwiched" between other approximations.

4. Discussion

We have established through a number of theoretical considerations and concrete examples that the accuracy of various Macaulay approximations can vary widely. Approximations that outperform in one case turn out to be unreliable in another case. The hyperbolic approximation, introduced in this paper, exhibited modest errors in most cases and thus the most reliability among the five approximations studied.

We can envision a variety of uses for the results presented here:

- To perform expeditious interest risk calculations by practitioners;

- As a study note to gain insight into risk management concepts that are tested in the actuarial examinations in the US and Europe;
- As potential areas of student research or as assigned projects that utilize real financial data in actuarial science classes taught by academics.

There is also potential to expand the scope of this study by incorporating non-flat yield curves, key rate durations, passage of time, and more complex financial instruments.

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