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# Optimal Design of Multi-Asset Options

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**Abstract:** The combination of stochastic derivative pricing models and downside risk measures often leads to the paradox (risk, return) = (−infinity, +infinity) in a portfolio choice problem. The construction of a portfolio of derivatives with high expected returns and very negative downside risk (henceforth "golden strategy") has only been studied if all the involved derivatives have the same underlying asset. This paper also considers multi-asset derivatives, gives practical methods to build multi-asset golden strategies for both the expected shortfall and the expectile risk measure, and shows that the use of multi-asset options makes the performance of the obtained golden strategy more efficient. Practical rules are given under the Black–Scholes–Merton multi-dimensional pricing model.

**Keywords:** multi-asset derivative; downside risk measure; unbounded market price of risk; golden strategy

**JEL Classification:** G13; G11; C61

## 1. Introduction

Since monetary tail risk measures became more popular at the end of the last century, they have been used to revisit many classical actuarial and financial topics. Particular attention has been devoted to the portfolio selection problem (Alexander et al. 2006; Stoyanov et al. 2007; or Mansini et al. 2007, to name a few), where the classical role of the standard deviation has been replaced by a tail risk measure. Nevertheless, theoretical approaches have shown that the portfolio selection problem may become unbounded if one accepts the assumptions of many arbitrage-free stochastic pricing models (Black–Scholes–Merton, Heston, etc.) and minimizes the downside risk under a minimum expected return. The paradoxical consequence is that one is able to create a sequence of investment strategies composed of derivative securities whose expected return tends to $+\infty$, whereas the downside risk tends to $-\infty$. Although there are former studies, an easy-to-understand theoretical exposition may be found in Balbás et al. (2019), where the authors present closed formulas to create the sequences above, even in a buy-and-hold framework and only involving European options and riskless assets. Later, Balbás et al. (2023a) provided us with methods to build these sequences in a more general buy-and-hold framework. "More general" means that the authors incorporated the classical frictions (market depth, bid–ask spread, additional commissions, etc.).

All the portfolios analyzed in the papers above contain derivatives with a unique underlying asset. An obvious *Question Q* arises: Can one construct more efficient sequences

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if one deals with multi-asset options? The answer to this question is the main objective of this paper. Since organized markets rarely quote multi-asset derivatives, one has to deal with derivatives that can be replicated by dynamically trading their underlying assets; so, we decided to present the approach in a complete market in order to simplify the mathematical exposition. Actually, Balbás et al. (2016) already reported empirical results indicating that three important international stock indices (the German $DAX$, the American $S\&P$ 500, and the European $EURO\ STOXX$ 50) may be outperformed by dynamically trading the corresponding index future contract. Moreover, the same paper also provided empirical evidence indicating that by trading commodity futures (Brent futures and Gold futures), one can outperform the three stock indices above. In both cases (stock indexes and commodities), the future contract was analyzed under the assumptions of the Black–Scholes–Merton model, which has encouraged us to retain this model in the case of multi-asset derivatives.

The outline of this paper is as follows: Notations and some theoretical background are presented in Section 2. The focus is on two tail risk measures, namely the expected shortfall (Rockafellar and Uryasev 2000) and the expectile (Newey and Powell 1987). The expected shortfall (or conditional value at risk) has been selected because it is very well known and reflects the downside risk in a very intuitive manner. The expectile is not so intuitive, but it is coherent (Artzner et al. 1999) and expectation-bounded (Rockafellar et al. 2006), and it is also elicitable, making it an easier back-testing implementation. Though the expected shortfall is not elicitable, practical back-testing procedures are also available due to their close connection to the value at risk, which is elicitable. In fact, the couple (value at risk, expected shortfall) is in some sense elicitable (see Embrechts et al. 2021 for further details). Finally, both the expected shortfall and the expectile are very closely related (Bellini et al. 2014; Tadese and Drapeau 2020), and the divergence of the expectile to $-\infty$ frequently implies the expected shortfall divergence.

The main notion to create the sequences above, that is, the notion of "golden strategy" of Balbás et al. (2019), is presented in Section 3. Roughly speaking, a golden strategy can be sold for a price that is strictly higher than the downside risk generated by this sale. It is also proved that the presence of golden strategies implies the absence of efficient portfolios in a return/risk approach. Indeed, every portfolio is beaten by the involved portfolio plus the sale of the golden strategy. Theorems 1 and 2 characterize the existence and practical estimation of golden strategies in a complete pricing model and for arbitrary coherent and expectation-bounded risk measures. These theorems are particularized for the expected shortfall (Theorem 3) and the expectile (Theorem 4) in Section 4. For the expected shortfall, one can give a closed formula providing the optimal golden strategy (if it exists), but an alternative closed formula is not achievable for the expectile. At any rate, the lack of any closed formula does not prevent the calculation of the optimal-expectile-linked golden strategy by means of tractable linear programming methods (Theorem 4). It is worth pointing out that the existence of golden strategies for the expectile implies this existence for the expected shortfall, but the converse may fail.

The multi-dimensional Black–Scholes–Merton model ($BSM$) was selected in Section 5. There are other models to price multi-asset derivatives, but $BSM$ is good enough and simplifies the mathematical exposition. As indicated above, the empirical evidence shows that $BSM$ was already adequate when dealing with derivatives with a unique underlying asset. Theorem 5 particularizes the closed formula of Theorem 3 for $BSM$, and therefore, it yields the optimal golden strategy, which is really a multi-asset derivative. This is an indisputable answer to *Question Q* above: the answer is "yes". If one deals with multi-asset derivatives, then the optimal golden strategy is more efficient than those obtained by combining derivatives with a unique underlying asset. Section 6 presents the main conclusions of this paper.

## 2. Preliminaries and Notation

Fix a future planning period $T$, a set $\mathcal{T}$ of trading dates such that $\{0, T\} \subset \mathcal{T} \subset [0, T]$, a filtered probability space $\left(\Omega, (\mathcal{F}_t)_{t \in \mathcal{T}}, \mathcal{F}, \mathbb{P}\right)$ composed of the set of states of nature $\Omega$, the filtration $(\mathcal{F}_t)_{t \in \mathcal{T}}$ yielding the arrival of information such that $\mathcal{F}_0 = \{\varnothing, \Omega\}$ and $\mathcal{F} = \mathcal{F}_T$, and the probability measure $\mathbb{P}$ on $\mathcal{F}$. If $1 \leq p < \infty$, then $L^p$ ($L^p(\mathcal{F}, \mathbb{P})$, if necessary) denotes the space of random variables $y$ such that $\mathbb{E}\left(|y|^p\right) < \infty$ endowed with the norm $\|y\| = \left(\mathbb{E}\left(|y|^p\right)\right)^{1/p}$, where $\mathbb{E}(.)$ represents "mathematical expectation". Similarly, $L^\infty$ ($L^\infty(\mathcal{F}, \mathbb{P})$, if necessary) denotes the space of essentially bounded random variables $y$, and it is endowed with the norm $\|y\| = Ess\_Sup(|y|)$, where $Ess\_Sup(.)$ represents "essential supremum". We deal with a complete financial market; that is, we consider a framework such that every (pay-off at $T$, or marketed claim at $T$) $y \in L^2$ can be replicated by means of a self-financing strategy composed of the finitely many available assets. The pricing rule $\Pi : L^2 \longrightarrow \mathbb{R}$ yields the initial (at $t = 0$) price $\Pi(y)$ of the marketed claim $y \in L^2$, and it is known that the absence of arbitrage implies the existence of a unique stochastic discount factor ($SDF$, Duffie 1988), that is, a unique $z_\Pi \in L^2$ such that

$$\mathbb{P}(z_\Pi > 0) = 1 \tag{1}$$

and

$$\Pi(y) = e^{-rT}\mathbb{E}(z_\Pi y) \tag{2}$$

holds for every $y \in L^2$, where $r$ denotes the continuously compounded riskless rate. In particular, $e^{-rT} = \Pi(1) = e^{-rT}\mathbb{E}(z_\Pi)$ trivially implies that

$$\mathbb{E}(z_\Pi) = 1. \tag{3}$$

$\varphi : L^2 \longrightarrow \mathbb{R}$ is a coherent (Artzner et al. 1999) and expectation-bounded (Rockafellar et al. 2006) risk measure; that is, $\varphi$ is sub-additive ($\varphi(y_1 + y_2) \leq \varphi(y_1) + \varphi(y_2)$ if $y_j \in L^2$, $j = 1, 2$), positively homogeneous ($\varphi(\varkappa y) = \varkappa\varphi(y)$ if $\varkappa \geq 0$ in $\mathbb{R}$ and $y \in L^2$), decreasing ($\varphi(y_1) \leq \varphi(y_2)$ if $y_j \in L^2$, $\mathbb{P}(y_1 \geq y_2) = 1$), translation-invariant ($\varphi(y + \varkappa) = \varphi(y) - \varkappa$ if $\varkappa \in \mathbb{R}$ and $y \in L^2$), and mean dominating ($\varphi(y_1) \geq -\mathbb{E}(y)$ if $y \in L^2$). With general convex analysis linked methods (Zalinescu 2002), it has been proved that the properties above imply the norm-continuity of $\varphi$ and are also equivalent to the fulfillment of

$$\varphi(y) = Max\left\{-\mathbb{E}(zy); z \in \partial_\varphi\right\}, \tag{4}$$

for every $y \in L^2$, where

$$\partial_\varphi = \left\{z \in L^2; -\mathbb{E}(zy) \leq \varphi(y)\ \forall y \in L^2\right\} \tag{5}$$

is the sub-gradient (or sub-gradient at $y = 0$) of $\varphi$ and satisfies

$$\left\{\begin{array}{l} 1 \in \partial_\varphi \\ \mathbb{E}(z) = 1,\ \forall z \in \partial_\varphi. \end{array}\right. \tag{6}$$

Moreover, the norm-continuity of $\varphi$ implies the weak-compactness of $\partial_\varphi$ in $L^2$ and therefore the lower semi-continuity of $\varphi$ if $L^2$ is endowed with the weak topology (see Kopp 1984; or Zalinescu 2002 for further details about all of these concepts).

There are many examples of risk measures satisfying the properties above (Artzner et al. 1999; Rockafellar and Uryasev 2000; Hamada and Sherris 2003; Rockafellar et al. 2006; Mansini et al. 2007; Ahmadi-Javid 2012; Chen and Hu 2018; etc.). Two particular cases play an important role in this paper, namely the expected shortfall with the confidence level $1-\beta^*$ for $0<\beta^*<1$ ($\varphi = ES_{1-\beta^*}$), and the expectile with parameter $\beta$ for $0<\beta<1/2$ ($\varphi = \mathcal{E}_\beta$). Although the special focus is on $ES_{1-\beta^*}$, some specific aspects of $\mathcal{E}_\beta$ are also analyzed. The properties of both measures below may be found in Balbás et al. (2023b), among many others. In particular, the set $\partial_\varphi$ of (5) becomes

$$\partial_{ES_{1-\beta^*}} = \left\{z \in L^2; \mathbb{E}(z) = 1,\ 0 \le z \le 1/\beta^*\right\} \tag{7}$$

and

$$\partial_{\mathcal{E}_\beta} = \left\{z \in L^2; \mathbb{E}(z) = 1,\ \xi \le z \le \xi\frac{1-\beta}{\beta},\ \xi \in \mathbb{R}\right\} \tag{8}$$

respectively. Moreover, taking expectations in (8), it is easy to see that $\xi \le \mathbb{E}(z) = 1 \le \xi(1-\beta)/\beta$, and therefore, (8) is equivalent to

$$\partial_{\mathcal{E}_\beta} = \left\{z \in L^2; \mathbb{E}(z) = 1,\ \xi \le z \le \xi\frac{1-\beta}{\beta},\ \xi \in [\beta/(1-\beta), 1]\right\}. \tag{9}$$

Evidently, (4) becomes

$$\begin{cases} Max\ -\mathbb{E}(zy) \\ \mathbb{E}(z) = 1 \\ z \in L^2,\ 0 \le z \le 1/\beta^* \end{cases} \tag{10}$$

for $ES_{1-\beta^*}$ and

$$\begin{cases} Max\ -\mathbb{E}(zy) \\ \mathbb{E}(z) = 1 \\ z \in L^2,\ \xi \in \mathbb{R},\ \xi \le z \le (\xi(1-\beta))/\beta \end{cases} \tag{11}$$

for $\mathcal{E}_\beta$. Both (10) and (11) are linear optimization problems. According to Balbás et al. (2023b), their duals do not reflect any duality gap and provide us with another representation of both $ES_{1-\beta^*}(y)$ and $\mathcal{E}_\beta(y)$. The duals are

$$\begin{cases} Min\ \lambda + \mathbb{E}(\lambda_M)/\beta^* \\ y = \lambda_m - \lambda_M - \lambda \\ \lambda_m \in L^2,\ \lambda_M \in L^2,\ 0 \le \lambda_m,\ 0 \le \lambda_M,\ \lambda \in \mathbb{R} \end{cases} \tag{12}$$

and

$$\begin{cases} Min\ \lambda \\ y = \lambda_m - \lambda_M - \lambda \\ \beta\mathbb{E}(\lambda_m) - (1-\beta)\mathbb{E}(\lambda_M) = 0 \\ \lambda_m \in L^2,\ \lambda_M \in L^2,\ 0 \le \lambda_m,\ 0 \le \lambda_M,\ \lambda \in \mathbb{R} \end{cases} \tag{13}$$

respectively. The complementary slackness conditions below, along with the feasibility, characterize the optimal solutions of both (10)–(12)

$$z\lambda_m = (1/\beta^* - z)\lambda_M = 0.$$

Similarly, the solutions of (11)–(13) are characterized by feasibility and the fulfillment of

$$(z-\xi)\lambda_m = \left(\xi\frac{1-\beta}{\beta} - z\right)\lambda_M = 0.$$

Furthermore, in both cases, one can show that a couple of primal and dual solutions satisfy

$$\begin{cases} \lambda_m = (y - \mathbb{E}(zy))^+ \\ \lambda_M = (\mathbb{E}(zy) - y)^+. \end{cases} \tag{14}$$

Lastly, the inequality

$$ES_{1-\beta^*}(y) \leq \mathcal{E}_\beta(y) + \frac{\beta}{\beta^*(1-2\beta)}\left(\mathcal{E}_\beta(y) + \mathbb{E}(y)\right) \tag{15}$$

holds for every $y \in L^2$, every $0 < \beta < 1/2$, and every $0 < \beta^* < 1$.

## 3. Golden Strategies

As it has been shown in Balbás et al. (2019), among others, the absence of arbitrage is compatible with the existence of marketed claims $y \in L^2$ such that

$$\varphi(-y) < \mathbb{E}(z_\Pi y). \tag{16}$$

Following these authors, let us use the term "golden strategy" or "$\varphi-$golden strategy" to refer to the strategy above. Since $\varphi$ is translation-invariant,

$$\varphi(-y + \mathbb{E}(z_\Pi y)) = \varphi(-y) - \mathbb{E}(z_\Pi y) < 0,$$

that is, the sale of $y$ along with the investment of the received price $e^{-rT}\mathbb{E}(z_\Pi y)$ in a riskless asset is a self-financing strategy leading to the pay-off $-y + \mathbb{E}(z_\Pi y)$, whose global risk is strictly negative. Consequently, for every marketed claim $u \in L^2$, one has

$$\mathbb{E}(z_\Pi(u - y + \mathbb{E}(z_\Pi y))) = \mathbb{E}(z_\Pi u),$$

$$\varphi(u - y + \mathbb{E}(z_\Pi y)) \leq \varphi(u) + \varphi(-y + \mathbb{E}(z_\Pi y)) < \varphi(u)$$

because $\varphi$ is sub-additive, and

$$\mathbb{E}(u - y + \mathbb{E}(z_\Pi y)) = \mathbb{E}(u) + \mathbb{E}(-y + \mathbb{E}(z_\Pi y)) > \mathbb{E}(u)$$

because $\varphi$ is mean dominating, and therefore, $\mathbb{E}(-y + \mathbb{E}(z_\Pi y)) = \mathbb{E}(-y) + \mathbb{E}(z_\Pi y) \geq -\varphi(-y) + \mathbb{E}(z_\Pi y) > 0$. In other words, every $u$ is outperformed by $u - y + \mathbb{E}(z_\Pi y)$ because this second strategy has an identical price, strictly higher expected pay-off (and thus strictly higher expected return), and strictly lower risk. $-y + \mathbb{E}(z_\Pi y)$ allows us to beat every position. Lastly, since $\varphi$ is positively homogeneous and mean dominating,

$$\begin{cases} \lim_{\varkappa \to +\infty} \varphi(\varkappa(-y + \mathbb{E}(z_\Pi y))) = \lim_{\varkappa \to +\infty} \varkappa\varphi((-y + \mathbb{E}(z_\Pi y))) - \infty \\ \lim_{\varkappa \to +\infty} \mathbb{E}(\varkappa(-y + \mathbb{E}(z_\Pi y))) \geq -\lim_{\varkappa \to +\infty} \varphi(\varkappa(-y + \mathbb{E}(z_\Pi y))) = +\infty, \end{cases}$$

that is, if $-y + \mathbb{E}(z_\Pi y)$ is repeated over and over, with no limit, then one can construct a sequence of self-financing strategies whose risk tends to $-\infty$, whereas its expected pay-off tends to $+\infty$. Henceforth, let us focus on the existence of $y$ satisfying (16).

**Theorem 1.** *Suppose that $z_\Pi \notin \partial_\varphi$. Then, the following hold:*

*$(a)$ There are golden strategies $y$ such that $y \geq a$ for every $a \in \mathbb{R}$. In particular, for $a = 0$, one has that the prohibition of short-sales does not impede the existence of golden strategies.*

*$(b)$ There are golden strategies $y$ such that $y \leq a$ for every $a \in \mathbb{R}$.*

**Proof.** $(a)$ Suppose for a few moments that $a = 0$, and consider the optimization problem

$$\begin{cases} Min\ \varphi(-y) - \mathbb{E}(z_\Pi y) \\ y \geq 0, \end{cases} \tag{17}$$

with $y \in L^2$ being the decision variable. As Balbás et al. (2016) performed for quite similar optimization problems, if (17) is bounded, then one can prove that the dual of (17) does not generate any duality gap and becomes

$$\begin{cases} Max\ 0 \\ \Lambda = z - z_\Pi,\ z \in \partial_\varphi,\ \Lambda \geq 0, \end{cases} \tag{18}$$

with $(z, \Lambda) \in L^2 \times L^2$ being the decision variable. Taking expectations in the constraint of (18), one has $\mathbb{E}(\Lambda) = 0$ (see (3) and (6)), and therefore, $\Lambda \geq 0$ leads to $\Lambda = 0$ and $z = z_\Pi$. This equality is unfeasible because $z_\Pi \notin \partial_\varphi$, and therefore, the feasible set of (18) becomes void. Accordingly, (17) becomes unbounded.

Consider now a general $a \in \mathbb{R}$. Take $y \in L^2$ such that $\varphi(-y) - \mathbb{E}(z_\Pi y) < 0$ and $y \geq 0$. Hence (recall that $\varphi$ is translation-invariant and see (3)),

$$\varphi(-(y + a)) - \mathbb{E}(z_\Pi(y + a)) = \varphi(-y) - \mathbb{E}(z_\Pi y) < 0.$$

$(b)$ is similar to $(a)$ if the constraint $y \geq 0$ of (17) is replaced with $y \leq 0$. □

Suppose that $z_\Pi \notin \partial_\varphi$. Then, the proof of Theorem 1 implies that (17) is unbounded, as it remains unbounded if the constraint $y \geq 0$ is removed or replaced by $y \geq a$ or $y \leq a$. It could also be replaced by $b \leq y \leq c$, but the change in variable

$$y' = \frac{y - a}{b - a}$$

replaces $b \leq y \leq c$ with $0 \leq y' \leq 1$. In other words, $0 \leq y \leq 1$ is as general as the most general constraint if one looks for self-financing essentially bounded marketed claims with non-positive risk.

**Theorem 2.** $(a)$ *Problem*

$$\begin{cases} Min\ \varphi(-y) - \mathbb{E}(z_\Pi y) \\ 0 \leq y \leq 1 \end{cases} \tag{19}$$

*is bounded and solvable, its optimal value is negative or zero, and it is strictly negative if and only if $z_\Pi \notin \partial_\varphi$, in which case the solution of (19) is a golden strategy. If $z_\Pi \in \partial_\varphi$, then there are no golden strategies, and the optimal value of (19) vanishes.*

$(b)$ *Problem*

$$\begin{cases} Min\ \mathbb{E}(\Lambda_M) \\ \Lambda_m - \Lambda_M = z - z_\Pi,\ z \in \partial_\varphi,\ \Lambda_m \geq 0,\ \Lambda_M \geq 0, \end{cases} \tag{20}$$

*is the dual of (19), with $(z, \Lambda_m, \Lambda_M) \in L^2 \times L^2 \times L^2$ being the decision variable. It is feasible and solvable, and the optimal values of both (19) and (20) have identical absolute values and opposite signs.*

$(c)$ *If $z_\Pi \notin \partial_\varphi$ and $(\tilde{z}, \tilde{\Lambda}_m, \tilde{\Lambda}_M)$ solves (20), then $\mathbb{P}(z_\Pi > \tilde{z}) > 0$ and $\mathbb{P}(\tilde{z} > z_\Pi) > 0$.*

($d$) *If $\tilde{y}$ is (19)-feasible and $(\tilde{z}, \tilde{\Lambda}_m, \tilde{\Lambda}_M)$ is (20)-feasible, then they solve the corresponding problem if and only if*

$$\begin{cases} \tilde{y}\tilde{\Lambda}_m = 0 \\ (\tilde{y} - 1)\tilde{\Lambda}_M = 0 \\ \mathbb{E}(\tilde{z}\tilde{y}) \geq \mathbb{E}(z\tilde{y}) \ \forall z \in \partial_\varphi. \end{cases} \tag{21}$$

($e$) *The solutions $\tilde{y}$ and $(\tilde{z}, \tilde{\Lambda}_m, \tilde{\Lambda}_M)$ of (19) and (20) satisfy*

$$\begin{cases} \tilde{\Lambda}_m = (\tilde{z} - z_\Pi)^+ \\ \tilde{\Lambda}_M = (z_\Pi - \tilde{z})^+ \\ \tilde{z} > z_\Pi \implies \tilde{y} = 0 \\ \tilde{z} < z_\Pi \implies \tilde{y} = 1 \\ 0 < \tilde{y} < 1 \implies \tilde{z} = z_\Pi, \end{cases} \tag{22}$$

*where the three implications hold for $\omega \in \Omega$ out of a $\mathbb{P}-$null set.*

**Proof.** The Alaoglu theorem (Luenberger 1969) implies that the set $0 \leq y \leq 1$ is weakly compact in $L^2$. Thus, the existence of the solution of (19) follows from the lower semi-continuity of $\varphi$ for the weak topology of $L^2$. Furthermore, since $y = 0$ is (19)-feasible, the optimal value can never be strictly positive. One can proceed as in Balbás et al. (2016) in order to prove that (20) is the dual problem of (19), and similar arguments show that (21) characterizes the solutions of (19) and (20). Take a solution $(\tilde{z}, \tilde{\Lambda}_m, \tilde{\Lambda}_M)$ of (20) whose existence is guaranteed by the usual primal–dual relationships (Luenberger 1969), suppose that $z_\Pi \notin \partial_\varphi$, and let us prove both $\mathbb{P}(z_\Pi > \tilde{z}) > 0$ and $\mathbb{P}(\tilde{z} > z_\Pi) > 0$. If $\tilde{z} \leq z_\Pi$, then $\mathbb{E}(\tilde{z}) = \mathbb{E}(z_\Pi) = 1$ (see (3) and (6)) leads to $\tilde{z} = z_\Pi$, which is a contradiction because $\tilde{z} \in \partial_\varphi$ and $z_\Pi \notin \partial_\varphi$. The proof of $\mathbb{P}(\tilde{z} > z_\Pi) > 0$ is analogous. Take again a solution $(\tilde{z}, \tilde{\Lambda}_m, \tilde{\Lambda}_M)$ of (20). If there are no golden strategies, then $\tilde{y} = 0$ is an obvious solution of (19), so $\tilde{\Lambda}_m = \tilde{z} - z_\Pi$ and $\tilde{\Lambda}_M = 0$ must be satisfied by the solution of (20) due to (21). Taking expectations, $\mathbb{E}(\tilde{\Lambda}_m) = \mathbb{E}(\tilde{z}) - \mathbb{E}(z_\Pi) = 1 - 1 = 0$ (see (3) and (6)), so $\tilde{\Lambda}_m \geq 0$ leads to $\tilde{\Lambda}_m = 0$. The constraints of (20) imply that $0 = \tilde{z} - z_\Pi$, that is, $z_\Pi \in \partial_\varphi$. Conversely, if $z_\Pi \in \partial_\varphi$, then one can take $(\tilde{z}, \tilde{\Lambda}_m, \tilde{\Lambda}_M) = (z_\Pi, 0, 0)$, so $\tilde{y} = 0$ solves (19), and there are no golden strategies. The constraints $\Lambda_m - \Lambda_M = z - z_\Pi$, $\Lambda_m \geq 0$, and $\Lambda_M \geq 0$ of (20) easily imply that the minimum of $\mathbb{E}(\Lambda_M)$ must satisfy $\tilde{\Lambda}_m = (\tilde{z} - z_\Pi)^+$ and $\tilde{\Lambda}_M = (z_\Pi - \tilde{z})^+$. Finally, the rest of the conditions in (22) trivially follow from $\tilde{\Lambda}_m = (\tilde{z} - z_\Pi)^+$, $\tilde{\Lambda}_M = (z_\Pi - \tilde{z})^+$, and (21). □

## 4. Focusing on the Expected Shortfall and the Expectile

As already said, $ES_{1-\beta^*}$ and $\mathcal{E}_\beta$ are two important examples of risk measures satisfying the imposed conditions. Let us focus on them. Henceforth, $\mathbb{1}_A : \Omega \longrightarrow \mathbb{R}$ will denote the usual indicator for every measurable set $A \in \mathcal{F}$, that is, $\mathbb{1}_A(\omega) = 1$ if $\omega \in A$ and $\mathbb{1}_A(\omega) = 0$ otherwise. Moreover, similar notation will apply if $(\Omega, \mathcal{F}, \mathbb{P})$ is replaced by another probability space.

**Theorem 3.** *Consider $0 < \beta^* < 1$ and $\rho = ES_{1-\beta^*}$.*

($a$) *There exist $ES_{1-\beta^*}-$golden strategies if and only if the inequality $\|z_\Pi\|_\infty = Ess\_Sup(z_\Pi) > 1/\beta^*$ holds.*

($b$) *There is a linear dual problem of (20) (bidual of (19)) given by*

$$\begin{cases} Max \ \mathbb{E}(z_\Pi y_1) - \mathbb{E}(y_2)/\beta^* + y_3 \\ y_2 \geq y_1 + y_3 \\ y_1 \leq 1 \\ y_1 \geq 0, \ y_2 \geq 0, \end{cases} \tag{23}$$

*$(y_1, y_2, y_3) \in L^2 \times L^2 \times \mathbb{R}$ being the decision variable. (23) is solvable, and there is no duality gap between (20) and (23).*

*$(c)$ If $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ solves (23), then $\tilde{y} = \tilde{y}_1$ solves (19).*

*$(d)$ If $\|z_\Pi\|_\infty > 1/\beta^*$, then $\tilde{y} = \mathbb{1}_{z_\Pi > 1/\beta^*}$ solves (19) and is an $ES_{1-\beta^*}-$golden strategy.*

**Proof.** $(a)$ This is an obvious consequence of (7) and Theorem 2$a$.

$(b)$ Bearing in mind (4) and (7), Problem (20) becomes the linear problem

$$\begin{cases} Min\ \mathbb{E}(\Lambda_M) \\ \Lambda_m - \Lambda_M = z - z_\Pi \\ z \le 1/\beta^* \\ \mathbb{E}(z) = 1 \\ z \ge 0,\ \Lambda_m \ge 0,\ \Lambda_M \ge 0. \end{cases} \tag{24}$$

Thus, according to the duality methods of Anderson and Nash (1987), (24) is the dual of (24). Although (23) is bounded, neither its solvability nor the absence of a duality gap with (24) are guaranteed. Nevertheless, both properties will be proved if one finds a (24)-feasible element and a (23)-feasible one that make the objectives of both problems identical. Consider a solution $\tilde{y}$ of (19) and a solution $\left(\tilde{z}, (\tilde{z} - z_\Pi)^+, (z_\Pi - \tilde{z})^+\right)$ of (24) (recall Theorem 2 and (22)). Consider also Problems ((10)↿ $-\tilde{y}$) and ((12)↿ $-\tilde{y}$), that is, Problems (10) and (12) once $y$ has been replaced by $-\tilde{y}$. The third condition in (21) shows that $\tilde{z}$ solves ((10)↿ $-\tilde{y}$). Finally, consider a solution $\left(\tilde{\lambda}, \tilde{\lambda}_m, \tilde{\lambda}_M\right)$ of ((12)↿ $-\tilde{y}$), and take

$$(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3) = \left(\tilde{y}, \tilde{\lambda}_M, -\tilde{\lambda}\right) \tag{25}$$

It is sufficient to verify the second constraint of (23) in order to show that (25) is (23)-feasible, since the rest of the restrictions trivially follow from the restrictions of (19) and ((12)↿ $-\tilde{y}$). One has that

$$\tilde{y}_1 + \tilde{y}_3 = -\tilde{\lambda} + \tilde{y} = \tilde{\lambda}_M - \tilde{\lambda}_m \le \tilde{\lambda}_M = \tilde{y}_2.$$

Hence, the solvability of (23) and the absence of duality gap trivially follow from

$$\begin{cases} \mathbb{E}(z_\Pi \tilde{y}_1) - \mathbb{E}(\tilde{y}_2)/\beta^* + \tilde{y}_3 = \mathbb{E}(z_\Pi \tilde{y}) - \mathbb{E}\left(\tilde{\lambda}_M\right)/\beta^* - \tilde{\lambda} \\ = \mathbb{E}(z_\Pi \tilde{y}) - \varphi(-\tilde{y}) = \mathbb{E}\left((z_\Pi - \tilde{z})^+\right), \end{cases}$$

where the second and third equalities are implied by the absence of duality gap between the pairs ((10)↿ $-\tilde{y}$)–((12)↿ $-\tilde{y}$) and (19)–(24).

$(c)$ Take the solutions $\left(\tilde{z}, (\tilde{z} - z_\Pi)^+, (z_\Pi - \tilde{z})^+\right)$ of (24) and $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3)$ of (23), and let us prove that $\tilde{y} = \tilde{y}_1$ satisfies (21). The complementary slackness conditions of linear programming (Anderson and Nash 1987) become

$$\begin{cases} \tilde{z}(\tilde{y}_2 - (\tilde{y}_1 + \tilde{y}_3)) = 0 \\ (\tilde{z} - z_\Pi)^+ \tilde{y}_1 = 0 \\ (z_\Pi - \tilde{z})^+ (1 - \tilde{y}_1) = 0 \\ \tilde{y}_2 (1/\beta^* - \tilde{z}) = 0. \end{cases} \tag{26}$$

The second equality of (26) implies the first one of (21), whereas the third equality of (26) implies the second one of (21). The first and fourth equalities of (26) lead to

$$\begin{cases} \mathbb{E}(\tilde{z}\tilde{y}_2) = \mathbb{E}(\tilde{z}(\tilde{y} + \tilde{y}_3)) \\ \mathbb{E}(\tilde{z}\tilde{y}_2) = \mathbb{E}(\tilde{y}_2)/\beta^* \end{cases}$$

i.e., (recall (6)), $\mathbb{E}(\tilde{z}\tilde{y}) = \mathbb{E}(\tilde{z}\tilde{y}_2) - \tilde{y}_3 = \mathbb{E}(\tilde{y}_2)/\beta^* - \tilde{y}_3$. It only remains to see that

$$\mathbb{E}(z\tilde{y}) \leq \mathbb{E}(\tilde{y}_2)/\beta^* - \tilde{y}_3 \tag{27}$$

for every $z \in \partial_{ES_{1-\beta^*}}$. Indeed, if $\tilde{y}_4 = \tilde{y}_2 - (\tilde{y} + \tilde{y}_3)$, then $(\tilde{y}_4, \tilde{y}_2, -\tilde{y}_3)$ is ((12)| $-\tilde{y}$)-feasible, and therefore, (27) holds because $z$ is ((10)| $-\tilde{y}$)-feasible.

$(d)$ According to Theorem 2, it is sufficient to show that $\tilde{y} = 1\!\!1_{z_\Pi > 1/\beta^*}$ solves (19). Thus, according to $(c)$, it is sufficient to show that $\left(1\!\!1_{z_\Pi > 1/\beta^*}, 1\!\!1_{z_\Pi > 1/\beta^*}, 0\right)$ solves (23). Thus, it is enough to find $\tilde{z} \in \partial_{ES_{1-\beta^*}}$ such that $\left(1\!\!1_{z_\Pi > 1/\beta^*}, 0, 0\right)$ and $\left(\tilde{z}, (\tilde{z} - z_\Pi)^+, (z_\Pi - \tilde{z})^+\right)$ satisfy (26). First of all, let us show that $\gamma = \mathbb{P}(z_\Pi > 1/\beta^*) < \beta^*$. Indeed, $\gamma \geq \beta^*$ would lead to a contradiction with (3)

$$\mathbb{E}(z_\Pi) = \int_\Omega z_\Pi(\omega)\mathbb{P}(d\omega) \geq \int_{z_\Pi > 1/\beta^*} z_\Pi(\omega)\mathbb{P}(d\omega) > \frac{\gamma}{\beta^*} \geq 1.$$

For $\alpha \in \mathbb{R}$, $\alpha \geq 0$, consider

$$z_\alpha(\omega) = \begin{cases} 1/\beta^*, & if \;\; z_\Pi(\omega) > 1/\beta^* \\ Min\{z_\Pi(\omega) + \alpha, 1/\beta^*\}, & otherwise. \end{cases}$$

Obviously, $0 \leq z_\alpha \leq 1/\beta^*$, so $z_\alpha \in \partial_{ES_{1-\beta^*}}$ if $\mathbb{E}(z_\alpha) = 1$ (recall (7)). The Dominated Convergence Theorem obviously implies that $[0, \infty) \ni \alpha \longrightarrow z_\alpha \in L^1$ is continuous and therefore so is $[0, \infty) \ni \alpha \longrightarrow \mathbb{E}(z_\alpha) \in \mathbb{R}$. Furthermore,

$$\mathbb{E}(z_0) = \int_{z_\Pi \leq 1/\beta^*} z_\Pi(\omega)\mathbb{P}(d\omega) + \frac{\gamma}{\beta^*} < \mathbb{E}(z_\Pi) = 1$$

and $\mathbb{E}\left(z_{1/\beta^*}\right) = 1/\beta^* > 1$, so the Bolzano Theorem implies the existence of $\alpha \in (0, 1/\beta^*)$ such that $\mathbb{E}(z_\alpha) = 1$ and $z_\alpha \in \partial_{ES_{1-\beta^*}}$. Take $\tilde{z} = z_\alpha$ and notice that $\left(\tilde{z}, (\tilde{z} - z_\Pi)^+, (z_\Pi - \tilde{z})^+\right)$ and $\left(1\!\!1_{z_\Pi > 1/\beta^*}, 1\!\!1_{z_\Pi > 1/\beta^*}, 0\right)$ obviously satisfy (26). □

**Corollary 1.** *If* $\rho = ES_{1-\beta^*}$ *and* $z_\Pi$ *is not essentially bounded, then* $\tilde{y} = 1\!\!1_{z_\Pi > 1/\beta^*}$ *solves (19) and is an* $ES_{1-\beta^*}-$*golden strategy.*

**Proof.** $\|z_\Pi\| = +\infty$ obviously implies $\|z_\Pi\|_\infty > 1/\beta^*$. □

**Remark 1.** *Notice that, under the conditions of Theorem 3 or Corollary 4, the digital (or binary) option* $\tilde{y} = 1\!\!1_{z_\Pi > 1/\beta^*}$ *is an optimal* $ES_{1-\beta^*}-$*golden strategy. This finding improves that of Balbás et al. (2016). Indeed, these authors did not deal with golden strategies but with capital requirements. Consequently, they did not find the optimal solution* $1\!\!1_{z_\Pi > 1/\beta^*}$ *but a more complex sub-optimal one. Nevertheless, though they dealt with a sub-optimal golden strategies with a single underlying asset, they reported empirical evidence illustrating that their strategy was able to outperform very important international stock indices. Accordingly, it is natural to assume that* $1\!\!1_{z_\Pi > 1/\beta^*}$ *will also outperform the involved indices because* $1\!\!1_{z_\Pi > 1/\beta^*}$ *maximizes the difference between the received price for its sale and the risk that this sale provokes.*

*Things are a little bit different for expectiles. Indeed, let us show that* $\mathcal{E}_\beta-$*golden strategies also exist under the absence of strictly positive lower bounds of* $z_\Pi$. *Accordingly, a general simple expression such as* $\tilde{y} = 1\!\!1_{z_\Pi > 1/\beta^*}$ *cannot be given for the expectile risk measure.*

**Theorem 4.** *Consider* $0 < \beta < 1/2$ *and* $\rho = \mathcal{E}_\beta$.

(*a*) *If* $\|z_\Pi\|_\infty > (1-\beta)/\beta$ *or* $Ess\_Inf(z_\Pi) < \beta/(1-\beta)$, *then there are* $\mathcal{E}_\beta-$*golden strategies.*

(*b*) *Problem (20) becomes the linear problem*

$$\begin{cases} Min\ \mathbb{E}(\Lambda_M) \\ \Lambda_m - \Lambda_M = z - z_\Pi \\ \xi \le z \le \xi \dfrac{1-\beta}{\beta} \\ \mathbb{E}(z) = 1 \\ z \in L^2,\ \Lambda_m \ge 0,\ \Lambda_M \ge 0,\ \xi \in \mathbb{R}, \end{cases} \tag{28}$$

*with* $(\xi, z, \Lambda_m, \Lambda_M) \in \mathbb{R} \times L^2 \times L^2 \times L^2$ *being the decision variable. Consider the optimization problem*

$$\begin{cases} Max\ \mathbb{E}(z_\Pi y_1) + y_4 \\ -y_1 = y_2 - y_3 + y_4 \\ \beta \mathbb{E}(y_2) = (1-\beta)\mathbb{E}(y_3) \\ y_1 \le 1 \\ y_1 \ge 0,\ y_2 \ge 0, y_3 \ge 0,\ y_4 \in \mathbb{R} \end{cases} \tag{29}$$

*with* $(y_1, y_2, y_3, y_4) \in L^2 \times L^2 \times L^2 \times \mathbb{R}$ *being the decision variable. (29) is the linear dual of (28) (or the bidual of (19)), it is bounded and solvable, and its optimal value equals the optimal value of (20) and (28).*

(*c*) *If* $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)$ *solves (29), then* $\tilde{y} = \tilde{y}_1$ *solves (19). If* $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)$ *solves (29), then* $\tilde{y} = \tilde{y}_1$ *is a* $\mathcal{E}_\beta-$*golden strategy if and only if* $\mathbb{E}(z_\Pi \tilde{y}_1) + \tilde{y}_4 > 0$.

**Proof.** (*a*) If $\|z_\Pi\|_\infty > (1-\beta)/\beta$ or $Ess\_Inf(z_\Pi) < \beta/(1-\beta)$, then $z_\Pi \notin \partial_{\mathcal{E}_\beta}$ due to (9), and Theorem 2*a* applies.

(*b*) Bearing in mind (4) and (8), Problem (20) becomes the linear problem (28), whose linear dual is (29) (Anderson and Nash 1987). As in Theorem 3*b*, consider a solution $\tilde{y}$ of (19) and a solution $\left(\tilde{\xi}, \tilde{z}, (\tilde{z} - z_\Pi)^+, (z_\Pi - \tilde{z})^+\right)$ of (28), and one must find a (29)-feasible element $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)$ such that $\mathbb{E}\left((z_\Pi - \tilde{z})^+\right) = \mathbb{E}(z_\Pi \tilde{y}_1) + \tilde{y}_4$. With similar notations as in the proof of Theorem 3, the third condition in (21) shows that $(\tilde{\xi}, \tilde{z})$ solves ((11)$-\tilde{y}$). Finally, consider a solution$(\tilde{\lambda}, \tilde{\lambda}_m, \tilde{\lambda}_M)$ of ((13)| $-\tilde{y}$), and take $\tilde{y}_1 = \tilde{y}$, $\tilde{y}_2 = \tilde{\lambda}_m$, $\tilde{y}_3 = \tilde{\lambda}_M$, and $\tilde{y}_4 = -\tilde{\lambda}$. The constraints of (19) and ((13)$-\tilde{y}$) show that $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)$ is (29)-feasible. Moreover, since $\tilde{\lambda}$ is the optimal value of (29),

$$\mathbb{E}(z_\Pi \tilde{y}_1) + \tilde{y}_4 = \mathbb{E}(z_\Pi \tilde{y}) - \tilde{\lambda} = \mathbb{E}(z_\Pi \tilde{y}) - \mathcal{E}_\beta(-\tilde{y}) = \mathbb{E}\left((z_\Pi - \tilde{z})^+\right),$$

where the last equality is implied by Theorem 2.

(*c*) Take the solutions $\left(\tilde{z}, (\tilde{z} - z_\Pi)^+, (z_\Pi - \tilde{z})^+\right)$ of (28) and $(\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4)$ of (29), and let us prove that $\tilde{y} = \tilde{y}_1$ satisfies (21). The complementary slackness conditions of linear programming become

$$\begin{cases} \tilde{y}_1 (\tilde{z} - z_\Pi)^+ = 0 \\ (1 - \tilde{y}_1)(z_\Pi - \tilde{z})^+ = 0 \\ \tilde{y}_2 (\tilde{z} - \tilde{\xi}) = 0 \\ \tilde{y}_3 \left(\tilde{\xi} \dfrac{1-\beta}{\beta} - \tilde{z}\right) = 0. \end{cases} \tag{30}$$

The first and second equalities of (30) and (21) coincide. Besides, for $z \in \partial_{\mathcal{E}_\beta}$, the first and second constraints of (29) lead to (recall (6))

$$\mathbb{E}(z\tilde{y}_1) = -\mathbb{E}(z\tilde{y}_2) + \mathbb{E}(z\tilde{y}_3) - \tilde{y}_4 = \mathbb{E}(z(\tilde{y}_3 - \tilde{y}_2)) - \tilde{y}_4.$$

Take $\xi > 0$ such that $(\xi, z)$ satisfy the conditions of (9). Since $y_2 \geq 0$ and $y_3 \geq 0$,

$$\begin{cases} \mathbb{E}(z\tilde{y}_3) \leq \xi \dfrac{1-\beta}{\beta} \mathbb{E}(\tilde{y}_3) \\ \mathbb{E}(z\tilde{y}_2) \geq \xi \mathbb{E}(\tilde{y}_2). \end{cases}$$

Hence,

$$\mathbb{E}(z(\tilde{y}_3 - \tilde{y}_2)) \leq \xi \left( \frac{1-\beta}{\beta} \mathbb{E}(\tilde{y}_3) - \mathbb{E}(\tilde{y}_2) \right) = 0$$

due to the second constraint of (29). Besides, if $z = \tilde{z}$, the third and fourth conditions in (30), along with the second constraint of (29), lead to

$$\mathbb{E}(\tilde{z}(\tilde{y}_3 - \tilde{y}_2)) = \tilde{\xi} \frac{1-\beta}{\beta} \mathbb{E}(\tilde{y}_3) - \tilde{\xi} \mathbb{E}(\tilde{y}_2) = 0.$$

Finally, the solution $\tilde{y} = \tilde{y}_1$ of (19) is a golden strategy if and only if the optimal objective value of (19) (of (29)) is strictly negative (positive) owing to Theorem 2. □

**Remark 2.** *Notice that there are important differences between Theorems 3 and 4. Although the optimal $ES_{1-\beta^*}$−golden strategy may be computed by solving the linear problem (23), it is actually enough to know $z_\Pi$, and one does not need to optimize any linear problem. Indeed, though Theorems 3(a), 3(b) and 3(c) were needed because they have to be used in order to prove Theorem 3(d), once $z_\Pi$ is known $\mathbb{1}_{z_\Pi > 1/\beta^*}$ will solve (19), and it will be non-null if and only if $\|z_\Pi\|_\infty > 1/\beta^*$, which is the unique case leading to the existence of $ES_{1-\beta^*}$−golden strategies.*

*In contrast, Theorem 4 does not provide us with any closed formula for the optimal $\mathcal{E}_\beta$−golden strategy, but it will be known if one optimizes the linear problem (29). It will often be an infinite-dimensional problem, but it is known that many continuous-time pricing models may be properly discretized (Duffie 1988), and therefore, the solution of (29) may be properly approximated by the solution of a finite-dimensional linear programming problem that may be solved by means of the simplex algorithm. Alternatively, one can deal with the infinite-dimensional problem (29) and the corresponding simplex-like algorithms of Anderson and Nash (1987). In such a case, the third and fourth expressions (implications) of (22) make it much easier to detect an initial "basic feasible solution" $y_0$, since $\tilde{z} > z_\Pi \implies \tilde{y} = 0$ leads to $\beta/(1-\beta) > z_\Pi \implies y_0 = 0$ and $\tilde{z} < z_\Pi \implies \tilde{y} = 1$ leads to $(1-\beta)/\beta < z_\Pi \implies y_0 = 1$ (recall (9)). Consequently, states of nature satisfying $\beta/(1-\beta) > z_\Pi$ cannot belong to the initial basis of a simplex-like algorithm, whereas states of nature satisfying $(1-\beta)/\beta < z_\Pi$ have to belong.*

## 5. Focusing on the Black–Scholes–Merton Multi-Dimensional Model

Let us focus on the *BSM* multi-dimensional model as a particular relevant case. Accordingly, first of all, let us summarize the most important properties of this model, which may be found in Contreras et al. (2016), among many others. There are alternative models to price multi-asset derivatives (Wu et al. 2023; Zhou et al. 2024; etc.), but let us focus on the most usual one and simplify the mathematical exposition.

### *5.1. Model Summary*

Consider a continuously compounded riskless rate $r$ and $n$ risky assets, $S_1, \ldots, S_n$, whose stochastic behavior is given by the Geometric Brownian Motions (*GBM*)

$$dS_j = S_j \Big( (\mu_j - \gamma_j) dt + \sigma_j dW_j^* \Big), \tag{31}$$

with $\mu_j$ being "drift", $\gamma_j$ being "dividend yield", $\sigma_j$ being "volatility" and $W_j^*$ being a Standard Brownian Motion ($SBM$), $j = 1, 2, \ldots, n$. The (symmetric) correlation matrix of $\{W_1^*, \ldots, W_n^*\}$ will be denoted by

$$\rho = \begin{pmatrix} 1, & \rho_{1,2}, & \ldots & \rho_{1,n} \\ \rho_{2,1}, & 1, & \ldots & \rho_{2,n} \\ \ldots & \ldots & \ldots & \ldots \\ \rho_{n,1}, & \rho_{n,2}, & \ldots & 1 \end{pmatrix}$$

and we assume that $\rho$ is regular (and therefore positive definite) in order to prevent the existence of redundant (replicable) securities in the set $\{S_1, \ldots, S_n\}$. If one fixes the time horizon $T$, it is known that the explicit solution of (31) becomes

$$S_j(T) = S_j(0) Exp\left(\left(\mu_j - \gamma_j - \frac{\sigma_j^2}{2}\right)T + \sigma_j \sqrt{T} W_j\right) \tag{32}$$

with $(W_1, \ldots, W_n)$ being a $n-$dimensional standard normal random variable whose correlation and covariance matrix equals $\rho$. In order to price and hedge European derivatives with maturity at $T$, notice that (32) allows us to simplify the probability space and suppose that

$$\Omega = \mathbb{R}_+^n = \{(\omega_1, \ldots, \omega_n) \in \mathbb{R}^n;\ \omega_j > 0,\ j = 1, \ldots, n\},$$

where $\mathcal{F}$ is the Borel $\sigma-$algebra of $\mathbb{R}_+^n$, and $\mathbb{P}$ is the probability measure induced on $\mathcal{F}$ by the log-normal random variables $\{S_1(T), \ldots, S_n(T)\}$. Obviously,

$$\mathbb{P}(S_j(T) \le \omega_j) = \mathbb{P}\left(W_j \le \frac{\log(\omega_j/S_j(0)) - \left(\mu_j - \gamma_j - \frac{\sigma_j^2}{2}\right)T}{\sigma_j \sqrt{T}}\right)$$

for $j = 1, \ldots, n$ and $\omega_j > 0$, and the joint cumulative distribution function of $\{S_1(T), \ldots, S_n(T)\}$ becomes

$$F\left((\omega_j)_{j=1}^n\right) = \mathbb{P}\left((W_j)_{j=1}^n \le \left(\frac{\log(\omega_j/S_j(0)) - \left(\mu_j - \gamma_j - \frac{\sigma_j^2}{2}\right)T}{\sigma_j \sqrt{T}}\right)_{j=1}^n\right).$$

Hence, the joint density function becomes

$$\begin{cases} \dfrac{\partial^n F}{\partial \omega_1 \ldots \partial \omega_n} = \\ f_\rho\left(\left(\dfrac{\log(\omega_j/S_j(0)) - \left(\mu_j - \gamma_j - \frac{\sigma_j^2}{2}\right)T}{\sigma_j \sqrt{T}}\right)_{j=1}^n\right) \sqrt{T}^{-n} \prod_{j=1}^n \left(\dfrac{S_j(0)}{\sigma_j \omega_j}\right) \end{cases}$$

where

$$f_\rho(u) = \frac{1}{\sqrt{2\pi}^n \sqrt{|\rho|}} Exp\left(-\frac{1}{2} u \rho^{-1} u'\right) \tag{33}$$

is the joint density function of $\{W_1, \ldots, W_n\}$, $|\rho|$ is the determinant of $\rho$, $u = (u_1, \ldots, u_n)$, and $A'$ denotes the transposed of an arbitrary matrix $A$. Moreover, if $\mathbb{L}_n$ denotes the

Lebesgue measure on $\mathcal{F}$, it is known that the Radon–Nikodym derivative of $\mathbb{P}$ with respect to $\mathbb{L}_n$ is the density function above, that is,

$$\begin{cases} \frac{d\mathbb{P}}{d\mathbb{L}_n} = \\ f_\rho\left(\left(\frac{\log(\omega_j/S_j(0)) - \left(\mu_j - \gamma_j - \frac{\sigma_j^2}{2}\right)T}{\sigma_j\sqrt{T}}\right)_{j=1}^n\right)\sqrt{T}^{-n}\prod_{j=1}^n\left(\frac{S_j(0)}{\sigma_j\omega_j}\right) \end{cases} \tag{34}$$

on $\mathbb{R}_+^n$.

### 5.2. The Stochastic Discount Factor

Consider the family of Arithmetic Brownian Motions ($ABM$)

$$\tilde{W}_j^* = W_j^* + \frac{\mu_j - r}{\sigma_j}t,$$

$j = 1, \ldots, n$, $t \geq 0$. Straightforward manipulations imply that (31) and (32) become

$$\begin{cases} dS_j = S_j\left((r - \gamma_j)dt + \sigma_j d\tilde{W}_j^*\right) \\ S_j(T) = S_j(0)Exp\left(\left(r - \gamma_j - \frac{\sigma_j^2}{2}\right)T + \sigma_j\sqrt{T}\tilde{W}_j\right) \end{cases} \tag{35}$$

Thus, both (31) and (32) remain the same if every $\mu_j$ is replaced by $r$ and every $W_j^*$ is replaced by $\tilde{W}_j^*$. It is known that the Girsanov Theorem guarantees the existence of an equivalent to $\mathbb{P}$ probability measure $\mathbb{Q}$ making $\tilde{W}_j^*$ a $SBM$ for $j = 1, \ldots, n$. Therefore, $(\tilde{W}_1, \ldots, \tilde{W}_n)$ becomes an $n-$dimensional standard normal random variable under $\mathbb{Q}$ whose correlation matrix is still $\rho$. Moreover, the current price of every marketed claim $P_T$ with maturity at $T$ will be $e^{-rT}\mathbb{E}(z_\Pi P_T) = e^{-rT}\mathbb{E}_\mathbb{Q}(P_T)$, $\mathbb{E}_\mathbb{Q}$ denoting "expectation under $\mathbb{Q}$". Proceeding as above,

$$\begin{cases} \frac{d\mathbb{Q}}{d\mathbb{L}_n} = \\ f_\rho\left(\left(\frac{\log(\omega_j/S_j(0)) - \left(r - \gamma_j - \frac{\sigma_j^2}{2}\right)T}{\sigma_j\sqrt{T}}\right)_{j=1}^n\right)\sqrt{T}^{-n}\prod_{j=1}^n\left(\frac{S_j(0)}{\sigma_j\omega_j}\right). \end{cases} \tag{36}$$

Hence, since the stochastic discount factor satisfies (Duffie 1988)

$$z_\Pi = \frac{d\mathbb{Q}}{d\mathbb{P}} = \left(\frac{d\mathbb{Q}}{d\mathbb{L}_n}\right)\left(\frac{d\mathbb{L}_n}{d\mathbb{P}}\right),$$

(34) and (36) lead to

$$z_{\Pi} = \frac{f_\rho\left(\left(\dfrac{\log\left(\omega_j/S_j(0)\right) - \left(r - \gamma_j - \dfrac{\sigma_j^2}{2}\right)T}{\sigma_j\sqrt{T}}\right)_{j=1}^{n}\right)}{f_\rho\left(\left(\dfrac{\log\left(\omega_j/S_j(0)\right) - \left(\mu_j - \gamma_j - \dfrac{\sigma_j^2}{2}\right)T}{\sigma_j\sqrt{T}}\right)_{j=1}^{n}\right)}$$

for every $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}_+^n$. Consequently, (33) leads to

$$z_{\Pi} = Exp\left(\frac{1}{2}\left(u\rho^{-1}u' - v\rho^{-1}v'\right)\right) \tag{37}$$

with

$$u_j = \frac{\log\left(\omega_j/S_j(0)\right) - \left(\mu_j - \gamma_j - \dfrac{\sigma_j^2}{2}\right)T}{\sigma_j\sqrt{T}}$$

$$v_j = \frac{\log\left(\omega_j/S_j(0)\right) - \left(r - \gamma_j - \dfrac{\sigma_j^2}{2}\right)T}{\sigma_j\sqrt{T}},$$

$j = 1, \ldots, n$. Bearing in mind that expressions inside the second parenthesis of (37) remain the same if $u_j$ is replaced by $-u_j$ and $v_j$ is replaced by $-v_j$, one can take

$$v_j = \frac{\log\left(S_j(0)/\omega_j\right) + \left(r - \gamma_j - \frac{\sigma_j^2}{2}\right)T}{\sigma_j\sqrt{T}} \tag{38}$$

$$u_j = \frac{\log\left(S_j(0)/\omega_j\right) + \left(\mu_j - \gamma_j - \dfrac{\sigma_j^2}{2}\right)T}{\sigma_j\sqrt{T}} = v_j + \frac{\mu_j - r}{\sigma_j}\sqrt{T} \tag{39}$$

$j = 1, \ldots, n$, $\omega_j > 0$. If one considers the Sharpe ratios

$$\mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_n) = \left(\frac{\mu_1 - r}{\sigma_1}, \ldots, \frac{\mu_n - r}{\sigma_n}\right),$$

then, bearing in mind that $\rho^{-1}$ is symmetric ($\rho^{-1} = \left(\rho^{-1}\right)'$), (39) implies that

$$\begin{cases} u\rho^{-1}u' - v\rho^{-1}v' = \left(v + \sqrt{T}\mathcal{R}\right)\rho^{-1}\left(v + \sqrt{T}\mathcal{R}\right)' - v\rho^{-1}v' \\ = 2\sqrt{T}v\rho^{-1}\mathcal{R}' + T\mathcal{R}\rho^{-1}\mathcal{R}', \end{cases}$$

and therefore, (37)–(39) imply that

$$\begin{cases} z_{\Pi} = Exp\left(\sqrt{T} v \rho^{-1} \mathcal{R}' + \frac{T}{2} \mathcal{R} \rho^{-1} \mathcal{R}'\right) \\ v = (v_1, \ldots v_n) \\ v_j = \frac{\log(S_j(0)/\omega_j) + \left(r - \gamma_j - \sigma_j^2/2\right)T}{\sigma_j \sqrt{T}}, \quad j = 1, \ldots, n, \quad \omega_j > 0 \\ \mathcal{R} = (\mathcal{R}_1, \ldots, \mathcal{R}_n) = \left(\frac{\mu_1 - r}{\sigma_1}, \ldots, \frac{\mu_n - r}{\sigma_n}\right), \quad j = 1, \ldots, n \end{cases} \tag{40}$$

which provides us with the $SDF$ of the model.

### 5.3. The Optimal Expected Shortfall-Linked Golden Strategy

(40) obviously implies that all the conditions imposed in Theorem 3 are satisfied unless $\mu_j = r$, $j = 1, \ldots, n$, in which case $z_{\Pi} = 1$ (or $\mathbb{Q} = \mathbb{P}$), and the model is risk-neutral. Let us suppose that $\mu_j \neq r$ for at least one risky asset. The equality

$$\|z_{\Pi}\| = Ess\_Sup(z_{\Pi}) = +\infty \tag{41}$$

easily follows from (40), and therefore, Corollary 1 implies the existence of $ES_{1-\beta^*}$−golden strategies for every $0 < \beta^* < 1$. More accurately, one has

**Theorem 5.** *Suppose that $\mu_j \neq r$ for at least one risky asset. Then, the following hold:*

*$(a)$ There are $ES_{1-\beta^*}$−golden strategies for every $0 < \beta^* < 1$, and $\tilde{y}_{\beta^*} = \mathbb{1}_{z_{\Pi} > 1/\beta^*}$ is the optimal one, where $z_{\Pi}$ is given by (40).*

*$(b)$ Consider the row matrix $\mathcal{R}^{(\rho} = \left(\mathcal{R}_1^{(\rho}, \ldots, \mathcal{R}_n^{(\rho}\right) = \mathcal{R}\rho^{-1}$. There exists $j \in \{1, \ldots, n\}$ such that $\mathcal{R}_j^{(\rho} \neq 0$. If $\mathcal{R}_j^{(\rho} > 0$, then $\tilde{y}_{\beta^*}(\omega) = 1$ if and only if*

$$\begin{cases} \log(\omega_j) < \\ \log(S_j(0)) + \left(r - \gamma_j - \sigma_j^2/2\right)T \\ + \frac{\sigma_j}{\mathcal{R}_j^{(\rho}} \sum_{i \neq j} \frac{\mathcal{R}_i^{(\rho}}{\sigma_i} (\log(S_i(0)/\omega_i) + (r - \gamma_i - \sigma_i^2/2)T) \\ + \frac{\sigma_j}{\mathcal{R}_j^{(\rho}} \left(\frac{T}{2} \mathcal{R}\rho^{-1}\mathcal{R}' + \log(\beta^*)\right) \end{cases} \tag{42}$$

*for $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}_+^n$. If $\mathcal{R}_j^{(\rho} < 0$, then $\tilde{y}_{\beta^*}(\omega) = 1$ if and only if*

$$\begin{cases} \log(\omega_j) > \\ \log(S_j(0)) + \left(r - \gamma_j - \sigma_j^2/2\right)T \\ + \frac{\sigma_j}{\mathcal{R}_j^{(\rho}} \sum_{i \neq j} \frac{\mathcal{R}_i^{(\rho}}{\sigma_i} (\log(S_i(0)/\omega_i) + (r - \gamma_i - \sigma_i^2/2)T) \\ + \frac{\sigma_j}{\mathcal{R}_j^{(\rho}} \left(\frac{T}{2} \mathcal{R}\rho^{-1}\mathcal{R}' + \log(\beta^*)\right) \end{cases} \tag{43}$$

*for* $\omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n_+$.

**Proof.** $(a)$ The obvious consequence of Corollary 1 and (41).

$(b)$ Obviously, $\mathcal{R}\rho^{-1} = 0 \implies \mathcal{R} = 0 \implies \mu_j = r,\ j = 1, \ldots, n$, against the assumptions. Suppose that $\mathcal{R}^{(\rho}_j > 0$.

$$\tilde{y}_{\beta^*}(\omega) = 1 \iff z_\Pi > 1/\beta^* \iff \log(z_\Pi) > -\log(\beta^*),$$

and therefore, (40) leads to

$$\tilde{y}_{\beta^*}(\omega) = 1 \iff v\left(\mathcal{R}^{(\rho}\right)' > -\left(\frac{\sqrt{T}}{2}\mathcal{R}^{(\rho}\mathcal{R}' + \frac{1}{\sqrt{T}}\log(\beta^*)\right).$$

The third equality in (40) and straightforward manipulations imply that (42) is equivalent to $\tilde{y}_{\beta^*}(\omega) = 1$. Besides, the proof of (43) is similar. □

**Remark 3.** *If one were dealing with future derivatives rather than spot ones, then it is known that (31), (32) and (35) become*

$$\begin{cases} dF_j = F_j\left((\mu_j - r)dt + \sigma_j dW^*_j\right) \\ F_j(T) = F_j(0)Exp\left(\left(\mu_j - r - \frac{\sigma_j^2}{2}\right)T + \sigma_j\sqrt{T}W_j\right) \\ dF_j = F_j\sigma_j d\tilde{W}^*_j \\ F_j(T) = F_j(0)Exp\left(\left(-\frac{\sigma_j^2}{2}\right)T + \sigma_j\sqrt{T}W_j\right) \end{cases}$$

*Thus, straightforward modifications of the arguments above imply that the right-hand side of (42) and (43) will become*

$$\begin{cases} \log(F_j(0)) - \left(\sigma_j^2/2\right)T \\ +\frac{\sigma_j}{\mathcal{R}^{(\rho}_j}\sum_{i\neq j}\frac{\mathcal{R}^{(\rho}_i}{\sigma_i}\left(\log(F_i(0)/\omega_i) - (\sigma_i^2/2)T\right) \\ +\frac{\sigma_j}{\mathcal{R}^{(\rho}_j}\left(\frac{T}{2}\mathcal{R}\rho^{-1}\mathcal{R}' + \log(\beta^*)\right) \end{cases} \tag{44}$$

*and one has the optimal* $ES_{1-\beta^*}$*−golden strategy for future derivatives.*

**Remark 4.** *If* $n = 1$*, then straightforward manipulations of (42) or (43) easily imply that, under the obvious notation,* $\tilde{y}_{\beta^*}$ *is the binary put (respectively, call) with strike*

$$k = S(0)e^{\left(\frac{\mu + r - \sigma^2}{2} - \gamma\right)T}(\beta^*)^{\sigma^2/(\mu - r)} \tag{45}$$

*if $\mu > r$ (respectively, $\mu < r$). Notice also that (45) leads to*

$$\beta^* = \left[ (k/S(0)) e^{\left( \frac{\sigma^2 - \mu - r}{2} + \gamma \right) T} \right]^{(\mu - r)/\sigma^2}, \tag{46}$$

*that is, if (46) generates a value $\beta^* \in (0,1)$, then, given the strike of a digital option (put if $\mu > r$, call if $\mu < r$), one can compute the level of confidence making this option an optimal $ES_{1-\beta^*}$−golden strategy. Besides, (45) and (46) become*

$$\begin{cases} k = F(0) e^{\left( \frac{\mu - r - \sigma^2}{2} \right) T} (\beta^*)^{\sigma^2/(\mu - r)} \\ \beta^* = \left[ (k/F(0)) e^{\left( \frac{\sigma^2 - \mu + r}{2} \right) T} \right]^{(\mu - r)/\sigma^2} \end{cases}$$

*for future options.*

**Remark 5.** *If $n = 2$, then*

$$\begin{cases} \rho^{-1} = \frac{1}{1 - \rho_{1,2}^2} \begin{pmatrix} 1, & -\rho_{1,2} \\ -\rho_{1,2}, & 1 \end{pmatrix} \\ \mathcal{R}^{(\rho} = \frac{1}{1 - \rho_{1,2}^2} (\mathcal{R}_1 - \rho_{1,2}\mathcal{R}_2, \mathcal{R}_2 - \rho_{1,2}\mathcal{R}_1)' \\ \mathcal{R}\rho^{-1}\mathcal{R}' = \frac{\mathcal{R}_1^2 + \mathcal{R}_2^2 - 2\rho_{1,2}\mathcal{R}_1\mathcal{R}_2}{1 - \rho_{1,2}^2}. \end{cases} \tag{47}$$

*Since there are several potential scenarios, let us shorten the exposition and consider the most common case $\mathcal{R}_2 > \mathcal{R}_1 > 0$. In order to simplify the notation, suppose also that one is dealing with future derivatives. (42), (44), and (47) easily lead to*

$$\frac{\omega_2}{F_2(0)} < C \left( \frac{\omega_1}{F_1(0)} \right)^{-\frac{\sigma_2(\mathcal{R}_1 - \rho_{1,2}\mathcal{R}_2)}{\sigma_1(\mathcal{R}_2 - \rho_{1,2}\mathcal{R}_1)}} (\beta^*)^{\frac{\sigma_2\left(1 - \rho_{1,2}^2\right)}{\mathcal{R}_2 - \rho_{1,2}\mathcal{R}_1}} \tag{48}$$

*where the parameter $C > 0$ depends on $(r, \mu_1, \mu_2, \sigma_1, \sigma_2, \rho_{1,2}, T)$; that is, $C$ is not affected by $\omega_j / F_j(0)$, $j = 1,2$. The subset of $\mathbb{R}_+^2$ generated by (48) clearly depends on the sign of $\mathcal{R}_1 - \rho_{1,2}\mathcal{R}_2$. If $\mathcal{R}_1 - \rho_{1,2}\mathcal{R}_2 > 0$ (for instance, if $\rho_{1,2}$ vanishes), then $\omega_2 / F_2(0)$ must be lying under a curve tending to infinity as $\omega_1 / F_1(0)$ tends to zero. If $\mathcal{R}_1 - \rho_{1,2}\mathcal{R}_2 = 0$, then $\omega_2 / F_2(0)$ must belong to the interval*

$$\left( 0, C (\beta^*)^{\frac{\sigma_2\left(1 - \rho_{1,2}^2\right)}{\mathcal{R}_2 - \rho_{1,2}\mathcal{R}_1}} \right),$$

*that is, $\tilde{y}$ is a digital put whose unique underlying asset is that with the highest Sharpe ratio. Lastly, if $\mathcal{R}_1 - \rho_{1,2}\mathcal{R}_2 < 0$, then $\omega_2/F_2(0)$ must be lying under a curve tending to zero as $\omega_1/F_1(0)$ tends to zero.*

**Remark 6.** *Remarks* 4 *and* 5 *show that it is worth involving several underlying securities in the same derivative. If $n = 2$, $\mathcal{R}_2 > \mathcal{R}_1 > 0$, and one separately deals with $S_1$ and $S_2$, then Remark* 4 *implies that the best choice is a couple of digital puts: one per asset. By contrast, if $S_1$ and $S_2$ are integrated, then (48) shows that one can beat the use of a digital put per security. In other words, Remarks* 4 *and* 5 *confirm that the answer to Question Q in the introduction is "yes". This is important because, as indicated in the introduction, the major objective of this paper is to verify that the use of multi-asset options improves the efficiency of a golden strategy.*

**Remark 7.** *As pointed out, the main purpose of this paper is theoretical, and any empirical study would significantly enlarge the content and would be beyond its scope. The empirical test remains to be investigated in the future. At any rate, it is worthwhile to recall that an empirical test involving derivatives with a single underlying asset was implemented in Balbás et al. (2016), and the results are clear: "the golden strategy beats very important international stock indices", and this finding remains true if the usual market imperfections and model limitations (market depth, bid–ask spread, absence of a constant riskless rate, absence of constant drifts and volatilities, etc.) are incorporated. It is also noteworthy that the authors tested an $ES_{1-\beta^*}$−golden strategy constructed with BSM which was not optimal; that is, they did not deal with $\mathbb{1}_{z_\Pi > 1/\beta^*}$. Consequently, there are two reasons to be optimistic with respect to the empirical performance of the multi-asset option of this paper. Firstly, as pointed out in Remark* 6*, the multi-asset option is better than a combination of options with a single underlying asset. Secondly, in this paper, the optimal golden strategy has been found.*

**Remark 8.** *Although empirical tests about the efficiency of $\mathbb{1}_{z_\Pi > 1/\beta^*}$ are left for future studies, one can summarize how they can be carried out. The methodology of Balbás et al. (2016) can be generalized. Indeed, since BSM is complete, these authors selected a quite liquid futures contract in order to replicate their golden derivative. For instance, if the selected underlying asset is the S&P* 500 *index, then their golden strategy was replicated by holding $\delta$ futures, where $\delta$ is the usual delta Greek of the golden strategy to be synthetically constructed. Once a day, they modified their position in the futures contract according to the modification of $\delta$. Under the present framework, that is, for multi-asset options, one has several deltas, that is, one delta per underlying security, so let us briefly summarize a simple way to estimate them all. In order to simplify the mathematical exposition, let us assume that there are only two underlying securities. With the obvious notation, the price of $\mathbb{1}_{z_\Pi > 1/\beta^*}$ takes the form of a double integral (recall (2))*

$$e^{-rT} \iint z_\Pi(\tilde{\omega}_1, \tilde{\omega}_2) \mathbb{1}_{z_\Pi(\tilde{\omega}_1, \tilde{\omega}_2) > 1/\beta^*} f_\rho(\omega_1, \omega_2) J(\omega_1, \omega_2) d\omega_1 d\omega_2, \tag{49}$$

*where $f_\rho$ is given by (33), $z_\Pi$ is given by (40), and $J$ is the determinant of the Jacobian matrix*

$$\begin{pmatrix} \partial\tilde{\omega}_1/\partial\omega_1, & \partial\tilde{\omega}_1/\partial\omega_2 \\ \partial\tilde{\omega}_2/\partial\omega_1, & \partial\tilde{\omega}_2/\partial\omega_2 \end{pmatrix}.$$

*The usual Cholesky decomposition (Gentle 1998) provides us with a linear change in variables transforming (49) into a new integral of the type*

$$k \iint z_\Pi(\tilde{\omega}'_1, \tilde{\omega}'_2) \mathbb{1}_{z_\Pi(\tilde{\omega}'_1, \tilde{\omega}'_2) > 1/\beta^*} f_I(\omega'_1, \omega'_2) \tilde{J}(\omega_1, \omega_2) d\omega'_1 d\omega'_2, \tag{50}$$

*for some constant k, $f_I$ similar to $f_\rho$ if $\rho$ is replaced by the identity matrix and $\tilde{J}$ generated by several involved Jacobians. Then, a second change in variables $\omega'_i = \Phi^{-1}(\omega''_i)$, $i = 1, 2$ transforms (50) into a new integral*

$$\int_0^1 \int_0^1 U(\tilde{\omega}''_1, \tilde{\omega}''_2) d\omega''_1 d\omega''_2,$$

*for some function $U$, with $\Phi$ being the cumulative distribution function of the standard normal distribution. $U(\tilde{\omega}''_1, \tilde{\omega}''_2)$ obviously depends on $(S_1(0), S_2(0))$ (recall (40)), and therefore,*

$$\delta_i = \int_0^1 \int_0^1 \frac{\partial U}{\partial S_i(0)} d\omega''_1 d\omega''_2,$$

*where $i = 1, 2$. Thus, $\delta_i$ is an integral that may be easily estimated by means of Monte Carlo simulations if $(\omega''_1, \omega''_2)$ is simulated as a couple of independent uniform distributions in the interval $(0, 1)$. Once $(\delta_1, \delta_2)$ is known, the generalization of Balbás et al. (2016) consists in holding $\delta_i$ futures of $S_i$, $i = 1, 2$ and rebalancing the number of futures once a day according to the dynamic evolution of $(\delta_1, \delta_2)$. The position in the riskless asset is fixed at the initial date $t = 0$.*

**Remark 9.** *The stochastic discount factor (40) may deserve a final comment. Indeed, in more general studies involving investment and consumption, the stochastic discount factor may be quite difficult to estimate because it may critically depend on various macroeconomic conditions (Duffie 1988). Nevertheless, the use of BSM has allowed us to overcome this difficulty. BSM does not involve consumption and is complete as a pricing model, which has implied that the closed formula (40) has been given for $z_\Pi$. In other words, the estimation of $z_\Pi$ has become simple in our particular setting.*

## 6. Conclusions

The existence of expected shortfall-linked and expectile-linked golden strategies has been deeply studied, and it has been pointed out that this existence often holds. These strategies are very important for practitioners because they allow us to create self-financing positions with negative risks. If a golden strategy is implemented jointly with another one, both risk and return are improved. Tractable (probably infinite-dimensional) linear programming problems have been presented to detect the expectile-linked golden strategies, and a closed formula has been given for the expected shortfall. This closed formula has been particularized for the Black–Scholes–Merton multi-dimensional model, and an important consequence has been obtained: the optimal golden strategy is a multi-asset option; that is, multi-asset-options allow us to beat portfolios composed of options with a single underlying asset.

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