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Spread Option Pricing Under Finite Liquidity Framework

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Abstract: This work explores a finite liquidity model to price spread options and assess the liquidity impact. We employ Kirk approximation for computing the spread option price and its delta. The latter is needed since the liquidity impact is caused by the delta hedging of a large investor. Our main contribution is a novel methodology to price spread options in this paradigm. Kirk approximation in conjunction with Monte Carlo simulations yields the spread option prices. Moreover, the antithetic and control variates variance reduction techniques improve the performance of our method. Numerical experiments reveal that the finite liquidity causes a liquidity value adjustment in option prices ranging from 0.53% to 2.81%. The effect of correlation on prices is also explored, and as expected the option price increases due to the diversification effect, but the liquidity impact decreases slightly.

Keywords: pricing spread options; finite liquidity market model; Kirk approximation formula

1. Introduction

This paper explores liquidity value adjustment (LVA) of pricing spread options in a finite liquidity market model. A Black Scholes setting is considered with liquidity costs on the trading of a risky asset, and the latter is driven by the delta hedging of a large trader.

A spread option is a financial instrument that gives the holder rights to exchange the price difference between two risky assets with a specific strike price at maturity. This financial is very popular in the commodity market, such as the oil market. The profit margin, called crack spread, is measured by the difference between the output revenue (petroleum product) and input cost (crude oil). Since it can be influenced by many factors like seasonality and economic situation, the crack spread option becomes an essential tool for refinery producers to secure their profit margins.

Let us go over the literature on pricing spread options. In the special case of zero strike, the spread option becomes an exchange option, which can be priced using a Black Scholes-type formula derived by Margrabe (1978). This is no longer the case for a spread option (with nonzero strike), and several approaches have been undertaken. One of these is the Kirk approximation, which offers an approximate expression for the price, see Kirk (1995), Li et al. (2008), Lo (2013), and Bjerksund and Stensland (2006). One advantage of this approach is that it extends to the multi-asset spread option (with three or more underlyings) and other types of options, see Li et al. (2010), Lau and Lo (2014), and Chen and Deng (2024). Other methods for pricing spread options rely on Monte Carlo simulations, see Korn and Zeytun (2013); numerical methods, see Heidarpour-Dehkordi and Christara (2018); Fourier inversion, see Hurd and Zhou (2010); and Copula methods, see Berton and Mercuri (2024). An overview of these methods is presented in the work of Carmona and Durrleman (2003).

The Black Scholes model assumes perfect liquidity, constant interest rate, constant volatility, and correlations. These assumptions can be relaxed to make the model more realistic, see Levendis and Maré (2022), Feng et al. (2014), Pasricha and He (2023), and Wang (2022). Our work relaxes the perfect liquidity assumption. The pioneer work in this direction is that of Almgren and Chriss (2000), and the follow-up works are those of Liu

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and Yong (2005) and Wilmott and Schönbucher (2000). The aforementioned papers deal with single asset options; for spread options, we mention Yazdanian et al. (2014), Yazdanian and Pirvu (2016), Zhang and Pirvu (2020), and Zhang and Pirvu (2021).

Let us point out a recent research stream on pricing options with liquidity risk: Pasricha and He (2023), Pasricha et al. (2022), He et al. (2024), He and Lin (2023), and He and Lin (2024). These works undertake a different approach to liquidity, assuming an exogenous, trade-independent liquidity impact.

Our paper continues the research on pricing spread options with liquidity impact started by Yazdanian et al. (2014), Yazdanian and Pirvu (2016), Zhang and Pirvu (2020), and Zhang and Pirvu (2021). The novelty is the use of Kirk approximation in conjunction with Monte Carlo simulations. We revisit the Kirk approximation and employ it to obtain the spread option delta necessary to assess the liquidity impact. Indeed, we consider the finite liquidity framework in which the price impact is caused by a large trader who is a delta hedger; thus, its delta enters the price dynamic of the illiquid underlying asset. The risk-neutral pricing formula available in this paradigm makes it possible to compute the spread option price by Monte Carlo simulations. One needs to simulate the illiquid asset price throughout time since its distribution is unknown due to the liquidity impact, and this is performed through the Euler–Maruyama scheme. The simulation results point to an option price liquidity value adjustment (LVA) ranging from 0.53% to 2.81%. The correlation effect on the LVA is explored, and it turns out that the LVA is decreased by a decrease in correlation. Our methodology extends naturally to the multi-asset spread option (with three or more underlyings), as it relies on the Kirk approximation, which is available in this setting.

Let us go over the contribution of this paper. We present a novel methodology to price the spread options in a finite liquidity framework. The illiquid asset price is nonlinear and is affected by the trading strategy of a large trader. This is the delta hedging in the full liquidity model, and due to the lack of a closed-form formula for it, we approximate it by means of Kirk approximation. Our approach is a combination of Monte Carlo simulations and Kirk approximation. Enhancements of Monte Carlo simulations, the antithetic and control variates variance reduction techniques, are employed in our numerical experiments. The LVA due to the liquidity impact is explored and analyzed.

The remainder of this paper is organized as follows: Section 2 presents the spread options, and Section 3 the full liquidity model. Section 4 presents the finite liquidity framework, and Section 5 presents the numerical results. Section 6 concludes the paper and points to future research.

2. Spread Options

Spread options are financial contracts that give the right to exchange assets for a strike price (the spread). In the following, we will focus on the use of spread options in commodity markets. Quality spread options are based on the differences between the prices of the grades of the same commodity; for instance, crack spread options and the heating of oil/crude oil or gasoline/crude oil. The rationale for buying a crack spread is to hedge the difference between their input costs and output prices. In the case of an oil crack spread, a typical buyer of this product will possess or buy crude oil and sell refined products, such as in the case of a petroleum refinery. Another possible use of an oil crack spread is to hedge against the equity value of a petroleum refinery. There are several factors which affect a crack spread, and they are related to their effects on the demand/supply. In the case of an oil crack spread, winter will make this product more valuable as the demand for heating oil increases; on the other hand, recession times will have the opposite effect as the demand for gasoline decreases.

3. The Full Liquidity Model

In this section, we revisit the full liquidity model and some methods for pricing exchange options and spread options.

3.1. Correlated Geometric Brownian Motion

Our financial market mode consists of a money market account and two risky underlying assets. The money market account accrues interest at the riskless rate $r > 0$. The price dynamics of the risky assets are modeled by two correlated Geometric Brownian Motions (GBMs). Let us start with two independent Brownian motions $W_1(t)$, $W_2(t)$ and define a pair of correlated Brownian motions as follows:

$\hat{W}_1(t) = W_1(t), \hat{W}_2(t) = \rho W_1(t) + \sqrt{1-\rho^2}W_2(t), d\hat{W}_1(t)d\hat{W}_2(t) = \rho dt$, where ρ is a correlation coefficient and $\rho \in [-1, 1]$.

The prices of the two risky underlying assets under the real-world probability $\mathbb{P}$ measure are:

$$S_1(t) = S_1(0)e^{(\mu_1 - q_1 - \frac{1}{2}\sigma_1^2)t + \sigma_1 \hat{W}_1(t)}$$
$$S_2(t) = S_2(0)e^{(\mu_2 - q_2 - \frac{1}{2}\sigma_2^2)t + \sigma_2 \hat{W}_2(t)}$$

where:

$q_i =$ the convenient yield (like dividends in the equity market)
$\mu_i =$ the deterministic drift for the corresponding asset
$\sigma_i =$ the deterministic volatility for the corresponding asset

They satisfy the following stochastic differential equations (SDEs):

$$dS_1(t) = \mu_1 S_1(t)dt + \sigma_1 S_1(t)d\hat{W}_1(t)$$
$$dS_2(t) = \mu_2 S_2(t)dt + \sigma_2 S_2(t)d\hat{W}_2(t)$$

Let $\mathbb{Q}$ be the risk-neutral probability measure and θ the market price of risk, which is given by

$$\theta = \begin{bmatrix} \frac{\mu_1 - r}{\sigma_1} \\ \frac{\mu_2 - r}{\sigma_2} \end{bmatrix}$$

Then, the two-dimensional Brownian motion under $\mathbb{Q}$, denoted $\hat{W}^{\mathbb{Q}}(t)$, is given by

$$d\hat{W}^{\mathbb{Q}}(t) = d\hat{W}(t) + \theta dt,$$

and

$$\begin{aligned} dS_1(t) &= rS_1(t)dt + \sigma_1 S_1(t)d\hat{W}_1^{\mathbb{Q}}(t) \\ dS_2(t) &= rS_2(t)dt + \sigma_2 S_2(t)d\hat{W}_2^{\mathbb{Q}}(t) \end{aligned} \tag{1}$$

3.2. Exchange Option and Margrabe Formula

The payoff of a European-style exchange option is $(S_1(T) - S_2(T))^+$; it basically gives the right but not the obligation to exchange an asset with price S_1 for an asset with price S_2 at a predetermined maturity T. By using the risk-neutral pricing formula, one can derive the price as a conditional expectation under the risk-neutral measure:

$$C(t, S_1(t), S_2(t)) = E_t^{\mathbb{Q}}[(S_1(T) - S_2(T))^+] \tag{2}$$

William Margrabe (see Yazdanian et al. (2014)) derived this expectation in closed form as

$$C(t, S_1(t), S_2(t)) = S_1(t)e^{-q_1\tau}N(d_1(t)) - S_2(t)e^{-q_2\tau}N(d_2(t)),$$

where:

$$
\begin{aligned}
\tau &= T - t \\
d_1(t) &= \frac{\log \frac{S_1(t)}{S_2(t)} + (q_2 - q_1 + \frac{\sigma^2}{2})}{\sigma\sqrt{\tau}} \\
d_2(t) &= d_1 - \sigma\sqrt{\tau} \\
\sigma &= \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \\
\rho &= \text{ correlation between } S_1 \text{ and } S_2,
\end{aligned}
$$

In Table 1 below, we present the prices of an exchange option for a set of parameters, which we use later on to test the convergence of the Monte Carlo simulation price of a spread option by setting $K = 0$.

Table 1. Exchange option price.

Parameters	**Values**
Convenient yield (q)	[0.03, 0.02]
Initial price	[110, 100]
Vol	[0.1, 0.15]
Correlation	1
Strike	0
τ	1
r	0.05
Price	8.821

3.3. Kirk Approximation of Spread Option

The payoff of a European-style spread option is $(S_1(T) - S_2(T) - K)^+$, where $K > 0$ is the strike price. It turns out that finding an analytical solution for the spread option is no longer possible, since the linear combination of log-normal processes (sum of GBMs) is not log-normally distributed.

Fortunately, there are several ways to approximate the price of a spread option; in this paper, we adopt Kirk approximation (see Kirk (1995)). Given two risky assets following correlated GBMs, the price at time t, given $S_1(t), S_2(t)$, denoted $V(t, S_1(t), S_2(t))$, has the following Kirk approximation:

$$
V(t, S_1(t), S_2(t)) \approx .C(t, S_1(t), S_2(t)) = e^{-r\tau}(F_1 N(d_1) - (F_2 + K)N(d_2)) \tag{3}
$$

where:

$$
\begin{aligned}
\tau &= T - t \\
F_i &= e^{(r-q_i)\tau} S_i(t) \text{ is the value of future contract of } S_i \\
d_1 &= \frac{\log \frac{F_1}{F_2+K} + \frac{\sigma^2}{2}}{\sigma\sqrt{\tau}} \\
d_2 &= d_1 + \sigma\sqrt{\tau} \\
\sigma &= \sqrt{\sigma_1^2 + \sigma_{eff}^2 - 2\rho\sigma_1\sigma_{eff}} \\
\sigma_{eff} &= \sigma_2\left(\frac{F_1}{F_2 + K}\right)
\end{aligned}
$$

The idea behind this approximation is to pretend that $S_2(T) + K$ is log-normally distributed. It has been shown that the Kirk approximation is a good approximation when the strike price K is far less than S_2; indeed, in such a case, $S_2(T) + K$ is close to log-normal (see Lo (2013) for more on this). In our case illustration, $S_2(t) = 100 >> K = 5$.

The reason why this approximation is appealing to us is that it yields an analytical formula of the value of the spread option, which enables us to calculate the corresponding Greeks, and this will be an essential part of the finite liquidity model considered later on.

In the following Table 2, we test the accuracy of the Kirk approximation by comparing it with the Monte Carlo simulation results.

Table 2. Kirk approximation value.

Parameters	**Values**
Convenient yield (q)	[0.03, 0.02]
Initial price	[110, 100]
Vol	[0.1, 0.15]
Correlation	1
Strike	5
τ	1
r	0.05
Price	4.442

3.4. Delta of Kirk Approximation

The Kirk approximation (3) leads to the following deltas:

$$\Delta_1 = \frac{\partial C}{\partial F_1} = e^{-r\tau} N(d_1) \tag{4}$$

$$\begin{gathered}\Delta_2 = \frac{\partial C}{\partial F_2} = \left(-e^{-r\tau} N(d_2) + e^{-r\tau} \phi(d_2)(F_2 + K)\sqrt{\tau}\frac{\partial \sigma}{\partial F_2} \right) \\ \text{where:} \\ \frac{\partial \sigma}{\partial F_2} = \left(\frac{\sigma_2 K}{\sigma} \right) \left(\frac{\sigma_2 \frac{F_2}{F_2+K} - \rho\sigma_1}{(F_2 + K)^2} \right)\end{gathered} \tag{5}$$

In our Monte Carlo simulations (Table 3), we set the number of scenarios m to 100,000 and the time step dt to 1 trading day so that n equals 252. We conclude that the Kirk approximation works well for the two-asset case since it is very close to the Monte Carlo result (the latter is accurate as it agrees with the result of Bjerksund and Stensland (2006)).

Table 3. Monte Carlo simulation.

Parameters	**Values**
Convenient yield (q)	[0.03, 0.02]
Initial price	[110, 100]
Vol	[0.1, 0.15]
Correlation	1
Strike	5
τ	1
r	0.05

Table 3. *Cont.*

Parameters	Values
m	100,000
dt	$\frac{1}{252}$
n	252
Price	4.456

3.5. Test of Accuracy of Delta from Kirk Approximation

Recall that in Section 3.4, we talked about the delta of the spread option derived from the Kirk approximation. Next, we test the accuracy of this theoretical approximation of the delta by comparing it with the Monte Carlo simulation result. It turns out that

$$\Delta_1 = e^{-r\tau} N(d_1) = 0.7767$$

$$\Delta_2 = \left(-e^{-r\tau} N(d_2) + e^{-r\tau} \phi(d_2)(F_2 + K)\sqrt{\tau} \frac{\partial \sigma}{\partial F_2} \right) = -0.7638$$

The finite difference method and the Monte Carlo simulation yield:

$$\frac{\partial C}{\partial S_1} = \frac{C(S_1 + h, S_2, K) - C(S_1 - h, S_2, K)}{2h}, \ \Delta_1 = \frac{\partial C}{\partial S_1} \frac{\partial S_1}{\partial F_1}$$

$$\frac{\partial C}{\partial S_2} = \frac{C(S_1, S_2 + h, K) - C(S_1, S_2 - h, K)}{2h}, \ \Delta_2 = \frac{\partial C}{\partial S_2} \frac{\partial S_2}{\partial F_2}$$

From the below Table 4, one can see that the results are almost the same; thus, the deltas based on Kirk approximation are an accurate approximation.

Table 4. Delta comparison.

h	**0.001**	**0.011**	**0.021**	**0.031**	**0.041**	**0.051**
Δ_1	0.7757	0.7752	0.7772	0.7737	0.7725	0.7758
Δ_2	−0.7686	−0.7662	−0.7632	−0.7668	−0.7669	−0.7649

4. Finite Liquidity Model

We know that one of the limitations of the Black Scholes model is the assumption of perfect liquidity. This is not the case in the real world, especially in the commodity market, as there are several factors that can affect the market supply and demand of an asset, thus causing movements in its equilibrium price, dictated by supply and demand. This in turn makes the asset price more volatile and nonlinear as it depends on the order size. In our model, we define the term "liquidity" as the ability to trade any amount of assets at the current market price without additional cost. On the other hand, we refer to a "finitely liquid asset" when the order size (buy or sell) of this asset impacts its price. We are going to explore in the following sections the effect of this price impact.

4.1. SDE under Finite Liquidity Framework

In our model, we define asset one as the illiquid (finitely liquid) asset, while asset two is perfectly liquid. This is the case for an oil crack spread, as the gasoline is a liquid asset while the crude oil is less liquid. Next, let us introduce our finite liquidity paradigm. We take the large trader approach, meaning that only the trades of a large trader affect the price, and we assume that the large trader is a delta hedger. The liquidity impact on the second asset is introduced through the SDEs governing the price dynamics. Given two independent Brownian motions $W_1(t), W_2(t)$, the differentials of the two asset prices are

$$dS_1(t) = \mu_1 S_1(t)dt + \sigma_1 S_1(t)dW_1(t) + \lambda(t, S_1(t))d\Delta_1(t, S_1(t), S_2(t))$$
$$dS_2(t) = \mu_2 S_2(t)dt + \sigma_2 S_2(t)(\rho dW_1(t) + \sqrt{1-\rho^2}dW_2(t))$$

where:

$$f(t, s_1, s_2) = \Delta F_1 = e^{-r\tau}N(d_1) \tag{6}$$

$$\lambda(t, s_1) = \begin{cases} \gamma(1 - e^{-\beta\tau^{\frac{3}{2}}}), & \text{if } \bar{S_1} < s_1 < \bar{S_2} \\ 0, & \text{otherwise} \end{cases}$$

As one can see, the second asset is perfectly liquid, as it does not have a price impact entering its dynamics, while the first asset is finitely liquid through the impact of $\lambda(t, S_1(t))d\Delta_1(t, S_1(t), S_2(t))$, which appears in the differential. Let $W_1^{\mathbb{Q}}(t), W_2^{\mathbb{Q}}(t)$ be the independent Brownian motions under the risk-neutral $\mathbb{Q}$ measure. For the SDE of asset one, we add a liquidity value adjustment term, and this is the product of a deterministic function lambda and the change in the trade of the large trader, $d\Delta_1$ (since the large trader is a delta hedger). The price increases when the large trader sells because of increased demand, and the opposite happens when they buy. The λ function amplifies or eases the liquidity impact when the underlying asset price S_1 is in some range $(\bar{S_1}, \bar{S_2})$.

Let us point out that the above SDE governing the price of asset one is not in canonical form, as $d\Delta_1(t, S_1(t), S_2(t))$ contains $dS_1(t)$. The way out of this predicament is presented in the next subsection.

4.2. Transformations of SDEs

The main objective is to transform the SDEs from Equation (6) to a canonical form (with only $dW^{\mathbb{Q}}(t)$ and the dt term on the left-hand side). This is accomplished as follows:

1. Derive $d\Delta_1(t, S_1(t), S_2(t))$ by using the Ito formula, which results in 2 quadratic variations and 1 cross variation term.
2. Plug the $d\Delta_1$ term back into dS_1.
3. Compute the quadratic variation and cross variation by using the temporary $dS_1(t)$.
4. Compute the quadratic variations and cross variations.

Following the above approach (for more details on it, see Yazdanian et al. (2014) and Yazdanian and Pirvu (2016)), one obtains the SDEs:

$$\begin{aligned} dS_1(t) &= rS_1(t)dt + \bar{\sigma}_{11}d\hat{W_1}^{\mathbb{Q}}(t) + \bar{\sigma}_{12}d\hat{W_2}^{\mathbb{Q}}(t) \\ dS_2(t) &= rS_2(t)dt + \sigma_2 d\hat{W_2}^{\mathbb{Q}}(t) \end{aligned} \tag{7}$$

where

$$\hat{W}_1^{\mathbb{Q}}(t) = W_1^{\mathbb{Q}}(t), \hat{W}_2^{\mathbb{Q}}(t) = \rho W_1^{\mathbb{Q}}(t) + \sqrt{1-\rho^2}W_2^{\mathbb{Q}}(t)$$

$$\bar{\sigma}_{11} = \frac{\sigma_1 S_1(t)}{1 - \frac{\partial\Delta_1}{\partial S_1}}, \bar{\sigma}_{12} = \frac{\frac{\partial\Delta_1}{\partial S_2}\sigma_2 S_2(t)}{1 - \frac{\partial\Delta_1}{\partial S_1}}$$

$$\frac{\partial\Delta_1}{\partial S_1} = \frac{\partial f}{\partial d_1}\frac{\partial d_1}{\partial F_1}\frac{\partial F_1}{\partial S_1} = N'(d_1)\frac{1}{F_1\sigma\sqrt{\tau}}e^{-q_1\tau}$$

$$\frac{\partial\Delta_1}{\partial S_2} = \frac{\partial f}{\partial d_1}\frac{\partial d_1}{\partial F_2}\frac{\partial F_2}{\partial S_2} = N'(d_1)\left(\frac{-d_2}{\sigma}\frac{\partial\sigma}{\partial F_2} - \frac{1}{\sigma\sqrt{\tau}(F_2+K)}\right)e^{-q_2\tau}$$

We call the above asset pricing model a partial feedback model because only a large trader affects the price of asset one.

5. Monte Carlo Simulations

The risk-neutral pricing formula makes it possible to compute the spread option price by Monte Carlo simulations. Some work is required to simulate $S_1(T)$, as its distribution is not known, unlike the perfect liquidity case. We use the Euler–Maruyama method (Algorithm 1) to discretize the two SDEs in Equation (7) so that the value of the $dS_i(t)$ term is approximated by the ΔS_i term, which represents the change in asset price within a small period of time Δt. In this way, we can simulate the future asset price by constructing a Markov chain, where the future price equals the sum of the simulated current price and the ΔS_i term.

$$\begin{aligned} S_{n+1} &= S_n + rS_n\Delta t + \bar{\sigma}(S_n, t_n)\Delta W_n^Q \\ \Delta t &= t_{n+1} - t_n \\ \Delta W_n^Q &= W_{n+1}^Q - W_n^Q \sim N(0, \sqrt{\Delta t}) \end{aligned}$$

Algorithm 1 Euler–Maruyama Method

procedure SET UP INITIAL PARAMETERS($S(0), \tau, vol, \rho, r, K, q, m, n, \gamma, \beta$)
    1. Initialization of Brownian Motion increments for two assets
    $\Delta W = np.random.multivariate\_normal * np.sqrt(dt)$
    2. Define all functions for partial derivatives f_{S_1}, f_{S_2}...
    3. Define lambda function
end procedure

procedure GENERATE MARKOV CHAIN($S_{n+1} = S_n + \Delta S$)
    for i in range(1, n+1) **do**
        update lambda function based on the previous result S_{i-1}
        update stochastic part based on the previous result S_{i-1}
        $S_i = S_{i-1} + rS_{i-1}\Delta t + \bar{\sigma}(S_{i-1}, t_{i-1})\Delta W_i^Q$
    end for
    Payoff = $S_1(T) - S_2(T) - K$
    filter Payoff[Payoff<0]=0
end procedure

procedure PRICING
    V = $\frac{e^{-r\tau}}{N}\sum_{i=1}^{N}(S_1(T) - S_2(T) - K)^+$
end procedure

5.1. Test of Accuracy

The accuracy of this approximation method can be tested by comparing the analytical solution for an exchange option (calculated from the Margrabe formula) and Monte Carlo simulations by setting γ and K equal to 0. From the result (Table 5), we can conclude that our approach works well.

Table 5. Test of accuracy.

Parameters	Values
Convenient yield (q)	[0.03, 0.02]
Initial price	[110, 100]
Vol	[0.1, 0.15]
Correlation	1
Strike	0
τ	1

Table 5. *Cont.*

Parameters	Values
r	0.05
m	100,000
dt	$\frac{1}{252}$
n	252
Monte Carlo price	8.8125
Analytical price	8.821

5.2. Model Performance by Naive Monte Carlo

We obtain the following results (Table 6) by running Monte Carlo simulations and by choosing parameter values for β, γ, and the range of liquidity impact.

Table 6. Performance comparison.

Parameters	Values
γ	0 and 0.2
β	100
$\bar{s_1}$	50
$\bar{S_1}$	200
m	10,000
dt	$\frac{1}{252}$
Perfect liquidity γ = 0	4.42363
Finite liquidity γ = 0.2	4.4937

From the above result, we can see a 1.58% increase in option price with the given parameters, due to the liquidity adjustment term, which makes the price of asset one more volatile, and this increases the need for protection against price variations given by the spread option. Moreover, naive Monte Carlo can be improved by variance reduction techniques such as antithetic and control variates, and this is our next topic.

5.3. Antithetic Variates

This variance reduction procedure works as follows:

1. Generate
$$S_i^+ = S_{i-1} + rS_{i-1}\Delta t + \bar{\sigma}(S_{i-1}, t_{i-1})\Delta W_i^Q$$
$$S_i^- = S_{i-1} + rS_{i-1}\Delta t - \bar{\sigma}(S_{i-1}, t_{i-1})\Delta W_i^Q$$
2. Based on the results from the above step, we generate a pair $S_i^+(T), S_i^-(T)$.
3. Obtain the antithetic payoff $\frac{S_i^+(T)+S_i^-(T)}{2}$.

5.4. Control Variates

Let $X = (S_1(T) - S_2(T) - K)^+$. This variance reduction procedure works as follows: find another random variable Y such that:

$$E[Y] = E[(S_1(T) - S_2(T) - K)^+],$$

and

$$Y = X + b[C - E[C]],$$

where C is the control variable whose expectation is known in closed form and b is a scalar. The optimal value of b is

$$b = \frac{-Cov(X,C)}{Var(C)},$$

and it can be found by naive Monte Carlo with a large number of scenarios. The choice of C is $(\frac{S_1(t)}{S_2(t)} - K^*)^+$, where $K^* = \frac{K}{S_2(0)}$, in a perfectly liquid model, being the payoff of a European call option whose expectation is known in closed form (being given by the Black Scholes formula).

Based on Table 7, one can see that both variance reduction methods give closed results, while the antithetic method gives the most narrow confidence interval, so we shall use the antithetic method for all the following Monte Carlo results.

Table 7. Variance reduction.

Method	Values	95% Confidence Interval
Naive MC	4.4937	(4.4720, 4.5154)
Antithetic MC	4.4810	(4.4786, 4.4833)
Control variate MC	4.4841	(4.4785, 4.4897)

5.5. Liquidity Impact on Price with Respect to Beta and Gamma

The liquidity impact, or LVA, is calculated as follows:

$$\frac{C_{Euler}(\tau = 0.5, \beta, \gamma) - C_{Kirk}(\tau = 0.5)}{C_{Kirk}(\tau = 0.5)},$$

where C_{Euler} denotes the price obtained through Monte Carlo simulations in the finite liquidity setting and C_{Kirk} is the price obtained via Kirk approximation in a perfectly liquid framework. Next, choose the τ to be 0.5 and the rest of the parameters to be the same as in the previous section. The results (see Figures 1–3) are calculated by antithetic Monte Carlo with n = 10,000. The correlation is set to 1.

One finding is that the LVA is increasing in γ and β. This fact is expected since the amplifying factor of the liquidity impact increases in γ and β. The increase is more pronounced if we increase both of them simultaneously; the increase ranges from 0.53% to 2.81%.

Next, we present the results (see Figures 4–6) when correlation is set to 0.75. As expected, the option price increases due to the diversification effect, but it is interesting to point out that liquidity impact on price decreases slightly.

gamma	0.2	0.4	0.6	0.8	1.0
beta					
10	4.441995	4.441995	4.441995	4.441995	4.441995
20	4.441995	4.441995	4.441995	4.441995	4.441995
30	4.441995	4.441995	4.441995	4.441995	4.441995
40	4.441995	4.441995	4.441995	4.441995	4.441995
50	4.441995	4.441995	4.441995	4.441995	4.441995

Figure 1. Kirk price, correlation 1.

gamma beta	0.2	0.4	0.6	0.8	1.0
10	4.465596	4.484250	4.500825	4.517826	4.538319
20	4.475936	4.491817	4.508115	4.529115	4.547559
30	4.473187	4.489345	4.511707	4.535626	4.555623
40	4.474969	4.496845	4.516119	4.539399	4.564114
50	4.470681	4.498814	4.517488	4.537060	4.566988

Figure 2. Finite liquidity price, correlation 1.

gamma beta	0.200000	0.400000	0.600000	0.800000	1.000000
10	0.53%	0.95%	1.32%	1.71%	2.17%
20	0.76%	1.12%	1.49%	1.96%	2.38%
30	0.70%	1.07%	1.57%	2.11%	2.56%
40	0.74%	1.23%	1.67%	2.19%	2.75%
50	0.65%	1.28%	1.70%	2.14%	2.81%

Figure 3. LVA, correlation 1.

gamma beta	0.2	0.4	0.6	0.8	1.0
10	6.263695	6.263695	6.263695	6.263695	6.263695
20	6.263695	6.263695	6.263695	6.263695	6.263695
30	6.263695	6.263695	6.263695	6.263695	6.263695
40	6.263695	6.263695	6.263695	6.263695	6.263695
50	6.263695	6.263695	6.263695	6.263695	6.263695

Figure 4. Kirk price, correlation 0.75.

gamma	0.2	0.4	0.6	0.8	1.0
beta					
10	6.288712	6.316967	6.334778	6.354461	6.366690
20	6.321152	6.303820	6.343456	6.374368	6.395610
30	6.302629	6.340690	6.346945	6.406314	6.407083
40	6.336495	6.292022	6.361099	6.354229	6.456710
50	6.285547	6.339930	6.346180	6.366877	6.423177

Figure 5. Finite liquidity price, correlation 0.75.

gamma	0.200000	0.400000	0.600000	0.800000	1.000000
beta					
10	0.40%	0.85%	1.13%	1.45%	1.64%
20	0.92%	0.64%	1.27%	1.77%	2.11%
30	0.62%	1.23%	1.33%	2.28%	2.29%
40	1.16%	0.45%	1.56%	1.45%	3.08%
50	0.35%	1.22%	1.32%	1.65%	2.55%

Figure 6. LVA, correlation 0.75.

5.6. Liquidity Impact on Price with Respect to Time to Maturity

The plot in Figure 7 shows the finite liquidity model performance of the spread option price with respect to τ with a fixed value of γ and β. One can see that the price calculated in the partial liquidity model is always higher than the price under the perfectly liquid setting.

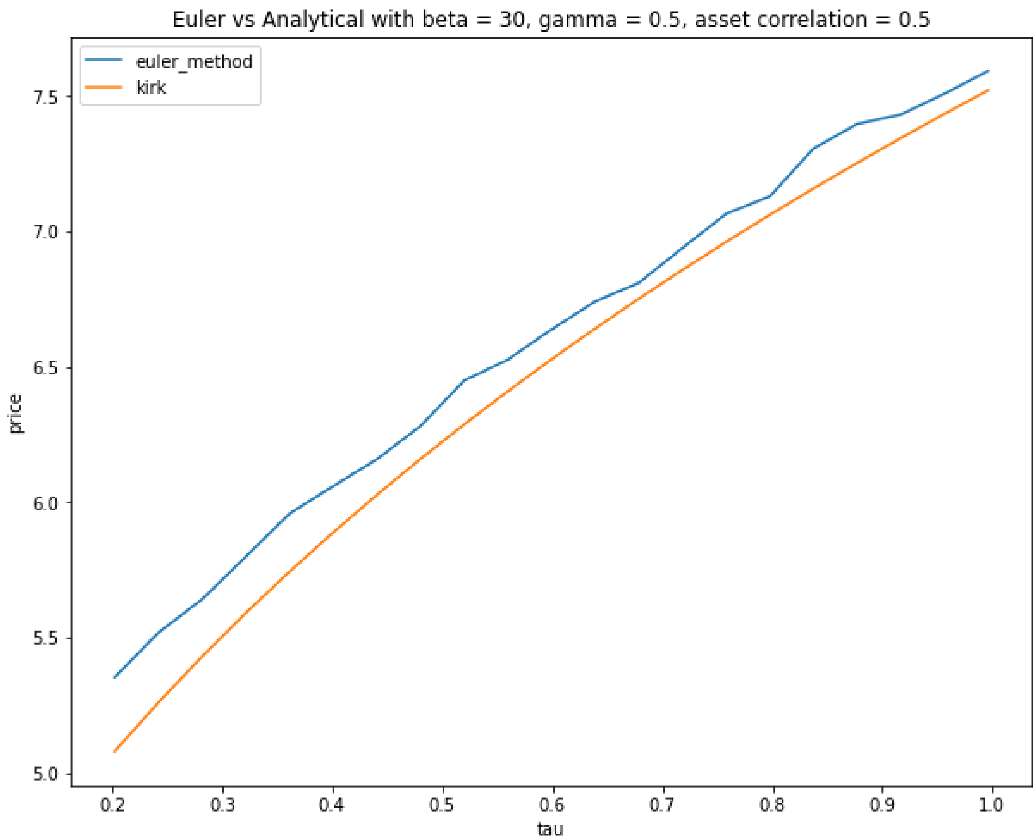

Figure 7. Liquidity impact on price with respect to tau.

6. Conclusions and Future Research

The finite liquidity framework is considered following the large trader perspective. The latter is a delta hedger and its hedge impacts the price of an underlying asset, and we refer to this as the liquidity impact. Our work designs a novel approach to price spread options in this finite liquidity setting. This is achieved by combining Kirk approximation with Monte Carlo simulations. This methodology is further improved by means of the antithetic and control variates variance reduction techniques. Numerical experiments reveal the LVA on option prices due to the liquidity impact on underlying asset price. The LVA increases when the liquidity impact amplifying factor, or the correlation of the two underlying assets, increases. Our approach extends naturally to multi-asset options given closed-form formulas or approximations available in the full liquidity model; there are some works available on this, such as those of Li et al. (2010), Lau and Lo (2014), and Chen and Deng (2024). We leave the implementation of this methodology to multi-asset options as a topic for future research.

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