

Quantile Credibility Models with Common Effects

Wei Wang ¹, Limin Wen ², Zhixin Yang ^{3,*} and Quan Yuan ³ ¹ Department of Financial Engineering, Ningbo University, Ningbo 315211, China; wangwei2@nbu.edu.cn² School of Mathematics and Statistics, Jiangxi Normal University, Nanchang 330022, China; wlmjxnu@163.com³ Department of Mathematical Sciences, Ball State University, Muncie, IN 47306, USA; qyuan@bsu.edu

* Correspondence: zyang6@bsu.edu

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Abstract: Different from classical Bühlmann and Bühlmann Straub credibility models in which independence between different risks are assumed, this paper takes dependence between risks into consideration and extends the classical Bühlmann model by introducing a common stochastic shock element. What is more, instead of relying on complete information of historical data, we aim to derive the premium using quantile of the available data. By the method of linear regression, we manage to obtain the quantile credibility premium with common effects. Our result is the generalization of existing results in credibility theory. Both quantile credibility model proposed by Pitselis (2013) and credibility premium for models with dependence induced by common effects obtained by Wen et al. (2009) are special cases of our model. Numerical simulations are also presented to illustrate the impact of quantile credibility with common effect.

Keywords: credibility premium; quantile model; common effect

1. Introduction

Credibility theory provides a fundamental approach for pricing insurance product. It was first introduced to actuarial science as a measure of credence to be attached to a particular body of experience for rate making purposes. A 0 credibility means the data is too small to be of any usage and 1 refers to the case that the data is fully credible for rate making. It is concerned with establishing measures of credibility and standards of full credibility. The earliest paper on the standard for full credibility was presented by Mowbray (1914), in the Proceedings of the Casualty Actuarial Society. When an estimate is to be made using newly acquired data, an important question that classical credibility theory answers is how to reasonably combine the data based on past experience and not fully credible new data to decide premiums to be charged. If there is enough recent data, full credibility is achieved; the prediction will be based on the recent data only accordingly. Otherwise, only partial credibility is attributed to the data and the prediction depends on the manual rate too. The problem with classical credibility is that it is based on arbitrary selection of the coverage probability and the accuracy parameter. Besides, the assumption of loss distribution has to be imposed ahead of time. Later on, a distribution free credibility estimation method is introduced by Bühlmann. It is an important breakthrough in the development of credibility theory and Bühlmann uses rigorous statistical framework of optimal prediction to study credibility premium. Specifically, let X_1, X_2, \dots, X_k denote k risk groups, $\Theta_1, \Theta_2, \dots, \Theta_k$ denote k random variables associated with the risk groups. It is assumed that the pair of random variables $(X_j, \Theta_j), j = 1, 2, \dots, k$ are independent across individuals (independent over risks) and for each j , $(X_{j1}, X_{j2}, \dots, X_{jN})$ are conditionally independent given Θ_j (conditionally independence over time). For selected risk group j , each X_j has the same unknown probability distribution for any time period, both for the experienced periods (X_{j1}, \dots, X_{jN}) , and for future outcomes $(X_{j(N+1)}, X_{j(N+2)}, \dots)$. Θ_j is constant through time for the risk group j .

According to Bühlmann, the average values of the mean and variance among risks are assumed to be known. The variance of the hypothetical means for the population is available as well. Bühlmann utilizes both the variability to be expected from observations and differences in the means among risks in the population to determine the possible future loss of risk. The credibility premium showed by Bühlmann is given as $\hat{\mu}(\Theta_j) = Z\bar{X}_j + (1 - Z)\mu$, where μ is the overall mean for k risk groups and Z is a proper weight to the sample mean \bar{X}_j of group j . The way of calculating the weight Z is actually an optimization problem: $\min E[(Z\bar{X}_j + (1 - Z)\mu - \mu(\Theta_j))^2]$ in which $\mu(\Theta_j)$ is the mean of the risk group j for different time periods. Note that in reality, the assumptions of independently identically distribution of random variables $\{X_{j1}, X_{j2}, \dots, X_{jN}, \dots\}$ are often violated. Bühlmann and Straub (1970) address this issue by assuming the means of the random variables are equal for the selected risk, but the process variances are inversely proportional to the size of the risk during each observation period and thus develop the credibility model with weights.

Realizing that more and more empirical studies have shown that under certain scenario cases, conditional dependence of claims on time have been observed to be more appropriate to reflect the reality, see for example, Bolancé and Bolancé (2003); Purcaru and Denuit (2002, 2003); Frees et al. (1999, 2001) and references therein. Under other situations, we find out that claims are dependent on risk categories. For example, house insurance for which geographic proximity of the insureds may result in exposures to common catastrophes. See e.g., Wu and Zhou (2006); Wang (1998); Wang et al. (1997) and the references therein. With regard to credibility pricing, there are few scholars who have examined such problems that claims depend on individual risks; the only available ones are Wen et al. (2009) and Yeo and Valdez (2006).

On the other hand, the application of quantile in actuarial science has drawn some scholars' attention. The classical examples are the quantile risk measure or tail value at risk. Reference Pitt (2006) shows the importance of modeling quantiles given the growing interest of regulators and others who use stochastic approaches to evaluate insurance liabilities and risk margins. Reference Pitselis (2013) examines the quantile credibility model after first introduced the idea of quantile into the framework of credibility at Insurance: Mathematics and Economics (IME) conference in 2007. In the field of rate-making, reference Kudryavtsev (2009) models quantile regression with safety loading and describes the advantages of the quantile regression approach. Besides, reference Pitselis (2016) and Pitselis (2017) discuss applications of quantile credibility in risk measure. For the credibility estimator, in addition to the classical ordinary least-square estimation, the quantile regression is proposed by reference Bozikas and Pitselis (2020) to address the non-normal error distributions and contaminated data due to outlier events. Reference Pitselis (2020) studies the credibility models for regression quantiles with hierarchical classifications.

Our interest is to study the quantile credibility model under common effects and the rest of paper is organized as follows. Section 2 presents the problem formulation. Section 3 derives the quantile credibility with common effects and some asymptotic behaviors are briefly discussed. In Section 4, numerical simulations are presented to demonstrate our results. Section 5 concludes the paper with some further remarks.

2. Formulation

In this section, we introduce the assumptions and notation we need in our model. We assume that $X = (X_1, X_2, \dots, X_k)$ denote k risk groups under observation and let $\Theta = (\Theta_1, \Theta_2, \dots, \Theta_k)$ denote k random variables associated with the risk groups. The k risk categories can be k individuals or things of the same nature. In our framework, we assume that there is a type of dependence among k individuals and this is delineated by a common effect random variable. For a given risk type $j, j = 1, 2, \dots, k$, the distribution of X_j is based on the risk parameter Θ_j and common effect random variable Λ . We use $\mathbb{1}_k = (1, 1, \dots, 1)'$ to denote a k dimensional column vector with 1 in all of the k entries and use \mathbb{I}_k to denote a $k \times k$ identity matrix. In particular, $e_j = (0, \dots, 1, \dots, 0)'$ is a standard unit column vector with the j th component being 1 and the rest of the components being 0. For the

sake of convenience, we also assume that for a given j , $X_j = (X_{j1}, X_{j2}, \dots, X_{jn})'$, that means the time horizons for any group j are the same. Furthermore, we assume that

1. The common effect random variable Λ has known expectation $E(\Lambda) = \mu_\Lambda$ and variance $Var(\Lambda) = \sigma_\Lambda^2$.
2. Given Λ , the random vectors (X_j, Θ_j) are mutually independent and identically distributed.
3. Given risk category j , the common effect Λ and parameter Θ_j , the claims $X_{j1}, X_{j2}, \dots, X_{jn}$ are conditionally independent and identically distributed (i.i.d.) with distribution function $F_j(X_j|\Theta_j, \Lambda)$.

The p th quantile of a risk variable X_j with cumulative distribution function $F_j(x)$ is defined as $\zeta_{pj} = F_{pj}^{-1} = \inf\{x, F_j(x|\Theta_j, \Lambda) \geq p\}$. A moment of reflections reveals that $F_j(x) \geq p$ if and only if $\zeta_{pj} \leq x$. Putting all the p th quantile for the k risk categories together, we have $\zeta_p = (\zeta_{p1}, \zeta_{p2}, \dots, \zeta_{pk})'$. When the distribution of X_j is not specified, the natural distribution free estimator of p th quantile, $\hat{\zeta}_{pj}$, is the empirical p th quantile, $\hat{\zeta}_{pj}$. Similarly we have $\hat{\zeta}_p = (\hat{\zeta}_{p1}, \hat{\zeta}_{p2}, \dots, \hat{\zeta}_{pk})'$. Given a sample $X_{j1}, X_{j2}, \dots, X_{jn}$ of a continuous random variable X_j , the empirical distribution function can be defined as

$$F_n(\zeta_{pj}) = \frac{\sum_{i=1}^n \mathbb{1}\{X_{ji} \leq \zeta_{pj}\}}{n}.$$

The corresponding empirical quantile function can be defined as

$$\hat{\zeta}_{pj} = n\left(\frac{i}{n} - p\right)X_{j(i-1)} + n\left(p - \frac{i-1}{n}\right)X_{j(i)} \text{ for } \frac{i-1}{n} \leq p \leq \frac{i}{n}, i = 1, 2, \dots, n.$$

More detailed information of quantile can be referred to [Herbert and Nagaraja \(2003\)](#) or [Parzen \(1979\)](#).

To proceed, we use the following assumptions to describe the center and dispersion of the observation $\hat{\zeta}_{pj}$.

4. $E[\hat{\zeta}_{pj}|\Theta_j, \Lambda] = \Xi_p(\Theta_j, \Lambda)$ and $Var(\hat{\zeta}_{pj}|\Theta_j, \Lambda) = \sigma_p^2(\Theta_j, \Lambda)$,
 $E[\sigma_p^2(\Theta_j, \Lambda)|\Lambda] = \sigma_{1p}^2(\Lambda)$ and $E[\sigma_{1p}^2(\Lambda)] = \sigma_{1p}^2$.

Having observed the risks for n years of experience, we want to estimate $\Xi_p(\Theta_j, \Lambda)$, which can be interpreted as the risk premium at p th quantile.

The structure parameters assume the following conditions:

5. $E[\Xi_p(\Theta_j, \Lambda)|\Lambda] = \Xi_p(\Lambda)$, $E[\Xi_p(\Lambda)] = \Xi_p$, $Var[\Xi_p(\Lambda)] = \sigma_{0p}^2$ and
 $Var[\Xi_p(\Theta_j, \Lambda)|\Lambda] = \sigma_{2p}^2(\Lambda)$, $E[\sigma_{2p}^2(\Lambda)] = \sigma_{2p}^2$.

To proceed, we first present the lemma below—it is a classical result in multi-variate analysis and more details can be found in [Wen et al. \(2009\)](#).

Lemma 1. Let $(X'_{1 \times p}, Y'_{1 \times p})$ be a random vector with expectation (μ'_X, μ'_Y) and covariance matrix

$$\Xi = \begin{pmatrix} \Xi_{XX} & \Xi_{XY} \\ \Xi_{YX} & \Xi_{YY} \end{pmatrix} \tag{1}$$

then,

1. $E(Y - A - BX)(Y - A - BX)'$ can be minimized by $A = \mu_Y - \Xi_{YX}\Xi_{XX}^{-1}\mu_X$ and $B = \Xi_{YX}\Xi_{XX}^{-1}$.
2. Assumed that $\mu_Y = B\mu_X$, then $E(Y - BX)(Y - BX)'$ can be minimized by

$$B = (\Xi_{YX} + \frac{(\mu_Y - \Xi_{YX}\Xi_{XX}^{-1}\mu_X)\mu'_X}{\mu'_X\Xi_{XX}^{-1}\mu_X})\Xi_{XX}^{-1}.$$

Classical Bühlmann credibility theory and Bühlmann Straub credibility theory are essentially optimization problems. Our main interest lies in the framework of quantile credibility, so we briefly

show its derivation as below. Note that quantile credibility is a special case of our model when assuming that Λ is a constant. Since there is no common effects we just need to consider risk groups for derivation.

$$Y = \Xi_p(\Theta), X = \hat{\xi}_p, \mu_Y = \mu_X = \Xi_p \mathbb{1}_k,$$

and

$$\sigma_p^2(\Theta) \mathbb{I}_k = \text{Var}(\hat{\xi}_p | \Theta), E[\sigma_p^2(\Theta)] = \sigma_{1p}^2 \mathbb{I}_k, \sigma_{0p}^2 \mathbb{I}_k = \text{Var}(\Xi_p(\Theta))$$

therefore,

$$\Xi_{XX} = \text{Var}(\hat{\xi}_p) = E\sigma_p^2(\Theta) + \text{Var}(\Xi_p(\Theta)) = (\sigma_{1p}^2 + \sigma_{0p}^2) \mathbb{I}_k.$$

and

$$\Xi_{YX} = \text{Cov}(\Xi_p(\Theta), \hat{\xi}_p) = (\Xi_p^2 + \sigma_{0p}^2 - \Xi_p^2) \mathbb{I}_k = \sigma_{0p}^2 \mathbb{I}_k.$$

According to the lemma above, we just need to combine what we have shown above:

$$\mu_Y + \Xi_{YX} \Xi_{XX}^{-1} (X - \mu_X) = \Xi_p \mathbb{1}_k + \frac{\sigma_{0p}^2}{\sigma_{1p}^2 + \sigma_{0p}^2} \mathbb{I}_k (\hat{\xi}_p - \Xi_p \mathbb{1}_k) = \frac{\sigma_{1p}^2}{\sigma_{1p}^2 + \sigma_{0p}^2} \Xi_p \mathbb{1}_k + \frac{\sigma_{0p}^2}{\sigma_{1p}^2 + \sigma_{0p}^2} \hat{\xi}_p \mathbb{1}_k$$

and thus reproduces the result established by [Pitselis \(2013\)](#).

3. Credibility Models

To carry out our premium calculations, we first present a couple of lemmas below.

Lemma 2. Assume the Assumptions (1)–(5) hold, then we have the following results:

1. $E[\hat{\xi}_p] = \Xi_p \mathbb{1}_k$
2. $\Xi_{\hat{\xi}_p \hat{\xi}_p} = \sigma_{1p}^2 \mathbb{I}_k + \sigma_{2p}^2 \mathbb{I}_k + \sigma_{0p}^2 \mathbb{1}_k \mathbb{1}_k'$
3. $\Xi_{\hat{\xi}_p \hat{\xi}_p}^{-1} = \frac{1}{\sigma_{1p}^2 + \sigma_{2p}^2} \left(\mathbb{I}_k - \frac{\sigma_{0p}^2 \mathbb{1}_k \mathbb{1}_k'}{\sigma_{1p}^2 + \sigma_{2p}^2 + k \sigma_{0p}^2} \right)$
4. $\Xi_{\hat{\xi}_p, \hat{\xi}_p} = \sigma_{2p}^2 e_j' + \sigma_{0p}^2 \mathbb{1}_k'$

Proof. For part (1), we have

$$\begin{aligned} E[\hat{\xi}_p] &= E[E[\hat{\xi}_p | \Theta, \Lambda]] \\ &= EE[(\hat{\xi}_{p1}, \hat{\xi}_{p2}, \dots, \hat{\xi}_{pk})' | \Theta, \Lambda] \\ &= E(\Xi_p(\Theta_1, \Lambda), \Xi_p(\Theta_2, \Lambda), \dots, \Xi_p(\Theta_k, \Lambda))' = (\Xi_p, \Xi_p, \dots, \Xi_p)' = \Xi_p \mathbb{1}_k. \end{aligned} \tag{2}$$

Regarding part (2), realizing that

$$\Xi_{\hat{\xi}_p, \hat{\xi}_p} = E[\text{Var}(\hat{\xi}_p | \Theta, \Lambda)] + \text{Var}[E(\hat{\xi}_p | \Theta, \Lambda)]. \tag{3}$$

$$\text{Var}(\hat{\xi}_p | \Theta, \Lambda)_{k \times k} = \begin{pmatrix} \sigma_p^2(\Theta_1, \Lambda) & 0 & \dots & 0 \\ 0 & \sigma_p^2(\Theta_2, \Lambda) & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sigma_p^2(\Theta_k, \Lambda) \end{pmatrix}. \tag{4}$$

Therefore,

$$E[\text{Var}(\hat{\xi}_p | \Theta, \Lambda)] = \sigma_{1p}^2 \mathbb{I}_k. \tag{5}$$

On the one hand,

$$\begin{aligned}
 \text{Var}(E(\hat{\xi}_p|\Theta, \Lambda)) &= E\text{Var}[E(\hat{\xi}_p|\Theta, \Lambda)|\Lambda] + \text{Var}E[E(\hat{\xi}_p|\Theta, \Lambda)|\Lambda] \\
 &= E[\text{Var}(\Xi_p(\Theta_1, \Lambda), \Xi_p(\Theta_2, \Lambda), \dots, \Xi_p(\Theta_k, \Lambda)|\Lambda)] \\
 &\quad + \text{Var}E[(\Xi_p(\Theta_1, \Lambda), \Xi_p(\Theta_2, \Lambda), \dots, \Xi_p(\Theta_k, \Lambda)|\Lambda)] \\
 &= E(\sigma_{2p}^2(\Lambda))\mathbb{I}_k + \text{Var}(\Xi_p(\Lambda)\mathbb{1}'_k) \\
 &= \sigma_{2p}^2\mathbb{I}_k + \sigma_{0p}^2\mathbb{1}_k\mathbb{1}'_k.
 \end{aligned}
 \tag{6}$$

Combining the results of Equations (5) and (6), we finish the proof of second part above. Based on the result of part (2), we can easily verify that

$$\Xi_{\hat{\xi}_p\hat{\xi}_p}^{-1} = \frac{1}{\sigma_{1p}^2 + \sigma_{2p}^2} (\mathbb{I}_k - \frac{\sigma_{0p}^2\mathbb{1}_k\mathbb{1}'_k}{\sigma_{1p}^2 + \sigma_{2p}^2 + k\sigma_{0p}^2})
 \tag{7}$$

Last part of the lemma can be shown as follows:

$$\begin{aligned}
 \Xi_{\hat{\xi}_{pj}\hat{\xi}_p} &= E[\text{Cov}(\hat{\xi}_{pj}, \hat{\xi}_p|\Theta, \Lambda)] + \text{Cov}[E(\hat{\xi}_{pj}|\Theta, \Lambda), E(\hat{\xi}_p|\Theta, \Lambda)] \\
 &= \text{Cov}(\Xi_p(\Theta_j, \Lambda), [\Xi_p(\Theta_1, \Lambda), \Xi_p(\Theta_2, \Lambda), \dots, \Xi_p(\Theta_k, \Lambda)]') \\
 &= E[\text{Cov}(\Xi_p(\Theta_j, \Lambda), (\Xi_p(\Theta_1, \Lambda), \Xi_p(\Theta_2, \Lambda), \dots, \Xi_p(\Theta_k, \Lambda))'|\Lambda)] \\
 &\quad + \text{Cov}(E(\Xi_p(\Theta_j, \Lambda)|\Lambda), E(\Xi_p(\Theta_1, \Lambda)|\Lambda), \dots, E(\Xi_p(\Theta_k, \Lambda)|\Lambda)) \\
 &= \sigma_{2p}^2 e'_j + \text{Cov}(\Xi_p(\Lambda), \Xi_p(\Lambda)\mathbb{1}_k) \\
 &= \sigma_{2p}^2 e'_j + \sigma_{0p}^2\mathbb{1}'_k.
 \end{aligned}
 \tag{8}$$

In our work, there is a common effects random variable Λ . This new feature makes the credibility premium calculation procedures remarkably different than the previous work. To proceed, we define $\tilde{\xi}_p = \frac{\sum_{j=1}^k \hat{\xi}_{pj}}{k}$ as the overall average p th quantile of claim experience of all individuals, note that we do not use the information of average claims \bar{X}_j for a certain individual j to make our analysis, we use p th quantile instead, this is the essential difference between our work and the previous ones on credibility premium with common effects. With all these preparations, we can proceed with presenting the main theory now. \square

Theorem 1. Under Assumptions (1)–(5), the best unbiased premium in the sense of least mean square prediction error is $\hat{\xi}_{pj}z_1 + z_2\tilde{\xi}_p + z_3\Xi_p$ where $\tilde{\xi}_p = \frac{\sum_{j=1}^k \hat{\xi}_{pj}}{k}$. z_1, z_2, z_3 are credibility factors satisfying $z_1 = \frac{\sigma_{2p}^2}{\sigma_{1p}^2 + \sigma_{2p}^2}$, $z_2 = \frac{\sigma_{0p}^2\sigma_{1p}^2k}{(\sigma_{1p}^2 + \sigma_{2p}^2)(\sigma_{1p}^2 + \sigma_{2p}^2 + k\sigma_{0p}^2)}$ and $z_3 = 1 - z_1 - z_2$.

Proof. In the context of quantile credibility, the following objective function needs to be minimized:

$$E[\Xi_p(\Theta_j, \Lambda) - c_0 - \sum_{j=1}^k c_{pj}\hat{\xi}_{pj}]^2$$

Using the results of the lemmas above, we can represent the estimate as below:

$$\begin{aligned}
 \hat{\xi}_{pj}^C &= \hat{c}_0 + \sum_{j=1}^k \hat{c}_{pj} \hat{\xi}_{pj} \\
 &= \Xi_p + \frac{\sigma_{2p}^2 e'_j + \sigma_{0p}^2 \mathbb{1}'_k}{\sigma_{1p}^2 + \sigma_{2p}^2} [\mathbb{1}_k - \frac{\sigma_{0p}^2 \mathbb{1}_k \mathbb{1}'_k}{\sigma_{1p}^2 + \sigma_{2p}^2 + k\sigma_{0p}^2}] (\hat{\xi}_p - \Xi_p \mathbb{1}_k) \\
 &= \Xi_p + \frac{1}{\sigma_{1p}^2 + \sigma_{2p}^2} [\sigma_{2p}^2 (\hat{\xi}_{pj} - \Xi_p) + \sigma_{0p}^2 k (\tilde{\xi}_p - \Xi_p) \\
 &\quad - \frac{\sigma_{0p}^2 (\sigma_{2p}^2 k + k^2 \sigma_{0p}^2)}{\sigma_{1p}^2 + \sigma_{2p}^2 + k\sigma_{0p}^2} (\tilde{\xi}_p - \Xi_p)] \\
 &= \Xi_p + \frac{1}{\sigma_{1p}^2 + \sigma_{2p}^2} [\sigma_{2p}^2 (\hat{\xi}_{pj} - \Xi_p) + \frac{\sigma_{0p}^2 \sigma_{1p}^2 k}{\sigma_{1p}^2 + \sigma_{2p}^2 + k\sigma_{0p}^2} (\tilde{\xi}_p - \Xi_p)] \\
 &= \frac{\sigma_{2p}^2}{\sigma_{1p}^2 + \sigma_{2p}^2} \hat{\xi}_{pj} + \frac{\sigma_{0p}^2 \sigma_{1p}^2 k}{(\sigma_{1p}^2 + \sigma_{2p}^2)(\sigma_{1p}^2 + k\sigma_{0p}^2 + \sigma_{2p}^2)} \tilde{\xi}_p + (1 - \frac{\sigma_{2p}^2}{\sigma_{1p}^2 + \sigma_{2p}^2} \\
 &\quad - \frac{\sigma_{0p}^2 \sigma_{1p}^2 k}{(\sigma_{1p}^2 + \sigma_{2p}^2)(\sigma_{1p}^2 + k\sigma_{0p}^2 + \sigma_{2p}^2)}) \Xi_p = z_1 \hat{\xi}_{pj} + z_2 \tilde{\xi}_p + z_3 \Xi_p,
 \end{aligned} \tag{9}$$

where $z_1 = \frac{\sigma_{2p}^2}{\sigma_{1p}^2 + \sigma_{2p}^2}$, $z_2 = \frac{\sigma_{0p}^2 \sigma_{1p}^2 k}{(\sigma_{1p}^2 + \sigma_{2p}^2)(\sigma_{1p}^2 + k\sigma_{0p}^2 + \sigma_{2p}^2)}$ and $z_3 = 1 - \frac{\sigma_{2p}^2}{\sigma_{1p}^2 + \sigma_{2p}^2} - \frac{\sigma_{0p}^2 \sigma_{1p}^2 k}{(\sigma_{1p}^2 + \sigma_{2p}^2)(\sigma_{1p}^2 + k\sigma_{0p}^2 + \sigma_{2p}^2)}$ □

Comparing our results with quantile credibility premium with no common effects in the work of Pitselis (2013), the credibility premium without common effect is

$$\hat{\xi}_{pj}^Q = \hat{\xi}_{pj} z_1 + (1 - z_1) \Xi_p,$$

from our calculation above, we can see that the credibility premium with common effect is

$$\hat{\xi}_{pj}^C = \hat{\xi}_{pj} z_1 + \tilde{\xi}_p z_2 + (1 - z_1 - z_2) \Xi_p,$$

where the credibility factor z_1 stays the same.

Remark 1. It can be verified that

$$\hat{\xi}_{pj}^C = \hat{\xi}_{pj}^Q + z_2 (\tilde{\xi}_p - \Xi_p), \text{ where } z_2 = \frac{\sigma_{0p}^2 \sigma_{1p}^2 k}{(\sigma_{1p}^2 + \sigma_{2p}^2)(\sigma_{1p}^2 + k\sigma_{0p}^2 + \sigma_{2p}^2)}.$$

1. If $\sigma_{0p}^2 = 0$, $\hat{\xi}_{pj}^C = \hat{\xi}_{pj}^Q$, then the result of quantile credibility without common effects is a special case of our results.
2. With the information of common effects, $\hat{\xi}_{pj}^C$ is better than $\hat{\xi}_{pj}$, since its expected prediction error is smaller.

Theorem 2. Under Assumptions (1)–(5), if we have no access to the historical mean Ξ_p , then the best unbiased premium in the sense of least mean square prediction error is

$$\hat{\xi}_{pj}^{\tilde{C}} = z_1 \hat{\xi}_{pj} + (1 - z_1) \tilde{\xi}_p.$$

Proof. According to Lemma 1, when we have no knowledge of Ξ_p , we can simplify part (2) of Lemma 1 and find out the estimate of Ξ_p as below:

$$\begin{aligned} \hat{\Xi}_p &= \frac{\Xi_p^2 \mathbb{1}'_k}{\Xi_p \mathbb{1}'_k \Xi_p^{-1} \Xi_p \mathbb{1}_k} \Xi_p^{-1} \hat{\xi}_p \\ &= \frac{\mathbb{1}'_k \Xi_p^{-1} \hat{\xi}_p}{\mathbb{1}'_k \Xi_p^{-1} \mathbb{1}_k}. \end{aligned} \tag{10}$$

On the one hand,

$$\begin{aligned} \mathbb{1}'_k \Xi_p^{-1} \hat{\xi}_p &= (1, 1, \dots, 1) \frac{1}{\sigma_{1p}^2 + \sigma_{2p}^2} (\mathbb{1}_k - \frac{\sigma_{0p}^2 \mathbb{1}_k \mathbb{1}'_k}{\sigma_{1p}^2 + \sigma_{2p}^2 + k\sigma_{0p}^2}) \hat{\xi}_p \\ &= \frac{1}{\sigma_{1p}^2 + \sigma_{2p}^2} (1 - \frac{k\sigma_{0p}^2}{\sigma_{1p}^2 + \sigma_{2p}^2 + k\sigma_{0p}^2}) \mathbb{1}'_k \hat{\xi}_p \\ &= \frac{1}{\sigma_{1p}^2 + \sigma_{2p}^2 + k\sigma_{0p}^2} \sum_{j=1}^k \hat{\xi}_{pj}. \end{aligned} \tag{11}$$

On the other hand,

$$\begin{aligned} \mathbb{1}'_k \Xi_p^{-1} \mathbb{1}_k &= (1, 1, \dots, 1) \frac{1}{\sigma_{1p}^2 + \sigma_{2p}^2} (\mathbb{1}_k - \frac{\sigma_{0p}^2 \mathbb{1}_k \mathbb{1}'_k}{\sigma_{1p}^2 + \sigma_{2p}^2 + k\sigma_{0p}^2}) (1, 1, \dots, 1)' \\ &= (1, 1, \dots, 1) \frac{1}{\sigma_{1p}^2 + \sigma_{2p}^2} (1 - \frac{k\sigma_{0p}^2}{\sigma_{1p}^2 + \sigma_{2p}^2 + k\sigma_{0p}^2}) \mathbb{1}_k \\ &= \frac{k}{\sigma_{1p}^2 + \sigma_{2p}^2} (1 - \frac{k\sigma_{0p}^2}{\sigma_{1p}^2 + \sigma_{2p}^2 + k\sigma_{0p}^2}) \\ &= \frac{k}{\sigma_{1p}^2 + \sigma_{2p}^2} \frac{\sigma_{1p}^2 + \sigma_{2p}^2}{\sigma_{1p}^2 + \sigma_{2p}^2 + k\sigma_{0p}^2} \\ &= \frac{k}{\sigma_{1p}^2 + \sigma_{2p}^2 + k\sigma_{0p}^2}. \end{aligned} \tag{12}$$

Putting together what we have above, we get

$$\hat{\Xi}_p = \frac{\sum_{j=1}^k \hat{\xi}_{pj}}{k} = \tilde{\xi}_p.$$

and therefore,

$$\hat{\tilde{C}}_{pj} = z_1 \hat{\xi}_{pj} + z_2 \tilde{\xi}_p + (1 - z_1 - z_2) \tilde{\xi}_p = z_1 \hat{\xi}_{pj} + (1 - z_1) \tilde{\xi}_p \tag{13}$$

□

4. Numerical Simulations

In this section, we illustrate our results numerically. We use the assumptions the same as that of [Yeo and Valdez \(2006\)](#) to compare the credibility premium. For the sake of consistency, we have chosen the same values for parameters. We generated $n = 1000$ groups of 10 year paths of claims for 10 different individuals with the common effects assumptions. The same as [Yeo and Valdez \(2006\)](#), we assumed that Θ and Λ are independent of each other.

$$X_j | \Theta_j, \Lambda \sim N(\Theta_j + \Lambda, \sigma_X^2)$$

where $\Theta_j \sim N(\mu_{\Theta_j}, \sigma_{\Theta_j}^2)$, $\mu_{\Theta_j} = 100$, $\sigma_{\Theta_j}^2 = 1024$ and $\Lambda \sim N(\mu_{\Lambda}, \sigma_{\Lambda}^2)$ with $\mu_{\Lambda} = 200$ and $\sigma_{\Lambda}^2 = 4096$, respectively. Table 1 is the detailed calculation of credibility premium for individuals for the first group, where Θ_j varies among 10 individuals but the common effect random variable Λ stays the

same for a given group. Descriptive statistics of the credibility premiums for ten individuals in this given group is given in Table 2.

Table 1. Information of the credibility premium of ten periods for the ten individuals in the first group.

Group	Contract	Observed Θ_1	observed Λ	$X_{1,1}$	$X_{1,2}$	$X_{1,3}$...	$X_{1,10}$
1	1	98.80	132.12	268.71	182.69	394.92	...	283.68
	2	175.68	132.12	296.85	347.79	370.99	...	374.21
	3	144.48	132.12	367.35	213.87	215.27	...	387.67
	4	143.35	132.12	337.74	443.31	332.42	...	371.85
	5	80.62	132.12	309.75	338.14	221.49	...	244.74
	6	95.73	132.12	314.40	317.58	374.51	...	265.73
	7	72.64	132.12	265.50	409.18	394.40	...	340.09
	8	137.50	132.12	181.41	215.33	370.74	...	332.11
	9	69.69	132.12	322.73	388.63	306.62	...	384.12
	10	111.81	132.12	329.49	326.14	272.04	...	323.35

Table 2. Descriptive statistics.

Contract	Minimum	Mean	Median	Maximum	Standard Deviation
1	182.69	313.23	309.06	396.30	67.05
2	132.85	318.74	332.38	323.02	81.35
3	184.44	292.43	317.26	387.66	75.83
4	247.74	350.54	338.39	443.31	66.23
5	194.31	277.10	278.14	339.05	49.66
6	265.73	340.68	339.77	401.97	39.56
7	84.68	280.31	302.79	451.73	129.96
8	181.41	290.28	271.71	458.11	81.69
9	106.62	305.30	319.45	388.63	84.48
10	210.88	295.46	306.24	348.69	45.80

In Table 3, we compare the results of credibility premiums with common effects with the quantile credibility models without common effects. The quantile that we use is median for the sake of convenience. Classical credibility examines the mean of each contract regardless of the shape of the distribution, while quantile credibility discusses the changes at different points of the distribution. From the results in Table 3, we can see that the discrepancy between mean and median results in different credibility premiums for the Bühlmann’s framework and the quantile credibility model. We can see that in Bühlmann’s credibility model with common effect, the most weight is put on an individual’s own experience and the least weight is put on the prior beliefs. For the quantile credibility model with common effect in which an individual’s and the rest of the group’s experience are integrated together, prior belief is also given least weight. However, looking at them separately, the individual’s own experience still has the most weight, but the least weight is put on the rest of group’s experiences, which is the new feature that quantile credibility brings in. Compared with the results in Pitselis (2013), where no common effects in embedded in quantile credibility premium, more weight is put on individual’s experience; the same outcome also applies to our results.

Table 3. Credibility premiums comparisons.

Contracts	1	2	3	...	9	10
Bühlmann credibility model with common effects Yeo and Valdez (2006)						
\bar{X}	219.05	278.03	263.44	...	175.14	229.06
\bar{X}_j	252.20	245.65	247.27	...	257.08	251.09
$\hat{\mu}(\Theta_j, \Lambda)$	230.90	267.90	258.74	...	203.36	237.18
Z_1	0.663	0.663	0.663	...	0.663	0.663
Z_2	0.323	0.323	0.323	...	0.323	0.323
Z_3	0.014	0.014	0.014	...	0.014	0.014
$\mu_\Lambda + \mu_\Theta = 300$	$\sigma_\Lambda^2 = 4096$	$\sigma_\Theta^2 = 1024$	$\sigma_x^2 = 6084$			
Quantile (median) credibility model with common effects, $p = 0.5$						
$\hat{\zeta}_{p,j}$	309.06	332.38	317.26	...	319.45	306.24
$\hat{\zeta}_p$	311.52	311.52	311.52	...	311.52	311.52
Z_1	0.43	0.43	0.43	...	0.43	0.43
Z_2	0.18	0.18	0.18	...	0.18	0.18
Z_3	0.39	0.39	0.39	...	0.39	0.39
$\Xi_p = 300$	$\sigma_{2,p}^2 = 836.01$	$\sigma_{1,p}^2 = 1126.77$	$\sigma_{0,p}^2 = 89.10$			

5. Further Remarks

To summarize, we examine the credibility theory in this work. Compared with classical Bühlmann model and the popular hierarchical credibility, our work extends their results by taking both the quantile of data and common effect of risk variables into consideration. To be more specific, we model the quantile credibility in which only quantile of the payments history is required. This feature of less dependence on the detailed data set implies a higher level of flexibility in the determination of premium. This is different from the classical Bühlmann model, where conditional independence is assumed, in this paper, we are interested in considering the dependence structure characterized by a stochastic latent risk parameter and studying the premium calculation accordingly. The phenomenon of dependence over risks have been recognized by people in both academia and industry; therefore, it is very meaningful to study the credibility model under this assumption. Some direct applications of our model are, for example, in certain homeowner insurance where geographic proximity of the insured results in exposures to common catastrophes, in some auto insurance in which one accident involves several insured and in some health insurance in which insured within the same working place are subject to the same type of infectious disease.

One direction of our future work is to incorporate the common shock into hierarchical credibility model in which a tree structure is embedded. We can examine the dependence of the individual risks within a certain level of the tree. Furthermore, there is also a possibility to model the common shock such that dependence of the risks among different levels of the tree could be examined. In addition, research have showed that there are satisfying applications of credibility theory in mortality modeling. For example, [Bozikas and Pitselis \(2020\)](#) use crossed classification credibility is applied to Lee–Carter model. The hierarchical credibility model has also been utilized to model multi-country mortality rate in [Tsai and Di Wu \(2020\)](#). Another interesting topic for future research is the application of quantile credibility with common effect in mortality modeling.

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