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Exploring Entropy-Based Portfolio Strategies: Empirical Analysis and Cryptocurrency Impact

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Abstract: This study addresses market concentration among major corporations, highlighting the utility of relative entropy for understanding diversification strategies. It introduces entropic value at risk (EVaR) as a coherent risk measure, which is an upper bound to the conditional value at risk (CVaR), and explores its generalization, relativistic value at risk (RLVaR), rooted in Kaniadakis entropy. Through extensive empirical analysis on both developed (i.e., S&P 500 and Euro Stoxx 50) and developing markets (i.e., BIST 100 and Bovespa), the study evaluates entropy-based criteria in portfolio selection, investigates model behavior across different market types, and assesses the impact of cryptocurrency introduction on portfolio performance and diversification. The key finding indicates that entropy measures effectively identify optimal portfolios, particularly in scenarios of heightened risk and increased concentration, crucial for mitigating negative net performances during low returns or high turnover. Bitcoin is primarily used for diversification and performance enhancement in the BIST 100 index, while its allocation in other markets remains minimal or non-existent, confirming the extreme concentration observed in stock markets dominated by a few leading stocks.

Keywords: portfolio optimization; entropy; value at risk; entropic value at risk; coherent risk measures; Bitcoin

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1. Introduction

Cryptocurrency is of particular importance for diversification purposes, because the stock market is currently witnessing an unprecedented level of dominance by giant corporations, particularly those in the technology sector. These corporations, often referred to as "the Magnificent Seven" (Phillips 2024), including Apple, Microsoft, Alphabet, Amazon, Nvidia, Tesla, and Meta, along with other significant players like Berkshire Hathaway and Eli Lilly, have increasingly become the driving force behind the stock market's performance. However, this concentration of power among a few key companies has raised concerns among analysts, who warn of potential risks associated with such dominance. The top 10 mega-cap companies in the S&P 500 currently represent nearly 35% of the index's total market capitalization (Shalett 2023). The current level of concentration, which has not been seen since the speculative period of the New Era[1] in June 2000 (Rekenthaler 2020), exposes investors to significant risk, especially if interest rates remain high and stock prices fall. Thus, while the current dominance of giant corporations in the stock market is not without precedent, its scale and potential implications deserve careful consideration and investigation.

Feng et al. (2018) asserts that cryptocurrencies exhibit left tail independence and cross tail independence with four selected stock indices, suggesting their potential to serve as significant diversifiers for the stock market, akin to gold. For Corbet et al. (2020), when investigating the period during the COVID-19 pandemic, digital assets not only provide diversification benefits for investors but also act as a safe haven similar to that of precious metals during historic crises. However, according to Gambarelli et al. (2023), the addition of a single cryptocurrency to a stock portfolio does not effectively hedge against market downturns and may increase the risk of short-term joint losses. Additionally, Orlando (2024) raised concerns about the speculative nature and susceptibility to manipulation of cryptocurrencies.

Entropy, derived from thermodynamics and information theory, is used in finance and economics to analyze uncertainty and dynamics in systems (Zhou et al. 2013) or scarcity (Kovalev 2016). In finance, entropy has been embraced as it quantifies disorder, randomness, and unpredictability, thereby assisting in risk assessment, portfolio optimization, and policy formulation. For instance, abnormal losses during the 2008 financial crisis, which were attributed to unexpected events (Jorion 2009; Orlando et al. 2021), can underscore the contribution of entropy as a measure of the unexpected (Taleb 2007; Orlando and Zimatore 2020; Bufalo et al. 2023). Along these lines, (Orlando and Lampart 2023) challenge conventional views by showing that decreasing prices align with increasing entropy, questioning the idea that diminished entropy in market crises suggests determinism. Moreover, their findings propose that bear markets tend to display higher entropy, signaling a greater probability of unexpected extreme events.

Entropy can also be viewed as a measure of distance and coherent entropy-based risk metrics are essential tools in minimum-risk portfolio selection, offering a robust framework for assessing and managing investment risk (Yin 2019). For instance, Philippatos and Wilson employed the mean-entropy model in their seminal contribution (Philippatos and Wilson 1972; Philippatos and Gressis 1975) utilizing entropy models and measures to represent an uncertain environment and derive optimal economic decisions. Entropy metrics exploit the principles of information theory to quantify uncertainty and complexity in financial markets, providing valuable insights into the risk–return trade-offs of different investment strategies (Pichler and Schlotter 2020). By maintaining coherence and flexibility, entropy-based risk measures enable investors to construct portfolios that balance risk and return efficiently, effectively accounting for tail risks and extreme events (Ardakani 2023). Relative entropy, pioneered Kullback and Leibler (1951), has gained considerable attention in portfolio selection. An emerging advancement in risk evaluation, EVaR, shows a dual representation, emphasizing its connection to relative entropy (Ahmadi et al. 2022). EVaR boasts several advantages: it is coherent (Artzner et al. 1999), strongly monotone, and convex, serving as an upper bound to CVaR, which, in turn, is an upper bound to value at risk (VaR). Additionally, a generalization of EVaR has emerged in the form of RLVaR (Cajas 2023), a coherent risk measure rooted in Kaniadakis entropy (Kaniadakis 2006), which surpasses EVaR.

In this study, we carry out an extensive empirical analysis, assessing various minimum-risk models using real-world datasets, which consist of stock indexes from both developed and emerging markets, as well as Bitcoin time series. The objective of our work is threefold: to determine whether entropy-based criteria outperform other models, to study how the considered models behave in developed markets compared to emerging ones, and to analyze the effect of the introduction of cryptocurrency on portfolio performance and diversification. Here, we highlight that during the observed period, Bitcoin demonstrates an unfavorable risk profile. This disadvantage plays a crucial role in portfolio optimization, diversification, and selection.

This article is structured as follows. Section 2 delves into the methodology and materials, detailing various definitions of entropy and explaining two portfolio entropy measures: the EVaR model and the RLVaR model. Following that, performance and risk measures are listed, and the dataset is presented. Finally, details of the simulations conducted are provided together with the portfolio optimization techniques adopted and the out-of-sample and sector analyses. Section 3 is divided into two parts: the first presents the results of the optimal portfolios in terms of risk and performance depending on the risk aversion, while the second provides the allocation analysis. In both cases, the inclusion of the crypto security is also considered. Finally, Section 4 concludes.

2. Methods and Materials

This section is dedicated to methods and materials. Concerning methods, Section 2.1 presents theoretical definitions of entropy, while Section 2.2 illustrates EVaR and RLVaR. Furthermore, Section 2.3 outlines how the different types of entropy are incorporated into the portfolio selection framework. Finally, Section 2.4 addresses various performance measures, including maximum drawdown, ulcer index, Sharpe ratio, Omega ratio, etc. Regarding materials, Section 2.5 describes the datasets, including mature markets (S&P500 and Euro Stoxx 50), emerging markets (Istanbul Exchange and Bovespa), and crypto assets (Bitcoin). Section 2.6 outlines the setup, including the hardware and software used, the optimization process, and the out-of-sample and sector analyses.

2.1. Definitions of Entropy

In this section, we explore some fundamental aspects of entropy, providing theoretical definitions as background knowledge.

2.1.1. Shannon Entropy

Shannon's seminal work introduced entropy in information theory as a measure of the expected uncertainty of a random variable across possible outcomes (Shannon 1948). This form of entropy is widely recognized as Shannon entropy, highlighting Shannon's deep impact on the field. Additionally, equivalences between Shannon entropy and Kolmogorov complexity have been identified (Leung-Yan-Cheong and Cover 1978). The concept of Shannon entropy has diverse applications across various fields including information theory (Gray 2011), cryptography (Koç and Özdemir 2023), machine learning (Aljalal et al. 2022), data mining (Holzinger et al. 2014), entanglement (Calixto et al. 2021), etc.

Let us first provide definitions in the discrete case.

Definition 1 (Shannon entropy)**.** *The entropy of a random variable X is*

$$\begin{aligned} H(X) &= E[-\log_b(p_X(x))] = \\ &= -\sum_{x\in\mathcal{X}} p_X(x)\log_b(p_X(x)), \end{aligned} \tag{1}$$

where b denotes the base of the logarithm, $p_X(x)$ denotes the probability mass function of X, and $\mathcal{X}$ identifies the support of X.

Typical choices of bases are 2, e, and 10, depending on the use. >From now on, throughout the paper, we will use e as a base, hence $\log_b(\cdot) = \ln(\cdot)$. >From the definition above, it is possible to derive some useful properties of the entropy. Consider a discrete random variable X with T states of nature, i.e., $X = \{x_1, x_2, \dots, x_T\}$. For the sake of notation, let us denote $\pi_t = p_X(x_t)$.

1. Continuity: the entropy is continuous with respect to the probabilities π_t.
2. Non-negativity: since $0 \le \pi_t \le 1 \quad \forall t = 1, \dots, T$, $H(X) \ge 0$.
3. Minimum: following from the previous property, $H(X) = 0$, i.e., it attains its minimum at $\pi_j = 1$ for a given j, and $\pi_t = 0$ for $t \ne j$.
4. Maximum: $H(X)$ is maximal when all the scenarios are equally likely, i.e., $\pi_t = \frac{1}{T}$. Thus, in general, $H(X) \le \ln(T)$.
5. Symmetry: the order of the outcomes x_t does not affect the value of $H(X)$.

6. Concavity: the entropy is concave in the probability mass function. Therefore,

$$H(\lambda p_X + (1-\lambda)q_X) \geq \lambda H(p_X) + (1-\lambda)H(q_X),$$

where p_X and q_X denote two probability distributions of X.

Let Y be another random variable. We can define the relationship between two random variables X and Y by means of concepts such as joint entropy conditional entropy.

2.1.2. Joint Entropy

Definition 2 (Joint entropy)**.** *The joint entropy of two discrete random variables X and Y is*

$$H(X,Y) = -\sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p_{X,Y}(x,y)\ln(p_{X,Y}(x,y)), \tag{2}$$

where $p_{X,Y}(x,y)$ is the joint probability mass function of X and Y.

The joint entropy measures the uncertainty of the two random variables X and Y considered together. However, there might be some instances in which a random variable, say X, is known. In this instance, the conditional entropy quantifies the entropy of a random variable Y (uncertain) while the other random variable X is known.

2.1.3. Conditional Entropy

Definition 3 (Conditional entropy)**.** *The conditional entropy of Y given X is*

$$\begin{aligned} H(Y|X) &= \sum_{x\in\mathcal{X}} p_X(x)H(Y|X=x) = \\ &= \sum_{x\in\mathcal{X}}\sum_{y\in\mathcal{Y}} p_{Y,X}(y,x)\ln(p_{Y|X=x}(y|x)), \end{aligned} \tag{3}$$

where $p_{Y|X=x}(y|x)$ is the conditional probability mass function of Y given $X = x$.

>From Bayes' theorem, the conditional entropy can be rewritten as

$$H(Y|X) = H(X,Y) - H(X).$$

>From Definitions 2 and 3, it is possible to derive the following additional properties of entropy.

1. Subadditivity: $H(X,Y) \leq H(X) + H(Y)$. The equality is obtained when X and Y are independent.
2. Comparison with individual entropies: $H(X,Y) \geq \max[H(X), H(Y)]$.

Shannon entropy measures the uncertainty in a random variable X distributed according to a distribution p_X. However, sometimes it might be useful to assume X is distributed according to a distribution q_X, even though the actual distribution is p_X. Consequently, Kullback and Leibler (1951) introduced the relative entropy, also known as Kullback–Leibler divergence.

2.1.4. Relative Entropy

Definition 4 (Relative entropy, Kullback–Leibler divergence)**.** *Given two probability distributions $p_X(x)$ and $q_X(x)$ over a discrete random variable X, the relative entropy of $p_X(x)$ with respect to $q_X(x)$ is*

$$D_{KL}(p_X||q_X) = \sum_{x\in\mathcal{X}} p_X(x)\ln\left(\frac{p_X(x)}{q_X(x)}\right). \tag{4}$$

The relative entropy measures the inefficiency of adopting the distribution $q_X(x)$ when the actual distribution is $p_X(x)$. Note that even though it computes how distant one distribution is from another, from a mathematical perspective it is not a distance. Indeed, the Kullback–Leibler divergence is not symmetric, i.e., in general, $D_{KL}(p_X||q_X) \neq D_{KL}(q_X||p_X)$, and it does not satisfy the triangle inequality.

2.1.5. Cross-Entropy

Definition 4 may be formulated using Shannon's entropy and cross-entropy. First, let us define the latter.

Definition 5 (Cross-entropy)**.** *Given two probability distributions $p_X(x)$ and $q_X(x)$ over a discrete random variable X, the cross-entropy of $q_X(x)$ relative to $p_X(x)$ is*

$$H(p_X, q_X) = -\sum_{x \in \mathcal{X}} p_X(x) \ln(q_X(x)). \tag{5}$$

Therefore, the relative entropy can be restated as follows:

$$D_{KL}(p_X||q_X) = H(p_X, q_X) - H(X)$$

Below, we report its main properties.

1. Non-negativity: due to the Gibbs' inequality, the cross-entropy of p_X and q_X is always greater than or equal to the Shannon's entropy of p_X, which translates into $D_{KL}(p_X||q_X) \geq 0$. The equality is obtained when the two distributions are identical.
2. Joint convexity: $D_{KL}(p_X||q_X)$ is convex both on p_X and on q_X.

Later, Kaniadakis (2002, 2006) proposed a generalization of Shannon's entropy, known as k-entropy or relativistic entropy.

Definition 6 (k-entropy, relativistic entropy)**.** *The k-entropy of a discrete random variable X is*

$$H_{\{k\}}(X) = -\sum_{x \in \mathcal{X}} p_X(x) \ln_{\{k\}}(p_X(x)), \tag{6}$$

where $\ln_{\{k\}}(x) = \frac{x^k - x^{-k}}{2k}$ is the k-logarithm function, and $k \in \mathbb{R}$.

The k-logarithm is one of the k-deformed functions included in (Kaniadakis 2001), to which we refer for a more in-depth analysis of the properties of such a function. For the scope of this work, we only report the following properties:

1. Asymptotic behavior with respect to k: $\ln_{\{0\}}(x) = \lim_{k\to 0} \ln_{\{k\}}(x) = \ln(x)$.
2. Strict increasing monotonicity: $\frac{\partial \ln_{\{k\}}(x)}{\partial x} > 0$.
3. Concavity: $\frac{\partial^2 \ln_{\{k\}}(x)}{\partial x^2} > 0$.

Therefore, it is easy to notice that Shannon's entropy can be considered as a special case of k-entropy, i.e.,

$$H_{\{0\}}(X) = \lim_{k\to 0} H_{\{k\}}(X) = H(X)$$

2.2. Risk Measures

Let ρ be a risk measure, i.e., a function that assigns a risk score to the random variable X over the feasible set χ. Below, we show some important properties of a risk measure $\rho : \chi \to \bar{\mathbb{R}}$, with $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, for every $X, Y \in \chi$:

(R1) Translation invariance: $\rho(X + \alpha) = \rho(X) + \alpha$ for every $\alpha \in \mathbb{R}$.
(R2) Positive homogeneity: $\rho(\lambda X) = \lambda\rho(X)$ for every $\lambda \geq 0$.
(R3) Subadditivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$.
(R4) Monotonicity: if $X \leq Y$, then $\rho(X) \leq \rho(Y)$.

Note that if a risk measure ρ is both positive homogeneous and subadditive, then it is convex on χ. If ρ satisfies all the properties listed above, then it is called coherent (see Artzner et al. (1999); Acerbi (2003)). Some of the most famous risk measures are not coherent: for example, variance (but not the standard deviation), which does not possess any of the aforementioned properties, and mean absolute deviation, which is not monotone and translation invariant. Another popular risk measure is the value at risk (VaR), where the VaR at a confidence level α is

$$\mathrm{VaR}_{1-\alpha}(X) = \inf\{t \in \mathbb{R} : P(X \leq t) \geq 1-\alpha\}, \quad \alpha \in [0,1]. \tag{7}$$

Such a measure does not satisfy the subadditivity property. Thus, in order to overcome this drawback, Rockafellar and Uryasev (2000) proposed the CVaR, also known as expected shortfall (ES) (see Acerbi and Tasche (2002)), defined as the worst α% values of X, i.e., mathematically,

$$\mathrm{CVaR}_{1-\alpha}(X) = \inf_{t \in \mathbb{R}} \left\{ t + \frac{1}{\alpha} E[X-t]_+ \right\}, \quad \alpha \in [0,1].$$

The CVaR satisfies all the properties from 1 to 4, and it is, therefore, coherent. Two other coherent risk measures are the EVaR, recently proposed by Ahmadi-Javid (2012); Ahmadi-Javid and Fallah-Tafti (2019), and the RLVaR, suggested by Cajas (2023). The next subsections will be dedicated to the description of these two models.

2.2.1. Entropic Value at Risk (EVaR)

The EVaR is a relatively new risk measure introduced by Ahmadi-Javid. Such a measure is derived from the Chernoff inequality (see Chernoff (1952)), which for a given random variable X reads

$$P(X \geq a) \leq e^{-za} M_X(z), \quad \forall z > 0,$$

where $M_X(z) = E[e^{zX}]$ is the moment-generating function of X, and a is a constant. Solving for a the equation $e^{-za} M_X(z) = \alpha$, with $\alpha \in (0,1)$, we have

$$a_X(\alpha, z) := \frac{1}{z} \ln\left(\frac{M_X}{\alpha}\right).$$

Therefore, for each $z > 0$, $a_X(\alpha, z) \leq VaR_{1-\alpha}(X)$ since $P(X \geq a_X(\alpha, z)) \leq \alpha$. Formally, the authors define the EVaR as

$$EVaR_{1-\alpha}(X) := \inf_{z>0} \left\{ \frac{1}{z} \ln\left(\frac{M_X}{\alpha}\right) \right\}, \tag{8}$$

which represents the smallest upper bound of VaR derived from the Chernoff inequality. As mentioned before, the EVaR satisfies all properties from (R1) to (R4), and it is, therefore, coherent. The dual representation (or robust representation) of the EVaR better shows its connection with entropy. Mathematically, we have

$$EVaR_{1-\alpha}(X) = \sup_{Q \in \mathfrak{I}} E_Q[X],$$

where $\mathfrak{I} = \{Q \ll P : D_{KL}(Q||P \leq -\ln(a))\}$. For the proof, we refer to Ahmadi-Javid (2012). For a given value α, the EVaR is an upper bound not only to the VaR but to the CVaR as well. Additionally, when $\alpha = 1$, $EVaR_0 = E[X]$, while $\lim_{\alpha \to 0} EVaR_{1-\alpha} = \operatorname{ess\,sup}(X)$, i.e., as the confidence level approaches 1, the EVaR tends to the maximal loss.

2.2.2. Relativistic Value at Risk

Let us start by introducing the concept of φ-divergence (also known as f-divergence; see Csiszar (1963); Morimoto (1963); Ali and Silvey (1966)), which, in turn, is based on divergence functions.

Definition 7 (Divergence functions)**.** *A convex function $\varphi : \chi \to \mathbb{R} \cup \{+\infty\}$ such that $\varphi(1) = 0$ is called a divergence function if*

$$\lim_{x \to \infty} \varphi(x) = \infty, \tag{9}$$

Definition 8 (φ-divergence)**.** *Given two probability distributions $p_X(x)$ and $q_X(x)$ over a discrete random variable X, the φ-divergence of $p_X(x)$ with respect to $q_X(x)$ is*

$$D_\varphi(p_X || q_X) = \sum_{x \in \mathcal{X}} p_X(x) \varphi\left(\frac{p_X(x)}{q_X(x)}\right). \tag{10}$$

Based on these two definitions, Dommel and Pichler (2020) define a new class of risk measures called φ-divergence risk measures.

Definition 9 (φ-divergence risk measures)**.** *Given a divergence function φ with convex conjugate ψ, the φ-divergence risk measure $\rho_{\varphi,\beta} : \chi \to \mathbb{R} \cup \{+\infty\}$ is defined as follows:*

$$\rho_{\varphi,\beta}(X) = \inf_{\substack{\mu \in \mathbb{R} \\ t > 0}} \left\{ t\left(\beta + \mu + E\left[\psi\left(\frac{X}{t} - \mu\right)\right]\right) \right\}, \tag{11}$$

where $\beta > 0$ is a risk aversion coefficient.

Remark 1 (Coherence of φ-divergence risk measures)**.** *The functional $\rho_{\varphi,\beta}(X)$ satisfies properties (R1)–(R4), and it is, therefore, coherent (see Dommel and Pichler (2020) for the proof).*

The class of φ-divergent risk measures always attains values between the expected value and the maximal loss, i.e.,

$$E[X] \leq \rho_{\varphi,\beta}(X) \leq \operatorname{ess\,sup}(X).$$

Dommel and Pichler (2020) also show the dual representations of such risk measures, which is the following:

$$\rho_{\varphi,\beta}(X) = \sup_{Z \in M_{\varphi,\beta}} E[Z'X],$$

where $M_{\varphi,\beta} := \{Z \geq 0, E[Z] = 1, E[\varphi(Z) \leq \beta]\}$. The relativistic value at risk is a special case of φ-divergent risk measures, where $\varphi(Z) = Z \odot \ln_{\{k\}}(Z)$, and $\beta = -\ln_{\{k\}}(\alpha T)$. Therefore, the dual representation of RLVaR is the following:

$$RLVaR^k_{1-\alpha}(X) = \sup_{Z \in M_{\varphi,\beta}} E[Z'X], \tag{12}$$

where $M_{\varphi,\beta} = \{Z \geq 0, E[Z] = 1, E[Z \ln_{\{k\}}(Z)] \leq \ln_{\{k\}}(\frac{1}{\alpha T})\}$, and k denotes the deformation parameter of the k-logarithm function, where, in this instance, $k \in (0, 1)$. Note that because of the asymptotic behavior of the k-logarithm function with respect to k, $\lim_{k \to 0} RLVaR^k_{1-\alpha}(X) \approx EVaR_{1-\alpha}(X)$; on the other hand, $\lim_{k \to 1} RLVaR^k_{1-\alpha}(X) \approx \operatorname{ess\,sup}(X)$. Finally, merging the results of Sections 2.2.1 and 2.2.2, for a given level α, the following inequalities are satisfied:

$$E[X] \leq VaR_{1-\alpha}(X) \leq CVaR_{1-\alpha}(X) \leq EVaR_{1-\alpha}(X) \leq RLVaR^k_{1-\alpha}(X) \leq \operatorname{ess\,sup}(X)$$

2.3. Entropy in Portfolio Selection

In this section, we describe how the various types of entropy discussed in Section 2.1 are integrated into the portfolio selection framework. Let p_{it} and $r_{it} = \frac{p_{it} - p_{i(t-1)}}{p_{i(t-1)}}$ denote, respectively, the realized price and the realized (linear) daily return of asset i at time t, with $i = 1, \ldots, n$ (where n denotes the number of assets), and $t = 1, \ldots, T$ (where T denotes the length of the time series of the returns). As often assumed in portfolio optimization (see, for example, Roman et al. (2013) and Carleo et al. (2017)), we assume equally likely scenarios, i.e., each realization has an equally likely probability of occurrence of $\frac{1}{T}$. Furthermore, we indicate with $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ the portfolio weights vector, and with $R_t(\boldsymbol{x}) = \sum_{i=1}^{n} r_{it} x_i$ the portfolio return at time t. Finally, we denote by $\mu_i = \frac{1}{T} \sum_{t=1}^{t} r_{it}$ the expected return of asset i. Here, we report the portfolio selection models based on the measures addressed in the previous section.

2.3.1. The Entropy Value-at-Risk (EVaR) Model

The first model to optimize the portfolio EVaR was that by Ahmadi-Javid and Fallah-Tafti (2019). The authors developed a convex program whose variables and constraints are independent of the sample size T. In this work, we use the reformulation by Cajas (2021), which is a convex programming problem that has the advantage of being efficiently solvable by several software. Thus, the EVaR minimization problem can be formulated as follows:

$$\begin{cases} \min\limits_{x,z,t,u} & t + z \ln\left(\dfrac{1}{\alpha T}\right) \\ \text{s. t.} & \\ & z \geq \sum\limits_{j=1}^{T} u_t \\ & \left(-\sum\limits_{i=1}^{n} r_{ij} x_i - t, z, u_t \right) \in K_{\text{exp}} \quad j = 1, \ldots, T \\ & \sum\limits_{i=1}^{n} \mu_i x_i \geq \eta \\ & \sum\limits_{i=1}^{n} x_i = 1 \\ & x_i \geq 0 \quad\quad\quad\quad\quad\quad i = 1, \ldots, n \end{cases} \tag{13}$$

where $K_{exp} = \left\{ (a,b,c) : b > 0, c \geq e^{\frac{a}{b}} \right\} \cup \left\{ (a,b,c) : a \leq 0, b = 0, c \geq 0 \right\}$ is the exponential cone, as defined in Chares (2009).

2.3.2. The Relativistic Value-at-Risk (RLVaR) Model

Here we show the relativistic value-at-risk model proposed by Cajas (2023). The duality theorem of Chares (2009) allows the author to express the problem in its primal formulation, which is the following minimization problem:

$$
\begin{cases}
\min\limits_{x,z,t,\psi,\theta,\varepsilon,\omega} & t + z \ln_{\{k\}}\left(\dfrac{1}{\alpha T}\right) + \sum\limits_{j=1}^{T}(\psi_j + \theta_j) \\
\text{s. t.} & \\
& -\sum_{i=1}^{n} r_{ij}x_i - t + \varepsilon_j + \omega_j \leq 0 \qquad\qquad j = 1, \ldots, T \\
& \left(z\left(\dfrac{1+k}{2k}\right), \psi_j\left(\dfrac{1+k}{k}\right), \varepsilon_j\right) \in \mathcal{P}_3^{\frac{1}{1+k}, \frac{k}{1+k}} \qquad j = 1, \ldots, T \\
& \left(\omega_j\left(\dfrac{1}{1-k}\right), \theta_j\left(\dfrac{1}{k}\right), -z\left(\dfrac{1}{2k}\right)\right) \in \mathcal{P}_3^{1-k,k} \quad j = 1, \ldots, T \\
& \sum\limits_{i=1}^{n} \mu_i x_i \geq \eta \\
& \sum\limits_{i=1}^{n} x_i = 1 \\
& x_i \geq 0 \qquad\qquad i = 1, \ldots, n \\
& z > 0
\end{cases}
\tag{14}
$$

where $\mathcal{P}_3^{\alpha,1-\alpha} = \{x \in \mathbb{R}^3 : x_1^\alpha x_2^{1-\alpha} \geq |x_3|, x_1, x_2 \geq 0\} \cup \{(a, b, c) : a \leq 0, b = 0, c \geq 0\}$ is the power cone (see Chares (2009)).

2.4. Performance Measures

The out-of-sample performance of portfolios described above is evaluated using several performance measures typically adopted in the literature (see, e.g., Bruni et al. (2017); Cesarone et al. (2022) and references therein). Let R^{out} be the out-of-sample return of the portfolio, and $W_t = W_{t-1}(1 + R^{\text{out}})$.

- **Mean** is the daily average portfolio return, i.e., $\mu^{\text{out}} = \mathbb{E}[R^{\text{out}}]$. Evidently, higher values are associated with higher performances. The annualized mean is calculated as $E[R_a^{\text{out}}] = (1 + E[R^{\text{out}}])^{252} - 1$, where 252 are the trading days.
- **Standard deviation** computed as $\sigma^{\text{out}} = \sqrt{\mathbb{E}[(R^{\text{out}} - \mu^{\text{out}})^2]}$. Since it measures the portfolio risk, lower values are preferable.
- **Maximum Drawdown** Denoting the drawdowns by

$$
dd_t = \frac{W_t - \max\limits_{1 \leq \tau \leq t} W_\tau}{\max\limits_{1 \leq \tau \leq t} W_\tau}, \quad t \in \{1, \ldots, T\},
$$

  MaxDD is defined as **MaxDD** $= \min\limits_{1 \leq t \leq T} dd_t$.
- **The Ulcer** index, i.e.,

$$
UI = \sqrt{\frac{\sum_{t=1}^{T} dd_t^2}{T}}.
$$

  This evaluates the depth and the duration of drawdowns in prices over the out-of-sample period. Lower ulcer values are associated with better portfolio performances.
- **The Sharpe** ratio measures the gain per unit risk and is defined as

$$
SR^{\text{out}} = \frac{\mu^{\text{out}} - r_f}{\sigma^{\text{out}}},
$$

  where we set the risk-free rate $r_f = 0\%$ for simplicity since it does not influence the ranking among the models analyzed (Bruni et al. 2017). The larger this value, the better the performance.

- **The Sortino** ratio is similar to the Sharpe ratio but uses another risk measure, i.e., the target downside deviation, $TDD = \sqrt{\mathbb{E}[(\min\{R^{\text{out}} - r_f, 0\})^2]}$. Therefore, the Sortino ratio is defined as
$$SoR = \frac{\mu^{\text{out}} - r_f}{TDD},$$
where $r_f = 0$. Similar to the Sharpe ratio, the higher it is, the better the portfolio performance.
- **The Omega** ratio is defined as
$$\Omega_\eta(x) = \frac{\int_{-\infty}^{\eta}(1 - F_{R_P^{\text{out}}}(r))\,dr}{\int_{\eta}^{\infty} F_{R_P^{\text{out}}}(r)\,dr} = \frac{\mathbb{E}[\max(0, R_P^{\text{out}}(x) - \eta)]}{\mathbb{E}[\min(0, R_P^{\text{out}}(x) - \eta)]},$$
where $F_{R_P^{\text{out}}}$ is the cumulative distribution function of the out-of-sample portfolio return and η_0. In a nutshell, Omega is the ratio between the sum of positive deviations of R_P^{out} from η and the sum of its negative deviations. Higher values of the Omega ratio are always preferred.
- **The Rachev** ratio measures the upside potential, comparing the right and left tails. Mathematically, it is computed as
$$\frac{CVaR_\alpha(r_f - R^{\text{out}})}{CVaR_\beta(R^{\text{out}} - r_f)},$$
where we choose $\alpha = \beta = 5\%$ and $r_f = 0$.
- **VaR**, with reference to Equation (7); the confidence level was set to $1 - \alpha = 95\%$.
- **The Herfindahl** index, is a synthetic index used to measure risk concentration, defined as follows:
$$HI = \sum_{i=1}^{n}(RCR_i)^2,$$
where RCR_i is the risk contribution of the i-th asset to the total portfolio risk:
$$\text{RCR}_i = \frac{\text{RC}_i^{\sigma^2}(x)}{\sigma_P^2(x)},$$
in this case $\text{RC}_i^{\sigma^2}(x)$ is the risk contribution of the i-th asset calculated through the variance. It ranges from $\frac{1}{n}$ (when the risk is evenly spread over all the assets) to 1 (when the risk is concentrated in only one asset). Therefore, lower values are preferable since they are associated with more diversified portfolios. We compute the average of the Herfindahl index over the number of rebalances Q, i.e.,
$$E[HI] = \frac{1}{Q}\sum_{q=1}^{Q} HI_q.$$
- **Turnover**, i.e.,
$$Turn = \frac{1}{Q}\sum_{q=1}^{Q}\sum_{i=1}^{n}|x_{q,i} - x_{q-1,i}|,$$
where $x_{q,i}$ is the portfolio weight of asset k after rebalancing, and $x_{q-1,i}$ is the portfolio weight before rebalancing at time q. Lower turnover values imply better portfolio performance. We point out that this definition of portfolio turnover is a proxy of the effective one, since it evaluates only the amount of trading generated by the models at each rebalance, without considering the trades due to changes in asset prices between one rebalance and the next. In the results section, we report the annual turnover.

- **Transaction cost** is the average expense incurred by investors when rebalancing their portfolio. It is calculated as the product of turnover ($Turn$) and the sum of the average half spread (hs) and the average one-way commission rate (c):

$$TC = Turn\,(hs + c)\,.$$

  We set $hs = 0.0032$ and $c = 0.0046$, as in Jones (2002).
- **Net performance** is calculated as $NP = E[R_a^{\text{out}}] - TC$.

2.5. Datasets

In this section, we provide details about the four real-world equity datasets summarized in Table 1. These datasets comprise daily prices adjusted for dividends and stock splits obtained from Refinitiv and are outlined in Table 1. The aim is to provide an overview of four distinct stock markets, encompassing both developed and developing economies. In the USA and Europe, there exists a significant level of interconnectedness. However, in the USA, the industrial base is relatively lower compared to Europe (although this may change in the near future due to energy chaos Caddle (2023); Ulrich (2023)). Conversely, the financial sector and advanced industries such as information technology and biopharmaceuticals are more developed in the USA. Regarding developing markets, Turkey and Brazil serve as contrasting examples. Turkey, being relatively less rich in natural resources, relies heavily on its industrial sector and strategic geographical position between the East and the West.

The time series from the aforementioned datasets have been paired with that of Bitcoin, with data obtained from the same time interval from CoinMarketCap (2024). As the crypto market operates continuously, Saturdays and Sundays were excluded to align with the trading hours of the stock markets under consideration.

Table 1. List of the daily datasets analyzed.

Index	Abbreviation	Country	# Assets	Time Interval
Standard & Poor's 500	SP500	USA	496	January 2021–December 2023
Euro Stoxx 50	STX50	EU	25	January 2021–December 2023
Istanbul Exchange	BIST100	TUR	71	January 2021–December 2023
Bovespa	BVSP	BRA	74	January 2021–December 2023

Here, we elaborate on the empirical analysis conducted on four real-world datasets along with one cryptocurrency asset. Specifically, building upon the methods and materials outlined above, Section 2.6 illustrates the methodology employed for the experimental setup and the portfolio strategies analyzed.

2.6. Simulations and Set-Up

In this section, we begin by outlining the setup and simulations for optimized entropy models alongside classical ones like mean-variance, mean-CVaR, and mean-MinMax, in addition to the equally weighted portfolio. Then, we proceed with describing the sensitivity analysis across various expected target return levels for each model. Next, we detail the methodology for out-of-sample analysis, followed by an explanation of our sector analysis approach.

2.6.1. Portfolio Optimization and Selection

The optimization problems 13 and 14 were compared with a selection of classical models available in the literature: mean-variance (see Markowitz (1952)), mean-CVaR (using the formula linearized by Rockafellar and Uryasev (2000)), and mean-MinMax (as problem 1 from Young (1998). The confidence level $(1-\alpha)$ was set at 95% ($\alpha = 5\%$). In order to choose the value of k of the RLVaR, we performed a sensitivity analysis with respect to this parameter. Specifically, we simulated a univariate normal distribution with 30,000 observations, and we computed the EVaR (with $\alpha = 5\%$) and the MaxLoss. Then, taking 99 equally spaced values of k between 0.01 and 0.99, we computed the RLVaR with the same confidence level as the EVaR for each k. Finally, we selected the value of that parameter such that the RLVaR target was halfway between the EVaR and the MaxLoss, i.e., $k = 0.28$, rounded to 0.3 (see Figure 1).

For each model, we considered three different levels of expected target return $\eta_\beta = \eta_{min} + \beta(\eta_{max} - \eta_{min})$, where $\beta = 0, \frac{1}{3}$, and $\frac{2}{3}$, providing different portfolio options based on the level of risk aversion. The results are displayed in Section 3.

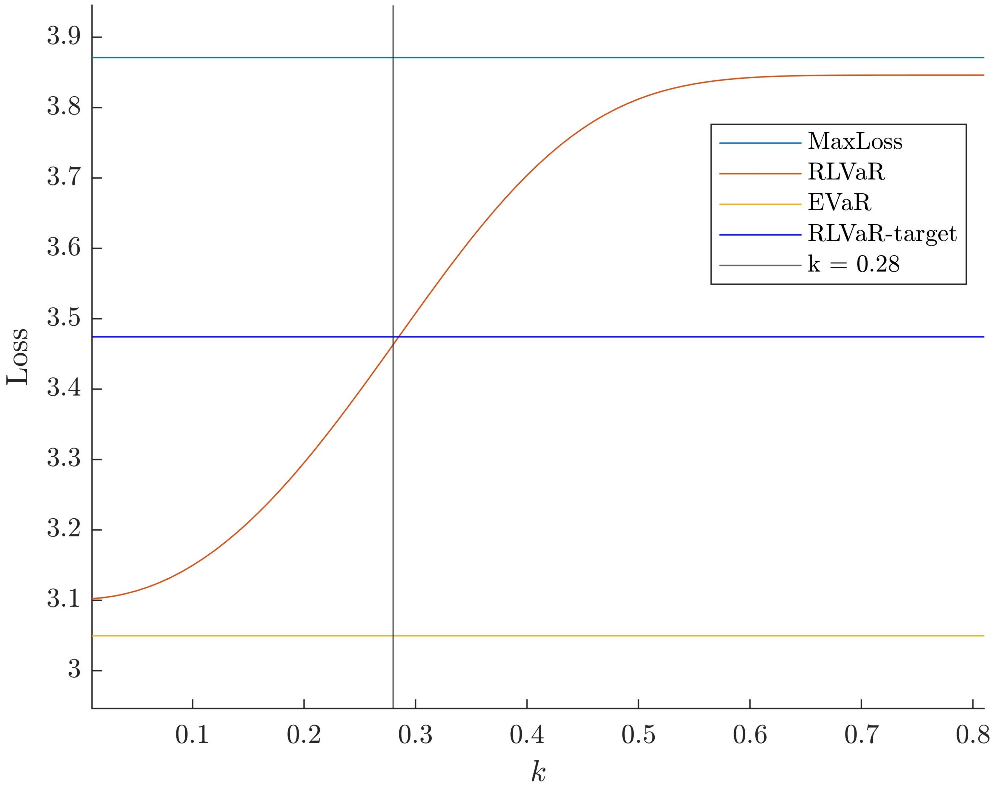

Figure 1. Selection of parameter k.

2.6.2. Hardware and Software

All procedures were implemented in MATLAB R2023b, with MOSEK support, and were run on a computer with an Intel(R) Xeon(R) CPU E5-2623 v4 @ 2.60 GHz processor and 64 GB of RAM.

2.6.3. Out-of-Sample Analysis

For the out-of-sample analysis, we employed a rolling time window approach for evaluation. Specifically, we enabled portfolio composition rebalancing within the holding period at consistent intervals. In our investigation, we designated a 2-year period (500 observations) for the sample window and 1 month (20 observations) for both the rebalancing interval and the holding period[2].

2.6.4. Sector Analysis

To perform a sector analysis of investment through each risk measure with constraint on expected return, on Refinitiv we obtained the sector composition of each index (see Table 2). Note the weights reported in the table reflect the financial market valuation of each sector and do not necessarily represent the actual weights on the economy.

As can be seen, there is a different sector composition in each market; the Brazilian market appears to be the most diversified.

Table 2. Sector composition.

GICS Sector	SP500	STX50	BIST100	Bovespa
Communication Services	4%	2%	2%	2%
Consumer Discretionary	11%	22%	11%	16%
Consumer Staples	8%	10%	10%	15%
Energy	5%	4%	2%	9%
Financials	14%	24%	12%	13%
Health Care	13%	6%	0%	5%
Industrials	16%	16%	25%	8%
Information Technology	13%	8%	4%	2%
Materials	6%	4%	19%	13%
Real Estate	6%	0%	3%	3%
Utilities	6%	4%	12%	14%

3. Empirical Findings

For each performance measure in Tables 3–11, we display the rank of the results with different colors. This includes a brief in-sample analysis of each dataset (Table 3), and the out-of-sample analysis of the five portfolio selection strategies analyzed (Tables 4–11).

More precisely, the colors range from deep green to deep red, where deep green represents the best performance while deep red the worst one. For the legend and abbreviations, see Appendix A.

3.1. Bitcoin versus Stock Markets

Let us begin by presenting some comparative data on the performance and risk profiles of the chosen stock markets and the crypto asset. Table 3 highlights the non-normality of the time-series data (Orlando and Bufalo 2021) and emphasizes that Bitcoin exhibits the least favorable risk profile. As we will explore in the following sections, this disadvantage plays a crucial role in portfolio optimization, diversification, and selection. Furthermore, Bitcoin presents challenges for its integration into traditional financial systems due to its unregulated nature. Despite offering benefits like faster transactions and privacy, concerns arise regarding its speculative nature and susceptibility to manipulation, prompting debates on its disruptive potential, speculative traits, and implications for illicit activities (Orlando 2024).

Table 3. Markets' and Bitcoin's performance.

	SP500	EuroStoxx50	BIST 100	Bovespa	Bitcoin
Mean	9.65%	9.67%	76.90%	5.92%	25.49%
SD	1.09%	1.12%	1.92%	1.22%	3.37%
Mdd	−25.43%	−25.50%	−22.26%	−26.50%	−76.63%
Ulcer	11.37%	9.54%	8.54%	14.11%	50.52%
Sharpe	3.36%	3.26%	11.79%	1.87%	2.67%
Sortino	4.79%	4.62%	17.12%	2.62%	3.90%
Omega	−109.63%	−109.58%	−140.84%	−105.01%	−108.16%
$\text{Rachev}_{5\%}$	93.90%	94.62%	91.87%	88.64%	104.61%
$\text{VaR}_{5\%}$	1.72%	1.81%	2.67%	2.10%	5.25%

3.2. Optimal Portfolios

In this section, five optimal portfolios according to the selected optimization problems (variance, CVaR, EVaR, RLVaR) and sensitivity factors are presented. In terms of considered markets, as mentioned in Section 2.5, the overview includes four distinct countries, spanning developed and developing economies. In the USA and Europe, there is significant interconnectedness. However, the USA has a relatively lower industrial base compared to Europe, but a more developed financial sector and advanced industries[3]. Recent developments, including the Inflation Reduction Act (IRA), the war in Ukraine, and the conflicts in the Middle East, significantly impact the competitiveness of Europe compared to the USA[4]. Turkey relies heavily on its industrial sector and strategic location, while Brazil, rich in natural resources, presents a different dynamic.

3.2.1. S&P 500

For the S&P 500, Table 12 indicates that the EVaR and RLVaR portfolios rank among the best across various risk levels η. However, even for η_0, the resulting portfolio is highly concentrated (6.25%, i.e 31 stocks), with the MaxLoss optimal portfolio investing in just 1.55% of the 500 stocks comprising the investable universe for $\eta_{2/3}$, thus confirming the high level of concentration of the USA stock market. Finally, in terms of turnover, the higher the η, the greater the turnover, confirming "existing empirical evidence that high turnover stock portfolios generate superior returns" (Dey 2005). In fact, as claimed by Cremers and Mei (2007), trading driven by systematic risk in returns drives turnover to the extent that it can explain 66% of systematic turnover. Therefore, portfolio rebalancing prompted by systematic risk emerges as a significant motive for stock trading.

Note that even when considering Bitcoin among the investable assets, diversification does not improve, as shown in Table 13. This is in line with the observations made in Section 3.1 regarding Bitcoin's risk profile compared to that of the S&P 500.

Table 4. Out-of-sample SP500.

η_{free}	**Var**	**CVaR**	**EVaR**	**RLVaR**	**MaxL**	$\eta_{1/3}$	**Var**	**CVaR**	**EVaR**	**RLVaR**	**MaxL**	$\eta_{2/3}$	**Var**	**CVaR**	**EVaR**	**RLVaR**	**MaxL**
Mean	−3.948%	−1.856%	−1.017%	−0.645%	0.292%	Mean	−0.705%	5.098%	11.269%	16.511%	19.138%	Mean	14.176%	22.603%	29.581%	25.051%	28.345%
NP	−5.437%	−4.884%	−2.167%	−1.506%	−0.860%	NP	−5.074%	0.133%	6.375%	11.489%	13.507%	NP	7.554%	15.384%	23.024%	17.546%	20.678%
SD	0.623%	0.668%	0.652%	0.670%	0.713%	SD	0.640%	0.739%	0.785%	0.804%	0.827%	SD	0.902%	0.983%	1.054%	1.104%	1.163%
Mdd	−13.210%	−11.953%	−11.762%	−12.368%	−13.207%	Mdd	−10.101%	−7.730%	−7.291%	−7.103%	−7.600%	Mdd	−8.945%	−10.863%	−10.098%	−12.101%	−12.103%
Ulcer	6.254%	6.168%	5.239%	5.427%	5.904%	Ulcer	4.701%	3.429%	2.824%	2.545%	2.435%	Ulcer	3.315%	3.642%	3.262%	3.769%	3.859%
Sharpe	−2.567%	−1.112%	−0.623%	−0.383%	0.162%	Sharpe	−0.439%	2.671%	5.402%	7.549%	8.401%	Sharpe	5.835%	8.226%	9.759%	8.041%	8.517%
Sortino	−3.486%	−1.530%	−0.871%	−0.537%	0.228%	Sortino	−0.602%	3.872%	7.639%	10.902%	12.221%	Sortino	8.751%	12.316%	14.821%	12.053%	12.940%
Omega	−93.698%	−97.201%	−98.432%	−99.042%	−100.403%	Omega	−98.920%	−106.964%	−114.898%	−121.780%	−124.798%	Omega	−115.927%	−123.608%	−128.525%	−123.612%	−125.470%
$Rachev_{5\%}$	86.012%	89.502%	89.173%	92.095%	95.326%	$Rachev_{5\%}$	92.305%	102.279%	86.450%	90.690%	92.529%	$Rachev_{5\%}$	115.715%	107.289%	108.593%	107.793%	110.817%
$VaR_{5\%}$	0.984%	1.124%	0.999%	0.992%	1.100%	$VaR_{5\%}$	1.031%	1.101%	1.177%	1.303%	1.309%	$VaR_{5\%}$	1.463%	1.486%	1.708%	1.794%	1.851%
E[HI]	7.836%	8.917%	10.788%	12.512%	14.546%	E[HI]	7.866%	14.611%	15.535%	18.009%	20.446%	E[HI]	22.736%	28.709%	33.373%	39.485%	44.126%
Turnover	190.953%	388.194%	147.333%	110.341%	147.651%	Turnover	560.042%	636.461%	627.472%	643.870%	721.971%	Turnover	848.999%	925.533%	840.543%	962.208%	982.912%
% stocks	6.37%	5.29%	4.52%	3.91%	3.06%	% stocks	6.12%	3.93%	3.48%	3.18%	3.02%	% stocks	2.65%	2.05%	1.70%	1.58%	1.55%

Table 5. Out-of-sample SP500 with Bitcoin.

η_{free}	**Var**	**CVaR**	**EVaR**	**RLVaR**	**MaxL**	$\eta_{1/3}$	**Var**	**CVaR**	**EVaR**	**RLVaR**	**MaxL**	$\eta_{2/3}$	**Var**	**CVaR**	**EVaR**	**RLVaR**	**MaxL**
Mean	−3.948%	−1.856%	−1.017%	−0.645%	0.292%	Mean	−0.705%	5.098%	11.269%	16.510%	19.138%	Mean	14.177%	22.603%	29.580%	25.051%	28.345%
NP	−5.437%	−4.884%	−2.166%	−1.506%	−0.859%	NP	−5.074%	0.133%	6.375%	11.488%	13.507%	NP	7.555%	15.384%	23.024%	17.546%	20.678%
SD	0.623%	0.668%	0.652%	0.670%	0.713%	SD	0.640%	0.739%	0.785%	0.804%	0.827%	SD	0.902%	0.983%	1.054%	1.104%	1.163%
Mdd	−13.211%	−11.953%	−11.761%	−12.368%	−13.207%	Mdd	−10.101%	−7.730%	−7.292%	−7.103%	−7.600%	Mdd	−8.944%	−10.863%	−10.098%	−12.101%	−12.103%
Ulcer	6.254%	6.168%	5.238%	5.427%	5.904%	Ulcer	4.701%	3.429%	2.824%	2.545%	2.435%	Ulcer	3.315%	3.642%	3.261%	3.769%	3.859%
Sharpe	−2.567%	−1.112%	−0.622%	−0.383%	0.162%	Sharpe	−0.439%	2.671%	5.403%	7.548%	8.401%	Sharpe	5.835%	8.226%	9.759%	8.041%	8.517%
Sortino	−3.486%	−1.531%	−0.871%	−0.537%	0.228%	Sortino	−0.602%	3.872%	7.639%	10.902%	12.221%	Sortino	8.752%	12.316%	14.821%	12.053%	12.940%
Omega	−93.699%	−97.201%	−98.432%	−99.042%	−100.403%	Omega	−98.920%	−106.964%	−114.899%	−121.780%	−124.799%	Omega	−115.928%	−123.608%	−128.525%	−123.612%	−125.470%
$Rachev_{5\%}$	86.011%	89.502%	89.174%	92.095%	95.327%	$Rachev_{5\%}$	92.306%	102.279%	86.451%	90.690%	92.529%	$Rachev_{5\%}$	115.722%	107.289%	108.593%	107.793%	110.817%
$VaR_{5\%}$	0.984%	1.124%	0.999%	0.992%	1.100%	$VaR_{5\%}$	1.031%	1.101%	1.177%	1.303%	1.309%	$VaR_{5\%}$	1.463%	1.486%	1.708%	1.794%	1.851%
E[HI]	7.836%	8.917%	10.789%	12.512%	14.546%	E[HI]	7.866%	14.611%	15.535%	18.009%	20.446%	E[HI]	22.736%	28.708%	33.373%	39.485%	44.127%
Turnover	190.950%	388.203%	147.334%	110.344%	147.631%	Turnover	560.041%	636.462%	627.474%	643.881%	721.975%	Turnover	848.966%	925.533%	840.542%	962.205%	982.908%
% stocks	6.35%	5.28%	4.51%	3.91%	3.05%	% stocks	6.10%	3.92%	3.47%	3.17%	3.02%	% stocks	2.67%	2.05%	1.69%	1.58%	1.54%

3.2.2. Euro Stoxx 50

For the Euro Stoxx 50, the scenario appears to differ from the USA market, with optimal portfolios varying: CVaR and EVaR perform best for η_0, VaR and EVaR for $\eta_{1/3}$, and RLVaR and MaxL for $\eta_{2/3}$ (see Table 6). In terms of diversification, considering the index composed of only 50 securities, the number of stocks considered varies between 20 (corresponding to VaR with η_0) and 9 (corresponding to MaxL with $\eta_{2/3}$). This once again confirms the concentration in the market as in the USA.

However, when considering Bitcoin, both performance and other indicators slightly improve, as observed in Table 7. This marginal improvement is consistent with the considerations made earlier regarding the risk profile of the cryptocurrency. Consequently, the lower performance of European stocks may still be attributed to the much lower turnover compared to the USA market.

Table 6. Out-of-sample EuroStoxx50.

η_{free}	**Var**	**CVaR**	**EVaR**	**RLVaR**	**MaxL**	$\eta_{1/3}$	**Var**	**CVaR**	**EVaR**	**RLVaR**	**MaxL**	$\eta_{2/3}$	**Var**	**CVaR**	**EVaR**	**RLVaR**	**MaxL**
Mean	9.288%	10.010%	9.905%	5.098%	4.423%	Mean	10.781%	7.978%	8.608%	6.346%	8.546%	Mean	15.378%	12.037%	14.521%	16.351%	19.047%
NP	8.700%	8.394%	9.243%	4.284%	3.941%	NP	8.964%	5.425%	6.164%	3.207%	4.772%	NP	11.935%	8.218%	11.559%	14.014%	15.715%
SD	0.720%	0.745%	0.760%	0.801%	0.831%	SD	0.767%	0.813%	0.776%	0.786%	0.815%	SD	0.928%	0.907%	0.931%	0.925%	0.936%
Mdd	−8.664%	−10.703%	−9.628%	−12.078%	−12.883%	Mdd	−8.655%	−12.574%	−9.654%	−10.280%	−9.854%	Mdd	−8.776%	−10.304%	−7.729%	−6.977%	−6.703%
Ulcer	4.803%	5.648%	4.771%	5.820%	5.812%	Ulcer	4.301%	5.621%	4.497%	4.451%	4.472%	Ulcer	3.887%	4.347%	3.453%	3.480%	3.020%
Sharpe	4.893%	5.082%	4.933%	2.464%	2.066%	Sharpe	5.295%	3.746%	4.220%	3.106%	3.992%	Sharpe	6.119%	4.975%	5.779%	6.497%	7.392%
Sortino	6.690%	7.003%	6.864%	3.356%	2.777%	Sortino	7.207%	4.895%	5.805%	4.306%	5.539%	Sortino	8.418%	6.808%	7.924%	9.051%	10.244%
Omega	−114.156%	−114.572%	−113.930%	−106.765%	−105.697%	Omega	−115.836%	−111.155%	−111.927%	−108.603%	−110.958%	Omega	−117.667%	−114.293%	−116.695%	−118.805%	−121.679%
$Rachev_{5\%}$	80.920%	82.333%	83.269%	82.618%	79.409%	$Rachev_{5\%}$	75.009%	73.516%	82.032%	87.877%	79.712%	$Rachev_{5\%}$	75.761%	75.150%	74.175%	78.748%	78.065%
$VaR_{5\%}$	1.076%	1.071%	1.132%	1.169%	1.294%	$VaR_{5\%}$	1.214%	1.268%	1.283%	1.299%	1.416%	$VaR_{5\%}$	1.488%	1.524%	1.497%	1.455%	1.499%
E[HI]	23.224%	26.616%	25.509%	26.983%	30.894%	E[HI]	27.563%	26.657%	28.297%	28.810%	33.216%	E[HI]	44.174%	46.364%	50.949%	52.050%	53.454%
Turnover	75.288%	207.091%	84.798%	104.332%	61.781%	Turnover	232.990%	327.415%	313.327%	402.473%	483.943%	Turnover	441.389%	489.518%	379.792%	299.575%	427.238%
% stocks	39.42%	31.09%	33.65%	28.85%	24.68%	% stocks	27.24%	28.21%	26.60%	25.00%	23.08%	% stocks	17.63%	17.63%	16.35%	17.63%	16.99%

Table 7. Out-of-sample EuroStoxx50 with Bitcoin.

η_{free}	**Var**	**CVaR**	**EVaR**	**RLVaR**	**MaxL**	$\eta_{1/3}$	**Var**	**CVaR**	**EVaR**	**RLVaR**	**MaxL**	$\eta_{2/3}$	**Var**	**CVaR**	**EVaR**	**RLVaR**	**MaxL**
Mean	14.951%	16.123%	15.606%	12.791%	14.409%	Mean	12.934%	10.581%	12.407%	10.746%	12.337%	Mean	14.989%	12.206%	14.863%	18.231%	20.950%
NP	14.375%	14.720%	15.118%	12.291%	14.392%	NP	11.072%	7.854%	10.050%	7.361%	7.701%	NP	11.532%	8.210%	11.842%	15.570%	16.771%
SD	0.687%	0.701%	0.699%	0.745%	0.748%	SD	0.749%	0.789%	0.749%	0.771%	0.794%	SD	0.920%	0.908%	0.936%	0.933%	0.966%
Mdd	−8.583%	−8.598%	−7.548%	−7.567%	−6.614%	Mdd	−8.695%	−11.177%	−8.307%	−8.406%	−7.264%	Mdd	−8.683%	−9.810%	−7.651%	−6.975%	−6.765%
Ulcer	4.165%	4.626%	3.792%	3.997%	3.153%	Ulcer	4.139%	5.225%	3.986%	3.674%	3.556%	Ulcer	3.947%	4.123%	3.417%	3.327%	2.892%
Sharpe	8.056%	8.459%	8.230%	6.414%	7.143%	Sharpe	6.446%	5.060%	6.194%	5.256%	5.819%	Sharpe	6.025%	5.032%	5.877%	7.122%	7.816%
Sortino	11.372%	12.197%	11.906%	9.032%	10.264%	Sortino	8.887%	6.747%	8.689%	7.434%	8.255%	Sortino	8.276%	6.894%	8.066%	9.933%	10.816%
Omega	−124.179%	−125.069%	−123.761%	−118.245%	−120.421%	Omega	−119.495%	−115.070%	−117.764%	−114.964%	−116.642%	Omega	−117.425%	−114.541%	−117.021%	−120.802%	−123.087%
$Rachev_{5\%}$	84.602%	90.162%	90.203%	84.700%	91.433%	$Rachev_{5\%}$	76.453%	75.507%	83.891%	91.035%	84.237%	$Rachev_{5\%}$	74.260%	74.468%	73.602%	77.731%	76.957%
$VaR_{5\%}$	1.051%	1.068%	1.051%	1.140%	1.170%	$VaR_{5\%}$	1.135%	1.285%	1.244%	1.357%	1.399%	$VaR_{5\%}$	1.576%	1.516%	1.525%	1.470%	1.507%
E[HI]	22.115%	25.283%	22.925%	26.990%	31.559%	E[HI]	26.423%	26.066%	26.842%	29.442%	33.790%	E[HI]	43.003%	46.446%	51.504%	52.441%	55.130%
Turnover	73.747%	179.929%	62.577%	64.085%	2.124%	Turnover	238.705%	349.578%	302.070%	433.981%	594.357%	Turnover	443.192%	512.327%	387.272%	341.200%	535.661%
% stocks	39.08%	32.31%	32.00%	29.85%	24.00%	% stocks	31.08%	29.54%	28.92%	27.38%	23.08%	% stocks	18.15%	18.46%	17.54%	19.08%	18.46%

3.2.3. BIST 100

In the Turkish stock market, the optimal portfolios are consistently stable with Var and CVaR for any level of η (see Table 8). However, this changes when Bitcoin is considered, as EVaR and RLVaR emerge as the best options, especially in the case of $\eta_{2/3}$ (see Table 9). As for the USA and European markets, both concentration and turnover increase with risk.

Table 8. Out-of-sample BIST100.

η_{free}	**Var**	**CVaR**	**EVaR**	**RLVaR**	**MaxL**	$\eta_{1/3}$	**Var**	**CVaR**	**EVaR**	**RLVaR**	**MaxL**	$\eta_{2/3}$	**Var**	**CVaR**	**EVaR**	**RLVaR**	**MaxL**
Mean	91.027%	88.750%	65.724%	74.119%	56.843%	Mean	77.144%	74.855%	73.722%	69.974%	69.347%	Mean	67.838%	64.504%	53.435%	49.643%	24.416%
NP	88.988%	85.308%	62.859%	71.416%	54.619%	NP	73.599%	68.593%	69.958%	65.568%	64.103%	NP	63.407%	58.285%	48.332%	43.565%	16.344%
SD	2.268%	2.344%	2.357%	2.455%	2.598%	SD	2.425%	2.582%	2.582%	2.663%	2.703%	SD	2.804%	2.894%	2.957%	3.067%	3.301%
Mdd	−24.858%	−26.079%	−20.971%	−19.414%	−22.140%	Mdd	−26.436%	−25.610%	−24.602%	−24.692%	−26.233%	Mdd	−33.578%	−29.730%	−31.797%	−32.988%	−34.887%
Ulcer	10.279%	10.078%	8.391%	8.006%	9.723%	Ulcer	12.042%	12.050%	10.108%	9.888%	10.752%	Ulcer	14.710%	12.147%	13.823%	14.201%	15.265%
Sharpe	11.341%	10.768%	8.514%	8.973%	6.881%	Sharpe	9.368%	8.599%	8.497%	7.914%	7.742%	Sharpe	7.336%	6.832%	5.750%	5.219%	2.627%
Sortino	17.054%	15.949%	12.786%	13.558%	10.427%	Sortino	13.997%	12.791%	12.901%	12.114%	11.994%	Sortino	11.090%	10.255%	8.677%	7.924%	4.004%
Omega	−135.761%	−133.570%	−125.742%	−126.899%	−119.835%	Omega	−128.352%	−125.070%	−124.771%	−123.021%	−122.455%	Omega	−121.256%	−119.216%	−116.035%	−114.534%	−107.190%
$Rachev_{5\%}$	98.123%	95.270%	100.050%	102.878%	106.124%	$Rachev_{5\%}$	102.059%	97.923%	101.412%	103.234%	107.156%	$Rachev_{5\%}$	110.312%	105.968%	105.138%	108.628%	114.401%
$VaR_{5\%}$	3.194%	3.144%	3.475%	3.646%	3.686%	$VaR_{5\%}$	3.606%	3.873%	3.931%	4.135%	3.959%	$VaR_{5\%}$	4.241%	4.518%	4.213%	4.561%	5.180%
E[HI]	8.094%	14.651%	21.417%	27.787%	34.489%	E[HI]	7.537%	15.974%	22.870%	30.089%	34.122%	E[HI]	17.625%	23.821%	33.234%	44.561%	60.137%
Turnover	261.504%	441.283%	367.316%	346.489%	285.190%	Turnover	454.396%	802.733%	482.607%	564.850%	672.270%	Turnover	568.085%	797.417%	654.122%	779.212%	1034.914%
% stocks	33.22%	15.16%	15.05%	9.84%	5.90%	% stocks	33.10%	17.25%	12.50%	9.14%	7.29%	% stocks	17.36%	12.27%	9.72%	7.75%	6.02%

Table 9. Out-of-sample BIST100 with Bitcoin.

η_{free}	Var	CVaR	EVaR	RLVaR	MaxL	$\eta_{1/3}$	Var	CVaR	EVaR	RLVaR	MaxL	$\eta_{2/3}$	Var	CVaR	EVaR	RLVaR	MaxL
Mean	101.242%	105.254%	108.556%	103.625%	96.156%	Mean	81.176%	79.556%	82.773%	84.389%	71.987%	Mean	62.083%	58.341%	47.091%	41.834%	28.259%
NP	99.327%	101.297%	106.184%	101.161%	94.076%	NP	77.966%	73.794%	78.479%	79.528%	66.441%	NP	57.778%	52.431%	41.014%	34.297%	17.904%
SD	1.956%	2.040%	1.862%	1.882%	1.887%	SD	2.211%	2.289%	2.238%	2.289%	2.310%	SD	2.685%	2.750%	2.786%	2.863%	2.997%
Mdd	−19.386%	−19.035%	−14.046%	−13.833%	−14.603%	Mdd	−21.461%	−19.560%	−20.471%	−18.671%	−19.834%	Mdd	−32.643%	−28.208%	−31.150%	−29.770%	−33.274%
Ulcer	5.780%	6.198%	4.201%	4.099%	4.272%	Ulcer	9.101%	7.000%	7.356%	6.982%	7.519%	Ulcer	14.204%	11.556%	13.081%	12.739%	14.104%
Sharpe	14.206%	14.011%	15.690%	15.016%	14.185%	Sharpe	10.681%	10.159%	10.707%	10.622%	9.325%	Sharpe	7.144%	6.638%	5.500%	4.847%	3.298%
Sortino	21.851%	21.267%	25.155%	24.271%	22.854%	Sortino	16.063%	15.106%	16.478%	16.478%	14.496%	Sortino	10.691%	9.904%	8.304%	7.346%	4.986%
Omega	−145.681%	−143.949%	−152.442%	−149.896%	−146.353%	Omega	−132.520%	−129.682%	−131.613%	−131.658%	−127.235%	Omega	−120.775%	−118.643%	−115.267%	−113.365%	−108.970%
$\text{Rachev}_{5\%}$	102.077%	96.922%	118.749%	121.547%	121.821%	$\text{Rachev}_{5\%}$	100.573%	97.223%	108.761%	109.563%	109.720%	$\text{Rachev}_{5\%}$	105.355%	106.653%	107.966%	112.146%	116.945%
$\text{VaR}_{5\%}$	2.659%	3.151%	2.263%	2.212%	2.373%	$\text{VaR}_{5\%}$	3.188%	3.421%	3.406%	3.339%	3.366%	$\text{VaR}_{5\%}$	4.009%	4.114%	3.775%	4.283%	4.691%
E[HI]	9.816%	17.508%	19.381%	24.293%	27.965%	E[HI]	7.567%	15.073%	20.285%	29.003%	33.418%	E[HI]	14.469%	22.668%	33.453%	43.982%	59.403%
Turnover	245.514%	507.362%	304.103%	315.935%	266.714%	Turnover	411.468%	738.748%	550.490%	623.202%	711.042%	Turnover	551.998%	757.748%	779.113%	966.237%	1327.583%
% stocks	33.90%	16.89%	15.64%	11.76%	10.05%	% stocks	32.19%	16.55%	14.61%	10.96%	9.13%	% stocks	20.32%	12.21%	10.50%	8.79%	7.31%

3.2.4. Bovespa

In Brazil, the best optimal portfolios change according to the level of risk. For η_0, Var and CVaR portfolios perform best. At $\eta_{1/3}$, EVaR and RLVaR portfolios are best, while at $\eta_{2/3}$, RLVaR and MaxL portfolios are preferred (see Table 10). The situation changes when Bitcoin is considered. For η_0, Var and CVaR portfolios perform best. At $\eta_{1/3}$ and $\eta_{2/3}$, EVaR and RLVaR portfolios are optimal, while at $\eta_{2/3}$, the MaxL portfolio performs poorly (see Table 11). In terms of concentration, the Bovespa contains 86 stocks (Topforeignstocks 2024), resulting in the number of considered securities ranging between 16 and 7. As for the other considered markets, turnover increases with higher risk.

Table 10. Out-of-sample Bovespa.

η_{free}	Var	CVaR	EVaR	RLVaR	MaxL	$\eta_{1/3}$	Var	CVaR	EVaR	RLVaR	MaxL	$\eta_{2/3}$	Var	CVaR	EVaR	RLVaR	MaxL
Mean	21.667%	20.952%	19.919%	19.246%	17.514%	Mean	23.204%	26.339%	30.950%	30.451%	28.808%	Mean	36.464%	33.908%	39.256%	39.679%	42.852%
NP	20.575%	18.202%	17.591%	17.027%	14.660%	NP	19.506%	22.132%	26.038%	24.709%	22.450%	NP	31.199%	27.702%	33.741%	34.139%	36.484%
SD	0.732%	0.768%	0.780%	0.798%	0.816%	SD	0.731%	0.752%	0.735%	0.742%	0.763%	SD	1.081%	1.089%	1.099%	1.100%	1.105%
Mdd	−6.505%	−7.769%	−8.437%	−9.624%	−10.004%	Mdd	−10.860%	−11.635%	−10.717%	−10.964%	−11.353%	Mdd	−17.605%	−20.991%	−22.005%	−21.794%	−21.686%
Ulcer	2.702%	3.416%	3.911%	4.273%	4.877%	Ulcer	4.334%	5.159%	4.113%	4.422%	4.608%	Ulcer	7.132%	9.073%	8.985%	8.678%	8.005%
Sharpe	10.641%	9.829%	9.250%	8.754%	7.852%	Sharpe	11.330%	12.351%	14.570%	14.226%	13.167%	Sharpe	11.423%	10.644%	11.965%	12.067%	12.816%
Sortino	16.685%	15.406%	14.350%	13.409%	12.136%	Sortino	17.056%	18.939%	22.687%	22.071%	20.488%	Sortino	16.828%	15.500%	17.795%	18.029%	19.377%
Omega	−131.731%	−128.157%	−126.588%	−125.312%	−122.264%	Omega	−133.403%	−137.165%	−144.542%	−143.014%	−138.941%	Omega	−133.837%	−131.522%	−135.765%	−135.915%	−138.646%
$\text{Rachev}_{5\%}$	116.755%	115.256%	120.027%	115.113%	118.985%	$\text{Rachev}_{5\%}$	98.521%	105.177%	114.253%	111.785%	111.595%	$\text{Rachev}_{5\%}$	92.459%	92.189%	98.127%	102.940%	104.161%
$\text{VaR}_{5\%}$	1.149%	1.076%	1.054%	1.150%	1.169%	$\text{VaR}_{5\%}$	1.241%	1.215%	1.203%	1.166%	1.223%	$\text{VaR}_{5\%}$	1.540%	1.617%	1.756%	1.714%	1.565%
E[HI]	15.477%	16.902%	14.696%	14.496%	15.346%	E[HI]	16.453%	21.624%	18.125%	16.889%	16.190%	E[HI]	35.107%	39.019%	37.344%	37.575%	37.989%
Turnover	139.998%	352.546%	298.509%	284.472%	365.820%	Turnover	474.165%	539.364%	629.693%	736.250%	815.108%	Turnover	675.021%	795.647%	707.108%	710.367%	816.400%
% stocks	18.65%	16.88%	18.02%	16.67%	16.15%	% stocks	18.96%	15.21%	14.69%	14.58%	14.58%	% stocks	10.63%	8.23%	8.33%	8.13%	8.65%

Table 11. Out-of-sample Bovespa with Bitcoin.

η_{free}	Var	CVaR	EVaR	RLVaR	MaxL	$\eta_{1/3}$	Var	CVaR	EVaR	RLVaR	MaxL	$\eta_{2/3}$	Var	CVaR	EVaR	RLVaR	MaxL
Mean	27.276%	26.015%	23.828%	24.594%	23.360%	Mean	26.942%	29.341%	34.808%	35.318%	32.627%	Mean	36.283%	33.783%	38.639%	37.916%	32.337%
NP	26.192%	23.159%	21.249%	21.855%	20.252%	NP	23.187%	24.623%	29.510%	29.799%	26.206%	NP	30.905%	27.661%	33.063%	32.236%	25.114%
SD	0.687%	0.758%	0.764%	0.783%	0.801%	SD	0.707%	0.744%	0.715%	0.714%	0.738%	SD	1.074%	1.082%	1.094%	1.103%	1.135%
Mdd	−5.476%	−6.355%	−7.512%	−8.330%	−9.128%	Mdd	−10.188%	−10.890%	−9.679%	−8.972%	−10.024%	Mdd	−17.507%	−20.889%	−22.050%	−21.969%	−22.499%
Ulcer	2.377%	2.731%	3.382%	3.440%	3.836%	Ulcer	4.003%	4.723%	3.644%	3.380%	4.051%	Ulcer	7.087%	9.030%	9.089%	8.937%	9.007%
Sharpe	13.947%	12.107%	11.104%	11.154%	10.404%	Sharpe	13.402%	13.722%	16.596%	16.809%	15.192%	Sharpe	11.441%	10.677%	11.862%	11.572%	9.802%
Sortino	22.023%	19.250%	17.250%	17.316%	16.113%	Sortino	20.300%	21.045%	26.092%	26.492%	23.814%	Sortino	16.835%	15.570%	17.594%	17.111%	14.309%
Omega	−142.435%	−135.867%	−132.699%	−132.718%	−129.986%	Omega	−141.190%	−142.392%	−152.249%	−152.544%	−146.326%	Omega	−133.964%	−131.569%	−135.457%	−134.477%	−128.640%
$\text{Rachev}_{5\%}$	112.350%	116.258%	116.503%	116.886%	115.517%	$\text{Rachev}_{5\%}$	98.458%	107.225%	115.644%	115.175%	111.817%	$\text{Rachev}_{5\%}$	92.537%	92.653%	96.997%	99.024%	94.299%
$\text{VaR}_{5\%}$	1.059%	1.082%	1.035%	1.065%	1.065%	$\text{VaR}_{5\%}$	1.196%	1.173%	1.129%	1.098%	1.188%	$\text{VaR}_{5\%}$	1.516%	1.596%	1.707%	1.694%	1.729%
E[HI]	14.895%	17.581%	15.313%	15.032%	16.120%	E[HI]	17.313%	21.312%	18.390%	17.163%	17.019%	E[HI]	35.928%	38.682%	37.952%	38.560%	37.407%
Turnover	139.005%	366.148%	330.674%	351.213%	398.432%	Turnover	481.431%	604.938%	679.282%	707.565%	823.236%	Turnover	689.512%	784.851%	714.889%	728.209%	925.982%
% stocks	20.47%	16.67%	19.24%	18.83%	18.11%	% stocks	19.44%	15.64%	16.05%	16.87%	16.05%	% stocks	10.70%	8.44%	8.64%	8.64%	8.85%

3.3. Allocation Analysis

In terms of sectors, Tables 12, 14, 16, and 18 depict the economic composition of the optimal portfolios for the above-mentioned equity markets without considering a crypto asset. Conversely, Tables 13, 15, 17, and 19 present the composition when the crypto investment is considered.

3.3.1. S&P 500

In the USA, the healthcare sector is highly preferred due to factors such as an aging population, advancements in biotechnology, and the impact of the pandemic (see Table 12). For example, Chen et al. (2015) find that money supply, economic growth, and stock returns from local markets contribute to creating bubbles in the healthcare sector of the U.S. and German stock markets, and two bubbles in the healthcare sector of the U.K. stock market. Additionally, Raghupathi and Raghupathi (2020) assert a positive correlation between healthcare expenditure and economic indicators such as income, GDP, and labor productivity. Moreover, Chen et al. (2018) discover that both healthcare funds and ETFs offer significantly positive average alpha and serve as a hedge against market downturn risk. They suggest that enhancing an all-stock portfolio like the S&P 500 index fund can be achieved by simply adding a value-weighted healthcare portfolio, resulting in both higher returns and lower standard deviation. Finally, Sun (2020) find that during the COVID-19 pandemic, larger-cap stocks in more than half of the sectors were affected differently from smaller-cap stocks, with smaller-cap stocks being hit harder.

Table 13 demonstrates that the allocation remains unchanged even when considering crypto assets. This observation aligns with previous findings regarding the performance and concentration of top stocks in the USA market.

Table 12. Sector investment averages of the SP500.

GICS Sector	Variance			Conditional Value at Risk			Entropic Value at Risk			Relativistic Value at Risk			MaxLoss		
η	free	1/3	2/3	free	1/3	2/3	free	1/3	2/3	free	1/3	2/3	free	1/3	2/3
Communication Services	14.91%	3.44%	0.00%	15.22%	4.39%	1.68%	8.64%	0.77%	0.00%	7.15%	0.04%	0.00%	6.43%	0.00%	0.00%
Consumer Discretionary	7.05%	7.68%	6.85%	6.93%	4.73%	2.99%	14.87%	8.73%	1.88%	15.30%	9.65%	1.66%	13.01%	10.71%	1.89%
Consumer Staples	24.87%	23.40%	7.20%	17.98%	18.67%	2.54%	7.56%	13.36%	3.64%	4.78%	11.18%	3.92%	2.77%	9.38%	4.17%
Energy	1.27%	7.44%	16.24%	3.54%	8.75%	19.78%	0.00%	6.29%	12.45%	0.00%	4.54%	10.76%	0.00%	3.60%	10.52%
Financials	9.48%	6.92%	4.72%	6.36%	4.38%	1.80%	16.99%	15.88%	5.06%	19.33%	19.13%	5.97%	29.22%	22.59%	4.80%
Health Care	31.34%	35.35%	47.48%	40.69%	38.34%	48.73%	35.00%	35.62%	51.86%	38.37%	35.20%	51.05%	39.10%	34.68%	52.00%
Industrials	8.20%	6.03%	1.21%	0.10%	0.50%	2.05%	0.08%	2.12%	1.81%	0.55%	1.59%	2.67%	1.15%	1.43%	2.30%
Information Technology	2.37%	6.73%	12.18%	1.24%	7.65%	15.66%	5.30%	13.21%	15.64%	4.19%	13.27%	14.39%	1.44%	12.26%	13.18%
Materials	0.52%	0.16%	3.26%	1.11%	0.78%	4.00%	11.57%	3.96%	7.66%	10.32%	5.41%	9.58%	6.88%	5.34%	11.13%
Real Estate	0.00%	0.00%	0.00%	0.13%	0.34%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Utilities	0.00%	2.84%	0.86%	6.69%	11.45%	0.76%	0.00%	0.06%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%

Table 13. Sector investment averages of the SP500 with Bitcoin.

GICS Sector	Variance			Conditional Value at Risk			Entropic Value at Risk			Relativistic Value at Risk			MaxLoss		
η	free	1/3	2/3	free	1/3	2/3	free	1/3	2/3	free	1/3	2/3	free	1/3	2/3
Communication Services	14.91%	3.44%	0.00%	15.22%	4.39%	1.68%	8.64%	0.77%	0.00%	7.15%	0.04%	0.00%	6.43%	0.00%	0.00%
Consumer Discretionary	7.05%	7.68%	6.85%	6.93%	4.73%	2.99%	14.87%	8.73%	1.88%	15.30%	9.65%	1.66%	13.01%	10.71%	1.89%
Consumer Staples	24.87%	23.40%	7.20%	17.98%	18.67%	2.54%	7.56%	13.36%	3.64%	4.78%	11.18%	3.92%	2.77%	9.38%	4.17%
Cripto	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Energy	1.27%	7.44%	16.24%	3.54%	8.75%	19.78%	0.00%	6.29%	12.45%	0.00%	4.54%	10.76%	0.00%	3.60%	10.52%
Financials	9.48%	6.92%	4.72%	6.36%	4.38%	1.80%	16.99%	15.88%	5.06%	19.33%	19.13%	5.97%	29.22%	22.59%	4.80%
Health Care	31.34%	35.35%	47.48%	40.69%	38.34%	48.73%	35.00%	35.62%	51.86%	38.37%	35.20%	51.05%	39.10%	34.68%	52.00%
Industrials	8.20%	6.03%	1.21%	0.10%	0.50%	2.05%	0.08%	2.12%	1.81%	0.55%	1.59%	2.67%	1.15%	1.43%	2.30%
Information Technology	2.37%	6.73%	12.18%	1.24%	7.65%	15.66%	5.30%	13.21%	15.64%	4.19%	13.27%	14.39%	1.44%	12.26%	13.18%
Materials	0.52%	0.16%	3.26%	1.11%	0.78%	4.00%	11.57%	3.96%	7.66%	10.32%	5.41%	9.58%	6.88%	5.34%	11.13%
Real Estate	0.00%	0.00%	0.00%	0.13%	0.34%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Utilities	0.00%	2.84%	0.86%	6.69%	11.45%	0.76%	0.00%	0.06%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%

3.3.2. Euro Stoxx 50

In Europe, given the impact of the COVID-19 pandemic, the burden of green energy transition on agriculture, industry, and citizens' bills, along with the energy crisis, the preferred allocation is that of consumer staples (see Table 14). These are the basic goods that people purchase to support their everyday lives. Levenda et al. (2021) demonstrate "that renewable energy transitions can leave the most vulnerable populations in worse-off scenarios, or at the very least, unable to benefit from the opportunity that transitions offer". Heard et al. (2017) affirm that the challenge and delay in identifying and implementing effective and comprehensive decarbonization pathways are underestimated. According to Amenta and Stagnaro (2022), in 2018 alone, the European Union (excluding the United Kingdom) spent EUR 73 billion to subsidize green energy production. These financial aids were covered by European energy consumers, mainly through withdrawals charged on electricity bills. In reviewing the European experience of subsidizing green electricity production, particularly from wind and solar photovoltaic energy, Amenta and Stagnaro (2022) raised three research questions: (1) Was it an effective environmental policy? (2) Was it an effective industrial policy? (3) Was it an effective social policy? The answer to all three questions is no. Regarding the COVID-19 pandemic, for Obst (2023), policymakers responded swiftly to the crisis with loose monetary and expansionary fiscal policies to mitigate its adverse effects and spur economic recovery. The European Central Bank (ECB) implemented programs like PEPP and APP, totaling over EUR 5 trillion, doubling its balance sheet by 2021's end. The European Commission (EC) introduced the EUR 800 billion NGEU recovery package to address EU disparities and fund digital and green projects. However, the predicted V-shaped recovery has not materialized in most EU states, with key indicators such as industrial production still below pre-crisis levels two years after the outbreak. While policy support was crucial, it may have contributed to higher inflation and raised concerns about debt sustainability and potential market distortions after support was withdrawn. Furthermore, while manufacturing remained relatively stable, energy-intensive industries experienced significant downturns. This raises questions about whether losses in the early stages of production will impact other parts of the value chain, leading to potential "cascading effects" as rising energy prices dampen economic outlook (Obst 2023).

Table 14. Sector investment averages of the EuroStoxx50.

GICS Sector	Variance			Conditional Value at Risk			Entropic Value at Risk			Relativistic Value at Risk			MaxLoss		
η	free	1/3	2/3	free	1/3	2/3	free	1/3	2/3	free	1/3	2/3	free	1/3	2/3
Consumer Discretionary	2.49%	10.04%	29.33%	1.88%	14.31%	31.82%	0.20%	16.32%	36.67%	0.00%	15.15%	39.00%	0.00%	13.50%	39.49%
Consumer Staples	39.72%	21.60%	5.65%	46.71%	25.03%	14.42%	52.86%	31.59%	16.42%	54.83%	40.04%	18.95%	57.54%	51.98%	17.88%
Energy	14.16%	22.04%	32.50%	7.48%	13.95%	24.47%	7.51%	16.65%	28.95%	5.67%	17.19%	28.99%	3.07%	20.47%	28.50%
Financials	0.15%	1.23%	0.53%	0.53%	0.00%	0.38%	0.00%	0.00%	0.00%	0.00%	0.40%	0.28%	0.00%	2.99%	3.04%
Health Care	18.29%	15.66%	4.06%	23.40%	23.86%	6.56%	20.36%	18.35%	4.81%	24.57%	13.47%	2.99%	25.82%	6.57%	2.49%
Industrials	21.74%	28.57%	27.93%	17.79%	17.77%	22.36%	5.83%	9.05%	11.99%	0.00%	2.83%	8.29%	0.00%	0.57%	6.53%
Information Technology	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Materials	3.44%	0.85%	0.00%	2.22%	5.08%	0.00%	13.24%	8.04%	1.15%	14.93%	10.91%	1.50%	13.57%	3.92%	2.07%

Table 15. Sector investment averages of the EuroStoxx50 with Bitcoin.

GICS Sector	Variance			Conditional Value at Risk			Entropic Value at Risk			Relativistic Value at Risk			MaxLoss		
η	free	1/3	2/3	free	1/3	2/3	free	1/3	2/3	free	1/3	2/3	free	1/3	2/3
Consumer Discretionary	2.85%	9.49%	28.24%	0.23%	13.88%	31.81%	0.00%	15.27%	36.81%	0.00%	15.68%	38.38%	0.00%	15.55%	40.41%
Consumer Staples	37.31%	21.25%	5.48%	45.64%	24.26%	14.04%	47.36%	27.95%	15.02%	47.21%	33.94%	16.17%	42.73%	39.93%	14.49%
Cripto	4.21%	2.26%	0.76%	4.04%	1.85%	0.73%	4.03%	3.16%	1.20%	4.22%	4.15%	1.65%	6.01%	5.48%	2.28%
Energy	14.02%	21.59%	31.94%	7.35%	13.35%	24.06%	11.95%	20.15%	29.37%	11.76%	22.36%	30.20%	13.05%	27.32%	29.69%
Financials	0.02%	1.24%	0.53%	0.37%	0.00%	0.33%	0.00%	0.00%	0.00%	0.00%	0.02%	0.11%	0.00%	1.35%	1.84%
Health Care	16.45%	15.13%	4.64%	19.47%	22.74%	6.94%	15.44%	15.10%	4.93%	20.20%	10.39%	2.84%	16.52%	5.43%	1.63%
Industrials	21.12%	28.04%	28.41%	18.52%	18.70%	21.80%	7.06%	11.26%	12.30%	0.00%	3.59%	9.88%	0.00%	1.27%	8.47%
Information Technology	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Materials	4.01%	1.00%	0.00%	4.38%	5.22%	0.28%	14.16%	7.11%	0.37%	16.61%	9.87%	0.78%	21.69%	3.67%	1.20%

Similarly to the USA, the introduction of a crypto asset does not alter the landscape of the European stock market, as shown in Table 15.

3.3.3. BIST 100

Turkey is the 19th largest economy in the world, with a GDP of approximately USD 906 billion. Despite ambitious reforms leading to significant growth and poverty reduction between 2006 and 2017, productivity growth has slowed in recent years (Hale 2023; Bank 2023). In 2022, Turkey's economy grew by 5.6%, a decrease from the previous year's 11.4%, with exports, investment, and manufacturing activity slowing down (Bank 2023). However, private consumption remained strong, expanding by 19.6%. Growth was led by the services sector (9.7%) and industry (3.3%). Despite significant forex interventions by the central bank, the Turkish lira depreciated by 30% in 2022 (Bank 2023). Turkey's industrial development significantly influences optimal portfolio allocations towards consumer staples, industrials, and materials sectors, with the latter being predominant in higher-risk portfolios (see Table 16).

Given the inherent risk of investing in the Turkish stock market, the introduction of a crypto asset offers a viable alternative for diversification and potentially enhancement in returns, as indicated in Table 17.

Table 16. Sector investment averages of the Bist100.

GICS Sector	Variance			Conditional Value at Risk			Entropic Value at Risk			Relativistic Value at Risk			MaxLoss		
η	free	1/3	2/3	free	1/3	2/3	free	1/3	2/3	free	1/3	2/3	free	1/3	2/3
Communication Services	0.42%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.42%	0.74%	0.00%	0.75%	1.05%	1.10%
Consumer Discretionary	13.33%	16.55%	11.14%	1.47%	6.62%	5.80%	0.66%	3.15%	0.00%	0.00%	2.25%	0.00%	2.01%	3.05%	0.00%
Consumer Staples	43.05%	22.26%	5.04%	44.93%	23.16%	2.28%	31.14%	21.71%	6.99%	23.53%	16.57%	4.85%	19.21%	12.41%	2.86%
Energy	0.00%	2.10%	0.42%	0.00%	0.00%	0.00%	0.00%	0.39%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Financials	9.02%	6.95%	2.75%	27.07%	13.88%	5.87%	15.90%	4.96%	0.00%	13.14%	1.06%	0.00%	0.00%	0.03%	0.00%
Industrials	9.63%	21.93%	46.53%	10.11%	27.90%	59.02%	5.13%	27.81%	51.84%	2.68%	31.34%	48.19%	5.64%	31.53%	42.71%
Materials	19.49%	23.69%	26.48%	15.97%	27.17%	24.62%	38.31%	41.61%	35.38%	48.52%	48.06%	42.12%	49.32%	51.88%	47.60%
Real Estate	0.30%	1.83%	3.44%	0.04%	0.65%	0.32%	0.00%	0.00%	4.16%	0.00%	0.00%	4.84%	0.00%	0.05%	5.73%
Utilities	4.75%	4.69%	4.20%	0.41%	0.61%	2.08%	8.87%	0.38%	1.63%	11.71%	0.00%	0.00%	23.08%	0.00%	0.00%

Table 17. Sector investment averages of the Bist100 with Bitcoin.

GICS Sector	Variance			Conditional Value at Risk			Entropic Value at Risk			Relativistic Value at Risk			MaxLoss		
η	free	1/3	2/3	free	1/3	2/3	free	1/3	2/3	free	1/3	2/3	free	1/3	2/3
Communication Services	0.45%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Consumer Discretionary	10.70%	14.85%	12.36%	9.14%	12.92%	4.61%	11.65%	8.02%	0.00%	15.23%	9.07%	0.00%	18.89%	5.99%	0.58%
Consumer Staples	35.74%	15.85%	6.35%	36.37%	8.55%	2.61%	18.74%	7.47%	5.15%	6.10%	6.42%	5.35%	2.74%	5.96%	2.98%
Cripto	17.73%	11.74%	1.42%	23.06%	16.45%	5.18%	28.37%	21.55%	8.04%	29.83%	21.90%	8.11%	31.61%	22.01%	8.93%
Energy	0.00%	2.07%	0.76%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Financials	9.69%	6.56%	3.55%	9.81%	8.19%	3.15%	15.91%	0.90%	0.00%	20.16%	0.05%	0.00%	21.12%	0.00%	0.69%
Industrials	7.23%	23.13%	43.08%	5.50%	32.62%	60.50%	5.88%	32.86%	50.52%	6.88%	27.45%	45.86%	7.37%	27.16%	39.46%
Materials	16.43%	20.88%	24.76%	15.86%	20.94%	21.92%	19.44%	28.40%	30.74%	21.80%	35.11%	36.62%	18.27%	38.88%	42.19%
Real Estate	0.19%	1.63%	3.32%	0.00%	0.00%	0.20%	0.00%	0.36%	3.64%	0.00%	0.00%	3.16%	0.00%	0.00%	5.03%
Utilities	1.83%	3.30%	4.41%	0.26%	0.33%	1.82%	0.00%	0.43%	1.91%	0.00%	0.00%	0.90%	0.00%	0.00%	0.14%

3.3.4. Bovespa

Brazil boasts abundant human and natural resources, including a diverse population and a vast territory with the world's largest rainforest and ample fresh water. It ranks as the ninth largest economy globally, with a GDP of around USD 500 billion. However, income inequality is pronounced, with the wealthiest 1 percent of the population receiving 10 percent of the country's income (Furtado 2004). As documented by Amann and Barrientos (2016), from the mid-1990s to the early 2010s, Brazil rose as a prominent figure among developing nations, gaining influence in both geopolitical and economic spheres. Its appeal stemmed from various factors, including productive agriculture, low reliance on non-renewable resources, a robust National Development Bank, stable macroeconomic policies, a high tax-to-GDP ratio supporting development, and innovative social programs like Bolsa Família. A notable aspect of Brazil's economic growth during this period was its emphasis on social inclusion.

Despite modest average annual growth rates of around 3%, Brazil achieved substantial reductions in poverty and inequality. The Gini coefficient, a measure of income inequality, notably declined from 0.6 to 0.53 between 1995 and 2011. This reduction was driven by significant income growth among lower-income groups compared to those at the top end of the income distribution. Brazil also demonstrated success in controlling inflation, which had previously risen to four digits in the early 1990s but dropped below 10% by the mid-1990s and remained relatively low thereafter. Additionally, the country achieved trade surpluses, fueled by increasing demand for key export commodities. This economic success has facilitated the accumulation of substantial international reserves, supported by an increase in foreign direct investment attracted by trade liberalization, privatization initiatives, and a growing domestic market. Since 2011, Brazil has faced a significant economic downturn, with GDP growth dropping sharply from 3.9% in 2011 to just 0.1% in 2014, followed by a contraction of 3.8% in 2015. This decline has been accompanied by high inflation, reaching 9.01% in 2015, leading authorities to reverse fiscal policies and raise interest rates to 14.25% by mid-2016, hampering economic recovery efforts (Amann and Barrientos 2016). This downturn challenges Brazil's previous reputation for 'growth with redistribution', as its historical growth patterns have been marked by volatile cycles of rapid growth followed by steep declines, earning the nickname "vôo de galhina" or "flight of the chicken". Moreover, Brazil's reliance on agricultural and mineral exports, once a strength during commodity booms, now poses a risk due to potential shifts in global demand trends (Amann and Barrientos 2016). For example, Nassif et al. (2015) used the Kaldor–Thirlwall model to analyze Brazil's productivity and economic development from 1970 to 2010 and found evidence of early deindustrialization and significant changes in the elasticity of demand for imports and exports at income during specific periods. To address its challenges, Brazil shifted away from neoliberalism and adopted a state-permeated capitalist approach (Nölke et al. 2014), aligning itself with the BRICs (Brazil, Russia, India, and China) (O'Neill 2001). This marked a significant long-term change in global economic power dynamics (Nölke et al. 2014). Therefore, the optimal portfolios for Bovespa allocate weights to utilities, materials, and energy (see Table 18). In particular, materials dominate in low-risk portfolios, while energy takes precedence in high-risk portfolios, effectively balancing each other out.

Table 18. Sector investment averages of the Bovespa.

GICS Sector	Variance			Conditional Value at Risk			Entropic Value at Risk			Relativistic Value at Risk			MaxLoss		
η	free	1/3	2/3	free	1/3	2/3	free	1/3	2/3	free	1/3	2/3	free	1/3	2/3
Communication Services	10.67%	8.14%	1.52%	10.87%	5.57%	0.00%	14.99%	5.19%	0.00%	14.39%	7.45%	0.00%	13.34%	6.82%	0.00%
Consumer Discretionary	0.01%	0.00%	0.00%	0.01%	0.00%	0.00%	0.00%	0.00%	0.00%	0.19%	0.00%	0.00%	0.16%	0.00%	0.00%
Consumer Staples	14.84%	6.77%	6.11%	9.82%	4.23%	8.01%	7.29%	4.30%	9.70%	7.93%	4.54%	10.64%	8.20%	5.29%	11.17%
Energy	0.67%	14.99%	41.68%	0.28%	11.54%	35.23%	0.00%	14.13%	41.77%	0.00%	13.97%	42.62%	0.00%	15.23%	43.51%
Financials	6.65%	19.66%	30.32%	2.50%	21.91%	28.71%	6.54%	16.79%	21.06%	8.38%	14.58%	18.05%	7.19%	12.30%	15.52%
Health Care	0.06%	1.45%	0.00%	0.05%	0.67%	0.01%	0.13%	0.87%	0.00%	0.00%	0.83%	0.00%	1.15%	0.04%	0.00%
Industrials	0.39%	1.19%	2.27%	4.79%	2.66%	2.24%	9.95%	4.35%	0.82%	10.86%	3.17%	0.83%	11.05%	1.92%	0.51%
Information Technology	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Materials	19.58%	10.52%	0.76%	23.70%	18.89%	1.36%	27.30%	20.51%	1.27%	28.03%	22.13%	1.68%	28.27%	22.92%	2.92%
Real Estate	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Utilities	47.12%	37.28%	17.33%	47.99%	34.52%	24.44%	33.81%	33.87%	25.38%	30.22%	33.34%	26.19%	30.62%	35.49%	26.37%

Table 19 illustrates a slight variation in allocation when considering crypto assets. This observation is consistent with earlier findings regarding the USA and European stock markets.

Table 19. Sector investment averages of the Bovespa with Bitcoin.

GICS Sector	Variance			Conditional Value at Risk			Entropic Value at Risk			Relativistic Value at Risk			MaxLoss		
η	free	1/3	2/3	free	1/3	2/3	free	1/3	2/3	free	1/3	2/3	free	1/3	2/3
Communication Services	9.27%	6.44%	1.32%	11.67%	6.03%	0.00%	16.17%	6.83%	0.00%	14.96%	9.15%	0.00%	12.98%	8.66%	0.00%
Consumer Discretionary	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.25%	0.00%	0.00%	0.47%	0.00%	0.00%
Consumer Staples	14.50%	6.92%	6.11%	9.72%	3.46%	7.89%	9.13%	4.55%	9.76%	10.78%	5.61%	10.83%	12.28%	7.20%	11.57%
Cripto	4.80%	3.84%	0.53%	1.80%	1.36%	0.00%	3.07%	2.48%	0.95%	3.27%	2.80%	1.18%	3.47%	3.14%	2.04%
Energy	1.10%	15.25%	41.61%	0.21%	11.35%	34.80%	0.17%	13.94%	41.81%	0.25%	13.59%	42.86%	0.38%	13.87%	42.79%
Financials	5.88%	19.34%	29.98%	3.10%	21.88%	28.96%	6.15%	17.50%	21.29%	8.07%	16.20%	18.89%	8.03%	15.36%	18.35%
Health Care	0.12%	1.47%	0.00%	0.00%	1.19%	0.00%	1.81%	1.52%	0.00%	2.26%	1.47%	0.00%	3.04%	0.16%	0.00%
Industrials	0.24%	1.13%	2.26%	4.80%	2.75%	2.13%	9.31%	4.48%	0.74%	10.59%	4.30%	0.69%	11.04%	4.84%	0.79%
Information Technology	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Materials	19.26%	10.14%	0.79%	23.57%	18.92%	1.35%	25.45%	19.40%	1.29%	25.14%	20.02%	1.89%	24.79%	20.56%	3.86%
Real Estate	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Utilities	44.85%	35.48%	17.40%	45.14%	33.06%	24.87%	28.74%	29.29%	24.18%	24.42%	26.87%	23.67%	23.52%	26.22%	20.60%

4. Conclusions

At the forefront of concern is the prevailing dominance of a handful of major corporations within the stock market landscape, prompting an in-depth exploration of the potential risks associated with such concentration. Relative entropy, a distance measure, has recently found broad application in portfolio selection because it aids in understanding diversification strategies. An intriguing and newly proposed risk measure, entropic value at risk (EVaR), possesses a dual representation that underlines its connection to relative entropy. EVaR boasts several advantages: it is coherent, strongly monotone, and convex, and serves as an upper bound to conditional value at risk (CVaR), which, in turn, is an upper bound to value at risk (VaR). Additionally, a generalization of EVaR has emerged in the form of relativistic value at risk (RLVaR), a coherent risk measure rooted in Kaniadakis entropy, which surpasses EVaR.

The objective of our work is threefold: firstly, to assess whether entropy-based criteria outperform other models; secondly, to investigate the behavior of the considered models in developed markets compared to emerging ones; and thirdly, to analyze the impact of cryptocurrency introduction on portfolio performance and diversification. Through extensive empirical analysis, we evaluated several minimum-risk models using real-world datasets comprising stock indexes such as the S&P 500 for the USA, Euro Stoxx 50 for Europe, BIST 100 for Turkey, and Bovespa for Brazil, along with Bitcoin. Our objective was threefold: first, to assess whether entropy-based criteria outperform other models; second, to examine the behavior of these models in developed versus emerging markets; and third, to analyze the impact of introducing cryptocurrency on portfolio performance and diversification. The results indicate that entropy measures help identify optimal portfolios, particularly when risk levels are elevated and portfolios become more concentrated (Shalett 2023). This becomes particularly crucial when returns are low and/or turnover is high, leading to negative net performances. Bitcoin, due to its unfavorable risk profile, is chosen for diversification and performance enhancement only in the case of the BIST 100. In other cases, either the optimal weight is zero (USA market) or is small. This validates the findings of Gambarelli et al. (2023), indicating that incorporating a single cryptocurrency into a stock portfolio fails to offer adequate hedging during market downturns and may escalate the risk of short-term joint losses. Finally, we confirm the extreme concentration of stock markets, where a few leading stocks dominate all others (Rekenthaler 2020, Phillips 2024).

A limitation of this analysis stems from the specific selection of stock markets (e.g., developed versus developing, large-cap versus small-cap stock indices, etc.), the sample period of the time series under consideration, and the frequency of the data. For instance, Orlando and Bufalo (2021) have shown a close connection between the sampling process and the distribution of returns. On the same line, Chiang and Doong (2001), in their analysis of the relationship between stock returns and time-varying volatility using a threshold autoregressive GARCH(1,1)-in-mean specification, find that while the null hypothesis of no asymmetric effect on conditional volatility is rejected for daily data, it cannot be rejected for monthly data. Another example of varying results based on data is related to the random walk hypothesis (RWH). While Fama (1970) found no evidence of patterns in stock prices when testing the RWH, Hong (1978), Cooper (1983), and Laurence (1986) provided support for it. Additionally, within the same stock market, the Athens Stock Exchange (ASE), Dockery et al. (2001) found overwhelming support for the acceptance of the random walk hypothesis using monthly data, indicating weak-form efficiency in the ASE. This finding contradicts previous studies on the ASE by Niarchos and Georgakopoulos (1986), who analyzed the influence of annual corporate profit reports on the ASE. Panagiotidis (2005), when analyzing three ASE indices, the FTSE/ASE20, the FTSE/ASE Mid 40, and the FTSE/ASE Small Cap, found strong evidence against the random walk hypothesis. Moreover, they discovered that the lower capitalization fraction of the market is more 'efficient', as past volatility does not assist in predicting future returns. That leads to varying results based on the consideration of large- and small-capitalization stock indices. In fact, as demonstrated by Hung et al. (2009), the weak-form efficient market hypothesis (EMH) for both large- and small-capitalization stock indices of TOPIX and FTSE reveals support for the EMH in large-cap stocks but rejection in small-cap ones. Similar results can be found in Varamini and Kalash (2008) for mutual funds, in Al-Khazali et al. (2016) for Islamic stock indices, in Petajisto (2017) for exchange-traded funds (ETFs), etc.

Future research could focus on analyzing the performance of entropy-based criteria in portfolio selection across a wider range of assets, including commodities, small caps, and ETFs. Additionally, investigating risk–return performances based on market momentum, such as during periods of decline and growth, would be valuable.

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Appendix A. Legend and Abbreviations

The following list provides a legend of abbreviations used in the article. Note that the holding period is 20 days.

- Mean = Expected annualized return
- NP = Annualized net performance
- SD = Standard deviation (daily)
- Mdd = Maximum drawdown (over the holding period)
- Ulcer = Ulcer index (over the holding period)
- Sharpe = Sharpe ratio (over the holding period)
- Sortino = Sortino ratio (over the holding period)
- Omega = Omega ratio (over the holding period)
- $\text{Rachev}_{5\%}$ = 5% Rachev ratio (over the holding period)
- $\text{VaR}_{5\%}$ = 5% Value at risk (over the holding period)
- E[HI] = Expected Herfindahl index (over the holding period)
- Turnover = Annualized turnover
- % stocks = Mean assets in portfolio (over the holding period)

Notes

[1] The New Era corresponds to the "infamously speculative" first 20 years of the third millennium, during which "so many stock-valuation charts behaved anomalously" (Shalett 2023). This period represents one of the greatest bull markets in history, reminiscent of the Roaring Twenties of the 20th century (Shiller 2021; Terzi 2021; Turner 2021).

[2] Transaction costs add complexity to portfolio management, as investors endeavor to minimize these costs while avoiding unwanted risks. Thus, any rebalancing strategy must strike a balance between reducing transaction costs and minimizing tracking error (Donohue and Yip 2003). Twenty observations correspond to monthly rebalancing, which seems to be the most practical option considering transaction costs and the necessity of having a time window for validation that is adequately tested for two years. For instance, Almadi et al. (2014) argue that with respect to the transaction-cost/rebalancing frontier, monthly rebalancing offers the most significant outperformance when unit transaction costs are below approximately 50 basis points, while dynamic portfolios based on annual rebalancing generally outperform benchmarks for unit transaction costs well above 400 basis points. Rattray et al. (2020) suggest that investors can also use monthly inflows and outflows to move back toward the target asset mix, and Chambers et al. (2021) documented that the Norwegian Government Pension Fund Global allocates monthly inflows to the asset class that exhibits the greatest underweighting compared to the benchmark.

[3] GDP in purchasing power parity (PPP) for the USA was USD 3,722,590 and for Europe USD 5,233,350 in 2017 dollars (Central Intelligence Agency 2024). As reported by the OECD (2023), between 2000 and 2010, USA manufacturing faced significant challenges. The number of manufacturing jobs, which had remained relatively stable at 17 million since 1965, plummeted by one third during that decade, dropping by 5.8 million to below 12 million in 2010 (rising only slightly to 12.3 million in 2016). While the 2007–2008 recession accelerated this decline, structural issues, not just financial ones, were also at play, including problems with capital investment, output, productivity, and trade deficits. Contrary to popular belief, the decline in manufacturing employment was not primarily due to productivity gains from robotics or automation; instead, the sector experienced a process of hollowing out. The USA advanced manufacturing effort stands out for its diverse range of technologies targeted by various institutes, contrasting with single-focus efforts seen in some countries. Rather than a singular focus, the USA approach encompasses materials, digital, bio, and nano technologies. An essential challenge lies in integrating these diverse strands into a cohesive system, with the future factory envisioned as a merging and connecting of various technologies. The formation of a network called ManufacturingUSA aims to address this challenge by bringing together the institutes' advanced technology strands into a new production system, marking a critical task for this innovative model to realize its potential.

[4] The enactment of the USA IRA in 2022 sparked concerns among trading partners regarding the perceived discriminatory nature of the law's domestic content requirements (Attinasi et al. (2024)). The war in Ukraine caused energy shortages and disruptions, particularly due to the loss of Russian oil and gas supplies that led to spikes in European energy prices (Knightley and Zhang 2024). Moreover, the conflict has led to a surge in orders for weapons and ammunition from European allies and the Pentagon, benefiting the USA defense sector. Federal Reserve data shows a 17.5% increase in industrial production in the USA defense and space sectors since the war began, with a significant portion of funding designated for Ukraine expected to return to the USA defense industrial base. Analysts describe this uptick in spending on USA military equipment as a "generational investment" (Fairless 2024). Middle East tensions disrupt trade, but they pose less of a threat to USA manufacturers compared to their European counterparts. The USA engages in relatively modest trade directly with the Middle East and North African nations, amounting to little more than $200 billion in total trade flows. Moreover, most USA trade with Asia occurs via the Pacific, mitigating the disruption faced by Europe in its trade with Asia through the Red Sea. Although European trade is increasingly redirected around the Horn of Africa rather than through the Suez Canal, resulting in longer shipping voyages and increased chartering costs, USA companies with a full supply chain in the USA or the Americas experience fewer issues (Knightley and Zhang 2024).

References

Acerbi, Carlo. 2003. Risk aversion and coherent risk measures: A spectral representation theorem. *Metodi statistici per la finanza e le assicurazioni* 1000–19. [CrossRef]

Acerbi, Carlo, and Dirk Tasche. 2002. Expected Shortfall: A Natural Coherent Alternative to Value at Risk. *Economic Notes* 31: 379–88. [CrossRef]

Ahmadi, Farid, Mohammad Pourmahmood Aghababa, and Hashem Kalbkhani. 2022. Identification of Chaos in Financial Time Series to Forecast Nonperforming Loan. *Mathematical Problems in Engineering* 2022: 2055655. [CrossRef]

Ahmadi-Javid, Amir. 2012. Entropic value-at-risk: A new coherent risk measure. *Journal of Optimization Theory and Applications* 155: 1105–23. [CrossRef]

Ahmadi-Javid, Amir, and Malihe Fallah-Tafti. 2019. Portfolio optimization with entropic value-at-risk. *European Journal of Operational Research* 279: 225–41. [CrossRef]

Al-Khazali, Osamah M., Guillaume Leduc, and Mohammad Saleh Alsayed. 2016. A Market Efficiency Comparison of Islamic and Non-Islamic Stock Indices. *Emerging Markets Finance and Trade* 52: 1587–605. [CrossRef]

Ali, Syed Mumtaz, and Samuel D Silvey. 1966. A general class of coefficients of divergence of one distribution from another. *Journal of the Royal Statistical Society: Series B (Methodological)* 28: 131–42. [CrossRef]

Aljalal, Majid, Saeed A. Aldosari, Marta Molinas, Khalil AlSharabi, and Fahd A. Alturki. 2022. Detection of Parkinson's disease from EEG signals using discrete wavelet transform, different entropy measures, and machine learning techniques. *Scientific Reports* 12: 22547. [CrossRef] [PubMed]

Almadi, Himanshu, David E. Rapach, and Anil Suri. 2014. Return predictability and dynamic asset allocation: How often should investors rebalance? *Journal of Portfolio Management* 40: 16. [CrossRef]

Amann, Edmund, and Armando Barrientos. 2016. Introduction—Is there a Brazilian development 'model'? *Quarterly Review of Economics and Finance* 62: 7–11. [CrossRef]

Amenta, Carlo, and Carlo Stagnaro. 2022. The Failures of the Entrepreneurial State: Subsidies to Renewable Energies in Europe. In *Questioning the Entrepreneurial State*. Cham: Springer, pp. 233–50. [CrossRef]

Ardakani, Omid M. 2023. Capturing information in extreme events. *Economics Letters* 231: 111301. [CrossRef]

Artzner, Philippe, Freddy Delbaen, Jean-Marc Eber, and David Heath. 1999. Coherent measures of risk. *Mathematical Finance* 9: 203–28. [CrossRef]

Attinasi, Maria Grazia, Lukas Boeckelmann, and Baptiste Meunier. 2024. Unfriendly friends: Trade and relocation effects of the US Inflation Reduction Act. Available online: https://cepr.org/voxeu/columns/unfriendly-friends-trade-and-relocation-effects-us-inflation-reduction-act (accessed on 28 April 2024).

Bruni, Renato, Francesco Cesarone, Andrea Scozzari, and Fabio Tardella. 2017. On exact and approximate stochastic dominance strategies for portfolio selection. *European Journal of Operational Research* 259: 322–29. [CrossRef]

Bufalo, Tatiana Cardoso, R. Bufalo, Lucas P. G. de Figueiredo, Qiuping A. Wang, and Fabio Lucio Alves. 2023. Assessing black swan events with the stochastic least action principle, Tsallis entropy and heavy-tailed distribution. *European Physical Journal Plus* 138: 282. [CrossRef]

Caddle, Peter. 2023. Two in Three German Companies Relocate Abroad Amid Energy Chaos. Available online: https://brusselssignal.eu/2023/11/two-in-three-german-companies-relocate-abroad-amid-energy-chaos (accessed on 22 March 2024).

Cajas, Dany. 2021. Entropic Portfolio Optimization: A Disciplined Convex Programming Framework. SSRN 3792520, pp. 1–22. Available online: https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3792520 (accessed on 1 March 2024).

Cajas, Dany. 2023. Portfolio Optimization of Relativistic Value at Risk. SSRN 4378498, pp. 1–29. Available online: https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4378498 (accessed on 1 March 2024).

Calixto, Manuel, Alberto Mayorgas, and Julio Guerrero. 2021. Entanglement and U (D)-spin squeezing in symmetric multi-quDit systems and applications to quantum phase transitions in Lipkin–Meshkov–Glick D-level atom models. *Quantum Information Processing* 20: 304. [CrossRef]

Carleo, Alessandra, Francesco Cesarone, Andrea Gheno, and Jacopo Maria Ricci. 2017. Approximating exact expected utility via portfolio efficient frontiers. *Decisions in Economics and Finance* 40: 115–43. [CrossRef]

Central Intelligence Agency. 2024. GDP—Composition, by Sector of Origin. Available online: https://web.archive.org/web/20190110065235/https://www.cia.gov/library/publications/resources/the-world-factbook/fields/214.html (accessed on 28 April 2024).

Cesarone, Francesco, Raffaello Cesetti, Giuseppe Orlando, Manuel Luis Martino, and Jacopo Maria Ricci. 2022. Comparing SSD-Efficient Portfolios with a Skewed Reference Distribution. *Mathematics* 11: 50. [CrossRef]

Chambers, David, Elroy Dimson, and Antti Ilmanen. 2021. The Norway model in perspective. *The Journal of Portfolio Management* 47: 178–87. [CrossRef]

Chares, Robert. 2009. Cones and Interior-Point Algorithms for Structured Convex Optimization Involving Powers and Exponentials. Available online: https://perso.uclouvain.be/francois.glineur/files/theses/Chares-PhD-thesis-2007.pdf (accessed on 28 December 2023).

Chen, Haiwei, James Estes, and William Pratt. 2018. Investing in the healthcare sector: Mutual funds or ETFs. *Managerial Finance* 44: 495–508. [CrossRef]

Chen, Mei-Ping, Yu-Hui Lin, Chun-Yao Tseng, and Wen-Yi Chen. 2015. Bubbles in health care: Evidence from the U.S., U.K., and German stock markets. *North American Journal of Economics and Finance* 31: 193–05. [CrossRef]

Chernoff, Herman. 1952. A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations. *The Annals of Mathematical Statistics* 23: 493–507. [CrossRef]

Chiang, Thomas C., and Shuh-Chyi Doong. 2001. Empirical Analysis of Stock Returns and Volatility: Evidence from Seven Asian Stock Markets Based on TAR-GARCH Model. *Review of Quantitative Finance and Accounting* 17: 301–18. [CrossRef]

CoinMarketCap. 2024. Cryptocurrency Prices, Charts And Market Capitalizations. Available online: https://coinmarketcap.com (accessed on 11 March 2024).

Cooper, JC. 1983. The Korean stock exchange: A qualitative and quantitative assessment. *The Investment Analyst* 70: 5–12.

Corbet, Shaen, Hou, Yang (Greg), Hu, Yang, Larkin, Charles and Oxley, Les. 2020. Any port in a storm: Cryptocurrency safe-havens during the COVID-19 pandemic. *Economics Letters* 194: 1–7. [CrossRef]

Cremers, K. J. Martijn, and Jianping Mei. 2007. Turning over Turnover. *Review of Financial Studies* 20: 1749–82. [CrossRef]

Csiszar, Imre. 1963. Eine information's theoretische ungleichung und ihre anwendung auf den beweis der ergodizitat von markoschen ketten. *Publications of the Mathematical Institute of the Hungarian Academy of Sciences* 8: 8–108.

Dey, Malay K. 2005. Turnover and return in global stock markets. *Emerging Markets Review* 6: 45–67. [CrossRef]

Dockery, Everton, D. Vergari, and F. Vergari. 2001. Explaining the behaviour of stock prices in an emerging market: An empirical analysis of the Greek stock market. *Managerial Finance* 27: 82–98. [CrossRef]

Dommel, Paul, and Alois Pichler. 2020. Convex risk measures based on divergence. *arXiv* arXiv:2003.07648.

Donohue, Christopher, and Kenneth Yip. 2003. Optimal portfolio rebalancing with transaction costs. *Journal of Portfolio Management* 29: 49. [CrossRef]

Fairless, Tom. 2024. How War in Europe Boosts the U.S. Economy. *Wall Street Journal*. Available online: https://www.wsj.com/economy/ukraine-war-europe-american-economy-654ca41b (accessed on 28 April 2024).

Fama, Eugene F. 1970. Efficient capital markets: A review of theory and empirical work. *Journal of Finance* 25: 383–417. [CrossRef]

Feng, Wenjun, Wang, Yiming, and Zhang, Zhengjun. 2018. Can cryptocurrencies be a safe haven: a tail risk perspective analysis. *Applied Economics* 50: 4745–62. https://doi/abs/10.1080/00036846.2018.1466993. [CrossRef]

Furtado, Angela. 2004. The World Bank and Brazil: A New Country Assistance Strategy. Available online: https://openknowledge.worldbank.org/entities/publication/291a993c-d944-5604-a559-ac3a85c66c5c (accessed on 22 March 2024).

Gambarelli, Luca, Marchi, Gianluca, and Muzzioli, Silvia. 2023. Hedging effectiveness of cryptocurrencies in the European stock market. *Journal of International Financial Markets, Institutions and Money* 84: 101757–79. [CrossRef]

Gray, Robert M. 2011. *Entropy and Information Theory*. New York: Springer Science & Business Media.

Hale, William. 2023. *The Political and Economic Development of Modern Turkey*. Andover: Taylor & Francis. [CrossRef]

Heard, B. P., B. W. Brook, T. M. L. Wigley, and C. J. A. Bradshaw. 2017. Burden of proof: A comprehensive review of the feasibility of 100% renewable-electricity systems. *Renewable and Sustainable Energy Reviews* 76: 1122–33. [CrossRef]

Holzinger, Andreas, Matthias Hörtenhuber, Christopher Mayer, Martin Bachler, Siegfried Wassertheurer, Armando J. Pinho, and David Koslicki. 2014. On Entropy-Based Data Mining. In *Interactive Knowledge Discovery and Data Mining in Biomedical Informatics*. Berlin: Springer, pp. 209–26. [CrossRef]

Hong, Hai. 1978. The random walk in stock markets: Theory and evidence. *Securities Industry Review* 4: 25–29.

Hung, Jui-Cheng, Yen-Hsien Lee, and Tung-Yueh Pai. 2009. Examining market efficiency for large- and small-capitalization of TOPIX and FTSE stock indices. *Applied Financial Economics* 19: 735–44. [CrossRef]

Jones, Charles M. 2002. A Century of Stock Market Liquidity and Trading Costs. Available online: https://papers.ssrn.com/sol3/papers.cfm?abstract_id=313681 (accessed on 23 March 2024).

Jorion, Philippe. 2009. Risk Management Lessons from the Credit Crisis. *European Financial Management* 15: 923–33. [CrossRef]

Kaniadakis, G. 2001. Non-linear kinetics underlying generalized statistics. *Physica A: Statistical Mechanics and Its Applications* 296: 405–25. [CrossRef]

Kaniadakis, Giorgio. 2002. Statistical mechanics in the context of special relativity. *Physical Review E* 66: 056125. [CrossRef]

Kaniadakis, G. 2006. Towards a relativistic statistical theory. *Physica A: Statistical Mechanics and Its Applications* 365: 17–23. [CrossRef]

Knightley, James, and Coco Zhang. 2024. *U.S. Manufacturing Outlook: Better Times Are Coming*. New York: Seeking Alpha. Available online: https://seekingalpha.com/article/4677549-us-manufacturing-outlook-better-times-coming (accessed on 23 March 2024).

Koç, Çetin Kaya, and Funda Özdemir. 2023. Development of Cryptography since Shannon. In *Handbook of Formal Analysis and Verification in Cryptography*. Boca Raton: CRC Press, pp. 1–56. [CrossRef]

Kovalev, Andrey V. 2016. Misuse of thermodynamic entropy in economics. *Energy* 100: 129–36. [CrossRef]

Kullback, Solomon, and Richard A Leibler. 1951. On information and sufficiency. *The Annals of Mathematical Statistics* 22: 79–86. [CrossRef]

Laurence, Martin M. 1986. Weak-form efficiency in the kuala lumpur and singapore stock markets. *Journal of Banking & Finance* 10: 431–45.

Leung-Yan-Cheong, S., and T. Cover. 1978. Some equivalences between Shannon entropy and Kolmogorov complexity. *IEEE Transactions on Information Theory* 24: 331–38. [CrossRef]

Levenda, A. M., I. Behrsin, and F. Disano. 2021. Renewable energy for whom? A global systematic review of the environmental justice implications of renewable energy technologies. *Energy Research & Social Science* 71: 101837. [CrossRef]

Markowitz, Harry. 1952. Portfolio selection. *The Journal of Finance* 7: 71–91.

Morimoto, Tetsuzo. 1963. Markov processes and the h-theorem. *Journal of the Physical Society of Japan* 18: 328–31. [CrossRef]

Nassif, André, Carmem Feijó, and Eliane Araújo. 2015. Structural change and economic development: Is Brazil catching up or falling behind? *Cambridge Journal of Economics* 39: 1307–32. [CrossRef]

Niarchos, Nikitas A., and Maxmilianos C. Georgakopoulos. 1986. The effect of annual corporate profit reports on the athens stock exchange: An empirical investigation. *Management International Review* 26: 64–72.

Nölke, Andreas, Tobias Ten Brink, Simone Claar, and Christian May. 2014. Domestic structures, foreign economic policies and global economic order: Implications from the rise of large emerging economies. *European Journal of International Relations* 21: 538–67. [CrossRef]

Obst, Thomas. 2023. Europe on the brink of recession? economic challenges and dangers of deindustrialization. In *Economic Policy in an Unstable Environment*. Edited by Katarzyna Kamińska, Andreas Bielig, and Jürgen Wandel. SGH Publishing House: Warsaw, Poland.

OECD. 2023. US Manufacturing Decline and the Rise of New Production Innovation Paradigms. Available online: https://www.oecd.org/unitedstates/us-manufacturing-decline-and-the-rise-of-new-production-innovation-paradigms.htm (accessed on 28 April 2024).

O'Neill, Jim. 2001. Building Better Global Economic BRICs. *Global Economics Paper* 66: 1–16. Available online: https://www.goldmansachs.com/intelligence/archive/archive-pdfs/build-better-brics.pdf (accessed on 12 March 2024).

Orlando, Giuseppe. 2024. Exploring Crypto & Blockchain in Finance: Present & Future. Available online: https://papers.ssrn.com/sol3/papers.cfm?abstract_id=4787238 (accessed on 9 April 2024).

Orlando, Giuseppe, and Giovanna Zimatore. 2020. Business cycle modeling between financial crises and black swans: Ornstein–Uhlenbeck stochastic process vs Kaldor deterministic chaotic model. *Chaos: An Interdisciplinary Journal of Nonlinear Science* 30: 083129. [CrossRef]

Orlando, Giuseppe, and Michele Bufalo. 2021. Empirical Evidences on the Interconnectedness between Sampling and Asset Returns' Distributions. *Risks* 9: 88. [CrossRef]

Orlando, Giuseppe, and Marek Lampart. 2023. Expecting the Unexpected: Entropy and Multifractal Systems in Finance. *Entropy* 25: 1527. [CrossRef]

Orlando, Giuseppe, Michele Bufalo, Henry Penikas, and Concetta Zurlo. 2021. *Modern Financial Engineering: Counterparty, Credit, Portfolio and Systemic Risks*. Singapore: World Scientific Publishing Company. [CrossRef]

Panagiotidis, Theodore. 2005. Market capitalization and efficiency. Does it matter? Evidence from the Athens Stock Exchange. *Applied Financial Economics* 15: 707–13. [CrossRef]

Petajisto, Antti. 2017. Inefficiencies in the Pricing of Exchange-Traded Funds. *Financial Analysts Journal* 73: 24–54. [CrossRef]

Philippatos, George C., and Charles J. Wilson. 1972. Entropy, market risk, and the selection of efficient portfolios. *Applied Economics* 4: 209–20. [CrossRef]

Philippatos, George C., and Nicolas Gressis. 1975. Conditions of Equivalence Among E-V, SSD, and E-H Portfolio Selection Criteria: The Case for Uniform, Normal and Lognormal Distributions. *Management Science* 21: 617–25. https://pubsonline.informs.org/doi/abs/10.1287/mnsc.21.6.617. [CrossRef]

Phillips, Matt. 2024. The S&P's top stocks haven't been this concentrated in 50 years. Available online: https://www.axios.com/2024/02/05/sp500-magnificent-seven-stock-market (accessed on 6 April 2024).

Pichler, Alois, and Ruben Schlotter. 2020. Entropy based risk measures. *European Journal of Operational Research* 285: 223–36. [CrossRef]

Raghupathi, Viju, and Wullianallur Raghupathi. 2020. Healthcare Expenditure and Economic Performance: Insights >From the United States Data. *Frontiers in Public Health* 8: 538294. [CrossRef] [PubMed]

Rattray, Sandy, Nick Granger, Campbell R. Harvey, and Otto Van Hemert. 2020. Strategic rebalancing. *Journal of Portfolio Management* 46: 10–31. [CrossRef]

Rekenthaler, John. 2020. The S&P 500 Grows Ever More Concentrated. Available online: https://www.morningstar.com/stocks/sp-500-grows-ever-more-concentrated (accessed on 11 March 2024).

Rockafellar, R. Tyrrell, and Stanislav Uryasev. 2000. Optimization of conditional value-at-risk. *Journal of Risk* 2: 21–41. [CrossRef]

Roman, Diana, Gautam Mitra, and Victor Zverovich. 2013. Enhanced indexation based on second-order stochastic dominance. *European Journal of Operational Research* 228: 273–81. [CrossRef]

Shalett, Lisa. 2023. Why Investing in the S&P 500 Isn't True Diversification. Available online: https://www.morganstanley.com/ideas/concentration-risk-high-s-and-p-500-q2-2023 (accessed on 11 March 2024).

Shannon, Claude Elwood. 1948. A mathematical theory of communication. *The Bell System Technical Journal* 27: 379–23. [CrossRef]

Shiller, Robert J. 2021. Looking Back at the First Roaring Twenties. *New York Times*, 16 April. Available online: https://www.nytimes.com/2021/04/16/business/roaring-twenties-stocks.html (accessed on 11 March 2024).

Sun, Lily. 2020. Analysis of stock market reactions to COVID-19 pandemic. Paper presented at the International Conference on Industrial Engineering and Operations Management, Detroit, MI, USA, August 9–11. pp. 526–33.

Taleb, Nassim Nicholas. 2007. Black Swans and the Domains of Statistics. *American Statistician* 61: 198–200. [CrossRef]

Terzi, Alessio. 2021. The 'Roaring Twenties': Revisiting the Evidence for Europe. Available online: https://cepr.org/voxeu/columns/roaring-twenties-revisiting-evidence-europe (accessed on 29 April 2024).

Topforeignstocks. 2024. The Complete List of Constituents of the Brazil Bovespa Index. Available online: https://topforeignstocks.com/indices/components-of-the-bovespa-index (accessed on 22 March 2024).

Turner, John. 2021. Are we about to experience another Roaring Twenties?—Economics Observatory. Available online: https://www.economicsobservatory.com/are-we-about-to-experience-another-roaring-twenties (accessed on 29 April 2024).

Ulrich, Klaus. 2023. *Is German Industry Migrating to the US?* Bonn: Deutsche Welle. Available online: https://www.dw.com/en/is-german-industry-migrating-to-the-us/a-65031130 (accessed on 29 November 2023).

Varamini, Hossein, and Svetlana Kalash. 2008. Testing Market Efficiency for Different Market Capitalization Funds. *American Journal of Business* 23: 17–28. [CrossRef]

World Bank. 2023. Overview—The World Bank in Türkiye. Available online: https://www.worldbank.org/en/country/turkey/overview#1 (accessed on 22 March 2024).

Yin, Dechao. 2019. Investment Decision Based on Entropy Theory. *Modern Economy* 10: 1211. [CrossRef]

Young, Martin R. 1998. A minimax portfolio selection rule with linear programming solution. *Management Science* 44: 673–83. [CrossRef]

Zhou, Rongxi, Ru Cai, and Guanqun Tong. 2013. Applications of Entropy in Finance: A Review. *Entropy* 15: 4909–31. [CrossRef]