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Characterization of Mean-Field Type $\mathcal{H}_-$ Index for Continuous-Time Stochastic Systems with Markov Jump

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Abstract: In this brief, we consider the mean-field type $\mathcal{H}_-$ index problem for stochastic Markovian jump systems. A sufficient condition is derived for stochastic Markovian jump systems with (x, u)-dependent noise based on generalized differential Riccati equations. Especially for stochastic Markovian jump systems with only x-dependent noise, a sufficient and necessary condition is developed to characterize $\mathcal{H}_-$ index larger than some $\xi > 0$. Finally, a numerical example is addressed to verify the effectiveness of our obtained results.

Keywords: mean field; $\mathcal{H}_-$ index; fault detection; Markovian jump; stochastic systems

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1. Introduction

It is universally acknowledged that almost all dynamical systems in practice are unavoidably affected by unknown inputs and faults that resulted from actuators, components, or sensors. The design of the fault diagnosis filter aims at improving the robustness against unknown inputs and sensitivity to fault. To achieve such a goal, some practical criteria have been applied, such as $\mathcal{H}_2$ norm, $\mathcal{H}_\infty$ norm, and $\mathcal{H}_-$ index [1–6]. Based on these criteria, some multi-objective optimization problems such as $\mathcal{H}_-/\mathcal{H}_\infty$, $\mathcal{H}_2/\mathcal{H}_\infty$ and $\mathcal{H}_\infty/\mathcal{H}_\infty$ have attracted the attention of many scholars [7–9]. $\mathcal{H}_\infty$ norm measures the unknown input's maximum influence on the residual signal. By comparison, the fault's minimum influence on residual signal is measured by $\mathcal{H}_-$ index, which is first introduced in [7]. In the past few years, the fault diagnosis filter design involving $\mathcal{H}_-/\mathcal{H}_\infty$ has attracted the attention of many scholars [10–17].

The $\mathcal{H}_-$ index was first defined in the frequency domain. The $\mathcal{H}_-$ index at zero frequency was defined and investigated in [2], which means the smallest singular value of non-zero. In [5], the generalized famous KYP lemma was used to characterize $\mathcal{H}_-$ index problem of finite frequency. The $\mathcal{H}_-$ index of all frequency range was investigated by matrix inequality and equality in [6], which denotes the minimum singular value. The $\mathcal{H}_-/\mathcal{H}_\infty$ fault detection of finite frequency and infinite frequency were characterized in [12,13]. However, in practical applications, the frequency domain is very limited. Corresponding $\mathcal{H}_-$ index problems of time domain have attracted many scholars' interest. In [9], the optimal solutions to robust $\mathcal{H}_-/\mathcal{H}_\infty$ problem in infinite and finite horizon were given for linear time-varying systems, which generalize corresponding resolutions to the time domain. In [11], the $\mathcal{H}_-$ index was attributed to the existence of the solution to differential Riccati equation for time-varying systems, and corresponding results were extended to systems whose initial condition was unknown. $\mathcal{H}_-/\mathcal{H}_\infty$-optimization was used to design the fault detection filter for nonlinear systems in discrete-time in [10]. For time-invariant systems, the method of matrix factorization was used to develop the $\mathcal{H}_-/\mathcal{H}_\infty$ FD problem in [14]. In [15], the FD observer design was formulated as an $\mathcal{H}_-/\mathcal{H}_\infty$ problem, which the solution was given via LMI formulation for the T-S fuzzy system. Fault detection for linear and nonlinear discrete-time systems were discussed in [16,17]. In [18,19], for discrete and continuous time-varying systems with Markov jump, corresponding finite horizon and infinite horizon $\mathcal{H}_-$ index were investigated via GDREs.

Considering that systems in the practical world are always affected by stochastic disturbances, many researchers have transferred their interest in stochastic control problems from determinate systems to stochastic systems [20–25]. Especially, interest in stochastic systems of mean-field type has been increasing. The mean-field theory is developed to study the collective behaviors resulting from individuals' mutual interactions in various physical and sociological dynamical systems [26,27]. To date, many results on finite and infinite horizon linear quadratic optimal control of mean-field stochastic systems have been presented, we refer the reader to [28–31] and references therein. For mean-field stochastic differential and difference equations, the $\mathcal{H}_\infty$ control and finite horizon mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control of mean-field stochastic systems were characterized in [32–35]. Refs. [36,37] investigated finite and infinite horizon Pareto-based optimality of mean-field stochastic systems.$\mathcal{H}_-$ index of mean-field stochastic system was characterized in [38].

In [19], the $\mathcal{H}_-$ index of classic stochastic differential equation with Markov jump was studied. The corresponding results will be generalized to the mean-field stochastic differential equation with the Markov jump in this paper. Since the expectations $\mathbf{E}x_t$ and $\mathbf{E}u_t$ appear in equations, the problem is not a simple generalization. The main contributions of this note are as follows: Based on existing research results, the definition and attribute of $\mathcal{H}_-$ index are extended to mean-field stochastic systems. The appearance of $\mathbf{E}x_t$, $\mathbf{E}u_t$ and Markov jumps in differential equations lead to more higher difficulty in mathematical deductions. This note illustrates the $\mathcal{H}_-$ index of mean-field stochastic continuous time-varying systems with Markovian jump in the finite horizon. The main result is about the necessary and sufficient condition of the $\mathcal{H}_-$ index greater than a given positive number, which is given by coupled generalized differential Riccati equation.

The remainder of the note is arranged as follows: The system considered in this article is formulated and some preliminary results are presented in Section 2, Section 3 gives the main results in terms of generalized differential Riccati equations. Numerical example is provided in Section 4 to show the effectiveness of our obtained results. Section 5 presents the conclusions of this article.

Notation. $\mathbf{R}^m$ is the set of all $m-$dimensional real vectors. $\widetilde{S} = \{1, 2, ..., s\}$. W' denotes the transpose of W. $W > 0$ ($W \geq 0$) means that W is positive definite (positive semi-definite) symmetric matrix. $\mathbf{R}^{n\times m}$: the set of all $n \times m$-dimensional real matrices. $I_{l\times l}$ is the $l \times l$ identity matrix. . $\mathbf{S}_m(\mathbf{R})$ is the set of all real symmetric matrices $\mathbf{R}^{m\times m}$. A *wide* (*square* or *tall*) system is the system whose inputs dimensions is more than (is equal to or less than) the outputs dimensions. $M^2_{\mathcal{F}}([0,T], \mathbf{R}^l)$ is the space of nonanticipative stochastic processes $x_t \in \mathbf{R}^l$ with respect to an increasing algebras $\mathcal{F}_t (t \geq 0)$ satisfying $\|x_t\|_{[0,T]} = \{\mathbf{E}\int_0^T \|x_t\|^2 dt\}^{1/2} = \{\mathbf{E}\int_0^T x_t x_t' dt\}^{1/2} < \infty$. $\mathcal{C}^{1,2}([0,T] \times \mathbf{R}^n; \mathbf{R})$ is the class of $\mathbf{R}$-valued functions $V(t,x)$ which are once continuously differentiable with respect to $t \in [0,T]$, and twice continuously differential with respect to $x \in \mathbf{R}^n$, except possibly at the point $x = 0$.

2. Preliminaries

In this section, a useful lemma will be given for the following stochastic Markov jump systems of the mean-field type in continuous-time:

$$\begin{cases} dx_t = [A_{t,\zeta_t} x_t + A^0_{t,\zeta_t}\mathbf{E}x_t + B_{t,\zeta_t} u_t + B^0_{t,\zeta_t}\mathbf{E}u_t]dt \\ \qquad +[C_{t,\zeta_t} x_t + C^0_{t,\zeta_t}\mathbf{E}x_t + D_{t,\zeta_t} u_t + D^0_{t,\zeta_t}\mathbf{E}u_t]dW_t, \\ y_t = K_{t,\zeta_t} x_t + K^0_{t,\zeta_t}\mathbf{E}x_t + F_{t,\zeta_t} u_t + F^0_{t,\zeta_t}\mathbf{E}u_t, \\ x_0 \in \mathbf{R}^n, t \in [0,T]. \end{cases} \quad (1)$$

where, $x_t \in \mathbf{R}^{n_2}$, $u_t \in M^2_{\mathcal{F}}([0,T], \mathbf{R}^{n_1})$ and $y_t \in \mathbf{R}^{n_3}$ are the system state, control input and regulated output, respectively. W_t is the one-dimensional standard Brownian motion, $\mathbf{E}$

denotes the expectation . $\{\zeta_t, t \geq 0\}$ is a continuous-time discrete-state Markov process, whose values is taken in $\widetilde{S}$ and has the transition probability described by

$$\mathcal{P}(\zeta_{t+\Delta t} = j|\zeta_t = i) = \begin{cases} g_{ij}\Delta t + o(\Delta t), & i \neq j, \\ 1 + g_{ii}\Delta t + o(\Delta t), & i = j, \end{cases} \tag{2}$$

where $\Delta t > 0, lim_{\Delta t \to 0} \frac{o(\Delta t)}{\Delta t} = 0$ and $g_{ij} \geq 0$ for $i \neq j, i, j \in \widetilde{S}$, determine the transition rate from mode i to mode j, and $\mathcal{G} = (g_{ij})_{s\times s}$ with $g_{ii} = -\sum_{j=1,j\neq i}^{S} g_{ij}$ for all $i \in \widetilde{S}$. A_{t,ζ_t}, A^0_{t,ζ_t}, B_{t,ζ_t}, B^0_{t,ζ_t}, C_{t,ζ_t}, C^0_{t,ζ_t}, D_{t,ζ_t}, D^0_{t,ζ_t}, K_{t,ζ_t}, K^0_{t,ζ_t}, F_{t,ζ_t} and F^0_{t,ζ_t} are corresponding weighted coefficient matrices. In different practical problems, their meanings are different. $A_{t,\zeta_t} = A_{t,i}$, $A^0_{t,\zeta_t} = A^0_{t,i}$, $B_{t,\zeta_t} = B_{t,i}$, $B^0_{t,\zeta_t} = B^0_{t,i}$, $C_{t,\zeta_t} = C_{t,i}$, $C^0_{t,\zeta_t} = C^0_{t,i}$, $D_{t,\zeta_t} = D_{t,i}$, $D^0_{t,\zeta_t} = D^0_{t,i}$, $K_{t,\zeta_t} = K_{t,i}$, $K^0_{t,\zeta_t} = K^0_{t,i}$, $F_{t,\zeta_t} = F_{t,i}$ and $F^0_{t,\zeta_t} = F^0_{t,i}$ when $\zeta_t = i$, are assumed to be continuous matrix-valued functions of proper dimensions. The process ζ_t and W_t are defined on filtered probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with the natural filter $\mathcal{F}_t = \{W_s, \zeta_s | 0 \leq s \leq t\}$, and ζ_t is independent of W_t. For any given $0 < T < \infty$ and $(u_t, x_0) \in M^2_{\mathcal{F}}([0, T], \mathbf{R}^{n_1}) \times \mathbf{R}^{n_2}$, the unique solution of (1) is denoted by $x_t = x_{(t,u;x_0,\zeta_0)} \in M^2_{\mathcal{F}}([0, T], \mathbf{R}^{n_2})$ with deterministic initial condition ζ_0, x_0.

Definition 1. *For system (1), the $\mathcal{H}_-$ index is defined as*

$$\|\Gamma\|_-^{[0,T]} := \inf_{u_t \in M^2_{\mathcal{F}}([0,T],\mathbf{R}^{n_1}), u_t \neq 0, \zeta_0 \in \widetilde{S}, x_0 = 0} \frac{\|y_t\|_{[0,T]}}{\|u_t\|_{[0,T]}}. \tag{3}$$

Remark 1. *When u_t and y_t denote fault signal and residual signal, respectively, the minimum sensitivity of system (1) from input u_t to output y_t is depicted as $\|\Gamma\|_-^{[0,T]}$. For wide system, $\|\mathbf{\Gamma}\|_-^{[0,T]} = 0$, so it is supposed that system (1) is a tall system or square system.*

For $\xi > 0$, $0 < T < \infty$ and $i \in \widetilde{S}$, we want to investigate the condition of the smallest sensitivity greater than ξ, i.e., $\|\Gamma\|_-^{[0,T]} > \xi$. Define

$$\widehat{J}_\xi^T(x_0, u) := \mathbf{E}\int_0^T [\|y_t\|^2 - \xi^2\|u_t\|^2]dt, \tag{4}$$

and

$$\widetilde{J}_\xi^T(x_0, i, u) := \mathbf{E}\bigg\{\int_0^T [\|y_t\|^2 - \xi^2\|u_t\|^2]dt|\zeta_0 = i\bigg\}, \tag{5}$$

then $\widehat{J}_\xi^T(0, u) > 0$ yields $\|\Gamma\|_-^{[0,T]} > \xi$.

Lemma 1. *[39] (Generalized Itô formula): Let $\alpha_{t,x,i}, \beta_{t,x,i} \in \mathbf{R}^n$ be $\mathcal{F}_t$-adapted process, $i \in \widetilde{S}$, $dx_t = \alpha_{t,x,\zeta_t}dt + \beta_{t,x,\zeta_t}dW_t$. Then for given $\Upsilon_{t,x,i} \in \mathcal{C}^{1,2}([0, T] \times \mathbf{R}^n; \mathbf{R}), i \in \widetilde{S}$, we have*

$$\begin{aligned} &\mathbf{E}\{\Upsilon_{T,x_T,\zeta_T} - \Upsilon_{k,x_k,\zeta_k}|\zeta_k = i\} \\ =&\mathbf{E}\{\int_k^T \Delta\Upsilon_{t,x_t,\zeta_t}dt|\zeta_k = i\}, \end{aligned}$$

where

$$\begin{aligned} \Delta\Upsilon_{t,x_t,i} =& \frac{\partial\Upsilon_{t,x_t,i}}{\partial t} + \alpha^T_{t,x_t,i}\frac{\partial\Upsilon_{t,x_t,i}}{\partial x} \\ &+ \frac{1}{2}\beta^T_{t,x_t,i}\frac{\partial^2\Upsilon_{t,x_t,i}}{\partial x^2}\beta_{t,x_t,i} + \sum_{j=1}^N g_{ij}\Upsilon_{t,x_t,j}. \end{aligned}$$

For system (1), by taking mathematical expectations, we can express $\mathbf{E}x_t$ and $x_t - \mathbf{E}x_t$ as following:

$$d\mathbf{E}x_t = [(A_{t,\zeta_t} + A^0_{t,\zeta_t})\mathbf{E}x_t + (B_{t,\zeta_t} + B^0_{t,\zeta_t})\mathbf{E}u_t]dt, \tag{6}$$

$$\begin{cases} d(x_t - \mathbf{E}x_t) = [A_{t,\zeta_t}(x_t - \mathbf{E}x_t) + B_{t,\zeta_t}(u_t - \mathbf{E}u_t)]dt \\ \quad +[C_{t,\zeta_t}(x_t - \mathbf{E}x_t) + (C_{t,\zeta_t} + C^0_{t,\zeta_t})\mathbf{E}x_t + D_{t,\zeta_t}(u_t - \mathbf{E}u_t) + (D_{t,\zeta_t} + D^0_{t,\zeta_t})\mathbf{E}u_t]dW_t. \end{cases} \tag{7}$$

Lemma 2. *For system (1), assume $\widetilde{\mathcal{P}}_t$ and $\widetilde{\mathcal{Q}}_t$ are differentiable, where*

$$\widetilde{\mathcal{P}}_t = [\widetilde{\mathcal{P}}_{t,1}, \widetilde{\mathcal{P}}_{t,2}, \dots, \widetilde{\mathcal{P}}_{t,s}],$$
$$\widetilde{\mathcal{Q}}_t = [\widetilde{\mathcal{Q}}_{t,1}, \widetilde{\mathcal{Q}}_{t,2}, \dots, \widetilde{\mathcal{Q}}_{t,s}],$$

$\widetilde{\mathcal{P}}_{t,i}, \widetilde{\mathcal{Q}}_{t,i} \in \mathbf{S}_n(\mathbf{R})$ *with* $i \in \widetilde{S}, t \in [0,T]$. *Let*

$$S_i(\widetilde{\mathcal{Q}}_{t,i}) = \begin{bmatrix} \dot{\widetilde{\mathcal{Q}}}_{t,i} + \mathcal{L}_i(\widetilde{\mathcal{Q}}_{t,i}) & \mathcal{H}_i(\widetilde{\mathcal{Q}}_{t,i}) \\ \mathcal{H}_i(\widetilde{\mathcal{Q}}_{t,i})' & \mathcal{M}_i^{\xi}(\widetilde{\mathcal{Q}}_{t,i}) \end{bmatrix}$$

and

$$\widehat{S}_i(\widetilde{\mathcal{Q}}_{t,i}, \widetilde{\mathcal{P}}_{t,i}) = \begin{bmatrix} \dot{\widetilde{\mathcal{P}}}_{t,i} + \widehat{\mathcal{L}}_i(\widetilde{\mathcal{Q}}_{t,i}, \widetilde{\mathcal{P}}_{t,i}) & \widehat{\mathcal{H}}_i(\widetilde{\mathcal{Q}}_{t,i}, \widetilde{\mathcal{P}}_{t,i}) \\ \widehat{\mathcal{H}}_i(\widetilde{\mathcal{Q}}_{t,i}, \widetilde{\mathcal{P}}_{t,i})' & \widehat{\mathcal{M}}_i^{\xi}(\widetilde{\mathcal{Q}}_{t,i}, \widetilde{\mathcal{P}}_{t,i}) \end{bmatrix},$$

where

$$\begin{aligned} \mathcal{L}_i(\widetilde{\mathcal{Q}}_{t,i}) &= A'_{t,i}\widetilde{\mathcal{Q}}_{t,i} + \widetilde{\mathcal{Q}}_{t,i}A_{t,i} + C'_{t,i}\widetilde{\mathcal{Q}}_{t,i}C_{t,i} + K'_{t,i}K_{t,i} + \sum_{j=1}^{s} g_{ij}\widetilde{\mathcal{Q}}_{t,j} \\ \mathcal{H}_i(\widetilde{\mathcal{Q}}_{t,i}) &= \widetilde{\mathcal{Q}}_{t,i}B_{t,i} + C'_{t,i}\widetilde{\mathcal{Q}}_{t,i}D_{t,i} + K'_{t,i}F_{t,i}, \\ \mathcal{M}_i^{\xi}(\widetilde{\mathcal{Q}}_{t,i}) &= D'_{t,i}\widetilde{\mathcal{Q}}_{t,i}D_{t,i} + F'_{t,i}F_{t,i} - \xi^2 I_{n_1}, \\ \widehat{\mathcal{L}}_i(\widetilde{\mathcal{Q}}_{t,i}, \widetilde{\mathcal{P}}_{t,i}) &= \overline{\mathcal{A}}'_{t,i}\widetilde{\mathcal{P}}_{t,i} + \widetilde{\mathcal{P}}_{t,i}\overline{\mathcal{A}}_{t,i} + \overline{\mathcal{C}}'_{t,i}\widetilde{\mathcal{Q}}_{t,i}\overline{\mathcal{C}}_{t,i} + \overline{\mathcal{K}}'_{t,i}\overline{\mathcal{K}}_{t,i} + \sum_{j=1}^{s} g_{ij}\widetilde{\mathcal{P}}_{t,j}, \\ \widehat{\mathcal{H}}_i(\widetilde{\mathcal{Q}}_{t,i}, \widetilde{\mathcal{P}}_{t,i}) &= \widetilde{\mathcal{P}}_{t,i}\overline{\mathcal{B}}_{t,i} + \overline{\mathcal{C}}'_{t,i}\widetilde{\mathcal{Q}}_{t,i}\overline{\mathcal{D}}_{t,i} + \overline{\mathcal{K}}'_{t,i}\overline{\mathcal{F}}_{t,i}, \\ \widehat{\mathcal{M}}_i^{\xi}(\widetilde{\mathcal{Q}}_{t,i}, \widetilde{\mathcal{P}}_{t,i}) &= \overline{\mathcal{D}}'_{t,i}\widetilde{\mathcal{Q}}_{t,i}\overline{\mathcal{D}}_{t,i} + \overline{\mathcal{F}}'_{t,i}\overline{\mathcal{F}}_{t,i} - \xi^2 I_{n_1}, \end{aligned}$$

and

$$\overline{\mathcal{A}}_{t,i} = A_{t,i} + A^0_{t,i},\ \overline{\mathcal{B}}_t = B_{t,i} + B^0_{t,i},\ \overline{\mathcal{C}}_{t,i} = C_{t,i} + C^0_{t,i},$$
$$\overline{\mathcal{D}}_{t,i} = D_{t,i} + D^0_{t,i}, \overline{\mathcal{K}}_{t,i} = K_{t,i} + K^0_{t,i}, \overline{\mathcal{F}}_{t,i} = F_{t,i} + F^0_{t,i}.$$

Then for $\forall x_0 \in \mathbf{R}^{n_2}$, $u_t \in M^2_{\mathcal{F}}([0,T], \mathbf{R}^{n_1})$, *we have*

$$\begin{aligned} \widetilde{J}^T_{\xi}(x_0, i, u) = {} & \mathbf{E}[(x_0 - \mathbf{E}x_0)'\widetilde{\mathcal{Q}}_{0,i}(x_0 - \mathbf{E}x_0)] + \mathbf{E}x'_0\widetilde{\mathcal{P}}_{0,i}\mathbf{E}x_0 \\ & - \mathbf{E}[(x_T - \mathbf{E}x_T)'\widetilde{\mathcal{Q}}_{T,\zeta_T}(x_T - \mathbf{E}x_T)|\zeta_0 = i] \\ & - \mathbf{E}[\mathbf{E}x'_T\widetilde{\mathcal{P}}_{T,\zeta_T}\mathbf{E}x_T|\zeta_0 = i] \\ & + \mathbf{E}\left\{\int_0^T \begin{bmatrix} x_t - \mathbf{E}x_t \\ u_t - \mathbf{E}u_t \end{bmatrix}' S_{\zeta_t}(\widetilde{\mathcal{Q}}_{t,\zeta_t}) \begin{bmatrix} x_t - \mathbf{E}x_t \\ u_t - \mathbf{E}u_t \end{bmatrix} dt|\zeta_0 = i\right\} \\ & + \left\{\int_0^T \begin{bmatrix} \mathbf{E}x_t \\ \mathbf{E}u_t \end{bmatrix}' \widehat{S}_{\zeta_t}(\widetilde{\mathcal{Q}}_{t,\zeta_t}, \widetilde{\mathcal{P}}_{t,\zeta_t}) \begin{bmatrix} \mathbf{E}x_t \\ \mathbf{E}u_t \end{bmatrix} dt|\zeta_0 = i\right\}. \end{aligned} \tag{8}$$

Proof. According to Lemma 1,

$$\mathbf{E}\{\Upsilon_{T,x_T,\zeta_T} - \Upsilon_{0,x_0,\zeta_0}|\zeta_0 = i\} = \mathbf{E}\{\int_0^T \Delta\Upsilon_{t,x_t,\zeta_t}dt|\zeta_0 = i\}.$$

Applying Lemma 1 for $Y_{t,x_t,\zeta_t} = \mathbf{E}x_t'\widetilde{\mathcal{P}}_{t,\zeta_t}\mathbf{E}x_t$ and $Y_{t,x_t,\zeta_t} = (x_t - \mathbf{E}x_t)'\widetilde{\mathcal{Q}}_{t,\zeta_t}(x_t - \mathbf{E}x_t)$, respectively, we obtain

$$\begin{aligned}
&[\mathbf{E}x_T'\widetilde{\mathcal{P}}_{T,\zeta_T}\mathbf{E}x_T|\zeta_0 = i] - \mathbf{E}x_0'\widetilde{\mathcal{P}}_{0,i}\mathbf{E}x_0 \\
&+ \mathbf{E}[(x_T - \mathbf{E}x_T)'\widetilde{\mathcal{Q}}_{T,\zeta_T}(x_T - \mathbf{E}x_T)|\zeta_0 = i] - (x_0 - \mathbf{E}x_0)'\widetilde{\mathcal{Q}}_{0,i}(x_0 - \mathbf{E}x_0) \\
=&\left\{\int_0^T \begin{bmatrix} \mathbf{E}x_t \\ \mathbf{E}u_t \end{bmatrix}' \widetilde{\mathbf{U}}(\widetilde{\mathcal{Q}}_{t,\zeta_t}, \widetilde{\mathcal{P}}_{t,\zeta_t}) \begin{bmatrix} \mathbf{E}x_t \\ \mathbf{E}u_t \end{bmatrix} dt|\zeta_0 = i\right\} \\
&+ \mathbf{E}\left\{\int_0^T \begin{bmatrix} x_t - \mathbf{E}x_t \\ u_t - \mathbf{E}u_t \end{bmatrix}' \mathbf{U}(\widetilde{\mathcal{Q}}_{t,\zeta_t}) \begin{bmatrix} x_t - \mathbf{E}x_t \\ u_t - \mathbf{E}u_t \end{bmatrix} dt|\zeta_0 = i\right\},
\end{aligned} \tag{9}$$

where

$$\mathbf{U}(\widetilde{\mathcal{Q}}_{t,\zeta_t}) = \begin{bmatrix} \dot{\widetilde{\mathcal{Q}}}_{t,\zeta_t} + C_{t,\zeta_t}'\widetilde{\mathcal{Q}}_{t,\zeta_t}C_{t,\zeta_t} + A_{t,\zeta_t}'\widetilde{\mathcal{Q}}_{t,\zeta_t} & \widetilde{\mathcal{Q}}_{t,\zeta_t}B_{t,\zeta_t} + D_{t,\zeta_t}'\widetilde{\mathcal{Q}}_{t,\zeta_t}C_{t,\zeta_t} \\ +\widetilde{\mathcal{Q}}_{t,\zeta_t}A_{t,\zeta_t} + \sum_{j=1}^{s} g_{\zeta_t j}\widetilde{\mathcal{Q}}_{t,j} & \\ B_{t,\zeta_t}'\widetilde{\mathcal{Q}}_{t,\zeta_t} + C_{t,\zeta_t}'\widetilde{\mathcal{Q}}_{t,\zeta_t}D_{t,\zeta_t} & D_{t,\zeta_t}'\widetilde{\mathcal{Q}}_{t,\zeta_t}D_{t,\zeta_t} \end{bmatrix},$$

$$\widetilde{\mathbf{U}}(\widetilde{\mathcal{Q}}_{t,\zeta_t}, \widetilde{\mathcal{P}}_{t,\zeta_t}) = \begin{bmatrix} \dot{\widetilde{\mathcal{P}}}_{t,\zeta_t} + \overline{\mathcal{A}}_{t,\zeta_t}'\widetilde{\mathcal{P}}_{t,\zeta_t} + \widetilde{\mathcal{P}}_{t,\zeta_t}\overline{\mathcal{A}}_{t,\zeta_t} & \overline{\mathcal{P}}_{t,\zeta_t}\widetilde{\mathcal{B}}_{t,\zeta_t} + \overline{\mathcal{D}}_{t,\zeta_t}'\widetilde{\mathcal{Q}}_{t,\zeta_t}\overline{\mathcal{C}}_{t,\zeta_t} \\ +\overline{\mathcal{C}}_{t,\zeta_t}'\widetilde{\mathcal{Q}}_{t,\zeta_t}\overline{\mathcal{C}}_{t,\zeta_t} + \sum_{\zeta_t=1}^{s} g_{\zeta_t j}\widetilde{\mathcal{P}}_{t,j} & \\ \overline{\mathcal{B}}_{t,\zeta_t}'\widetilde{\mathcal{P}}_{t,\zeta_t} + \overline{\mathcal{C}}_{t,\zeta_t}'\widetilde{\mathcal{Q}}_{t,\zeta_t}\overline{\mathcal{D}}_{t,\zeta_t} & \overline{\mathcal{D}}_{t,\zeta_t}'\widetilde{\mathcal{Q}}_{t,\zeta_t}\overline{\mathcal{D}}_{t,\zeta_t} \end{bmatrix}.$$

In addition,

$$\begin{aligned}
&\mathbf{E}\left\{\int_0^T [\|y_t\|^2 - \xi^2\|u_t\|^2]dt|\zeta_0 = i\right\} \\
=& \left\{\int_0^T \begin{bmatrix} \mathbf{E}x_t \\ \mathbf{E}u_t \end{bmatrix}' \widetilde{V} \begin{bmatrix} \mathbf{E}x_t \\ \mathbf{E}u_t \end{bmatrix} dt|\zeta_0 = i\right\} + \mathbf{E}\left\{\int_0^T \begin{bmatrix} x_t - \mathbf{E}x_t \\ u_t - \mathbf{E}u_t \end{bmatrix}' V \begin{bmatrix} x_t - \mathbf{E}x_t \\ u_t - \mathbf{E}u_t \end{bmatrix} dt|\zeta_0 = i\right\}
\end{aligned} \tag{10}$$

with

$$\widetilde{V} = \begin{bmatrix} \overline{\mathcal{K}}_{t,\zeta_t}'\overline{\mathcal{K}}_{t,\zeta_t} & \overline{\mathcal{K}}_{t,\zeta_t}'\overline{\mathcal{F}}_{t,\zeta_t} \\ \overline{\mathcal{F}}_{t,\zeta_t}'\overline{\mathcal{K}}_{t,\zeta_t} & \overline{\mathcal{F}}_{t,\zeta_t}'\overline{\mathcal{F}}_{t,\zeta_t} - \xi^2 I_{n_1} \end{bmatrix}$$

and

$$V = \begin{bmatrix} K_{t,\zeta_t}'K_{t,\zeta_t} & K_{t,\zeta_t}'F_{t,\zeta_t} \\ F_{t,\zeta_t}'K_{t,\zeta_t} & F_{t,\zeta_t}'F_{t,\zeta_t} - \xi^2 I_{n_1} \end{bmatrix}.$$

According to (9) and (10), we can obtain (8), the proof is end. □

3. Finite Horizon Mean-Field Type Stochastic $\mathcal{H}_-$ Index

The mean-field type stochastic $\mathcal{H}_-$ index will be investigated in this section, the sufficient and necessary condition of $\|\Gamma\|_-^{[0,T]} > \xi$, which means $\widehat{J}_\xi^T(0,u) > 0$, $\forall u_t \in M_{\mathcal{F}}^2([0,T], \mathbf{R}^{n_1}), u_t \neq 0, \zeta_0 \in \widetilde{S}$.

Theorem 1. *If for a given $\xi > 0$ there exist $\widetilde{\mathcal{Q}}_t^\xi = [\widetilde{\mathcal{Q}}_{t,1}^\xi, \widetilde{\mathcal{Q}}_{t,2}^\xi, \dots, \widetilde{\mathcal{Q}}_{t,s}^\xi]$ and $\widetilde{\mathcal{P}}_t^\xi = [\widetilde{\mathcal{P}}_{t,1}^\xi, \widetilde{\mathcal{P}}_{t,2}^\xi, \dots, \widetilde{\mathcal{P}}_{t,s}^\xi]$, $t \in [0,T]$, such that the following GDREs are fulfilled*

$$\begin{cases}
\mathcal{H}_i(\widetilde{\mathcal{Q}}_{t,i})\mathcal{M}_i^\xi(\widetilde{\mathcal{Q}}_{t,i})^{-1}\mathcal{H}_i(\widetilde{\mathcal{Q}}_{t,i})' = \dot{\widetilde{\mathcal{Q}}}_{t,i} + \mathcal{L}_i(\widetilde{\mathcal{Q}}_{t,i}), \\
\widehat{\mathcal{H}}_i(\widetilde{\mathcal{Q}}_{t,i}, \widetilde{\mathcal{P}}_{t,i})\widehat{\mathcal{M}}_i^\xi(\widetilde{\mathcal{Q}}_{t,i}, \widetilde{\mathcal{P}}_{t,i})^{-1}\widehat{\mathcal{H}}_i(\widetilde{\mathcal{Q}}_{t,i}, \widetilde{\mathcal{P}}_{t,i})' = \dot{\widetilde{\mathcal{P}}}_{t,i} + \widehat{\mathcal{L}}_i(\widetilde{\mathcal{Q}}_{t,i}, \widetilde{\mathcal{P}}_{t,i}), \\
\mathcal{M}_i^\xi(\widetilde{\mathcal{Q}}_{t,i}) > 0,\ \widehat{\mathcal{M}}_i^\xi(\widetilde{\mathcal{Q}}_{t,i}, \widetilde{\mathcal{P}}_{t,i}) > 0, \\
\widetilde{\mathcal{Q}}_{T,i} = \widetilde{\mathcal{P}}_{T,i} = 0, i = 1, \dots, s,
\end{cases} \tag{11}$$

then $\|\Gamma\|_-^{[0,T]} > \xi$.

Proof. In view of $\widetilde{\mathcal{Q}}_{T,i} = \widetilde{\mathcal{P}}_{T,i} = 0$, by Lemma 2, for any $u_t \in M^2_{\mathcal{F}}([0,T], \mathbf{R}^{n_1})$ with $u_t \neq 0$, $x_0 = 0$, it can be obtained that

$$\widehat{J}^T_\xi(0,i,u) = \mathbf{E}\left\{\int_0^T \begin{bmatrix} x_t - \mathbf{E}x_t \\ u_t - \mathbf{E}u_t \end{bmatrix}' S_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t}) \begin{bmatrix} x_t - \mathbf{E}x_t \\ u_t - \mathbf{E}u_t \end{bmatrix} dt \Big| \zeta_0 = i\right\} + \left\{\int_0^T \begin{bmatrix} \mathbf{E}x_t \\ \mathbf{E}u_t \end{bmatrix}' \widehat{S}_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t}, \widetilde{\mathcal{P}}^\xi_{t,\zeta_t}) \begin{bmatrix} \mathbf{E}x_t \\ \mathbf{E}u_t \end{bmatrix} dt \Big| \zeta_0 = i\right\}.$$

By (11) and the technique of completing squares, it follows that

$$\begin{aligned}
\widehat{J}^T_\xi(0,u) &= \mathbf{E}\int_0^T \begin{bmatrix} x_t - \mathbf{E}x_t \\ u_t - \mathbf{E}u_t \end{bmatrix}' S_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t}) \begin{bmatrix} x_t - \mathbf{E}x_t \\ u_t - \mathbf{E}u_t \end{bmatrix} dt \\
&+ \int_0^T \begin{bmatrix} \mathbf{E}x_t \\ \mathbf{E}u_t \end{bmatrix}' \widehat{S}_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t}, \widetilde{\mathcal{P}}^\xi_{t,\zeta_t}) \begin{bmatrix} \mathbf{E}x_t \\ \mathbf{E}u_t \end{bmatrix} dt \\
&= \mathbf{E}\int_0^T (x_t - \mathbf{E}x_t)' \Big[\dot{\widetilde{\mathcal{Q}}}^\xi_{t,\zeta_t} + \mathcal{L}_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t}) - \mathcal{H}_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t}) \mathcal{M}^\xi_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t})^{-1} \mathcal{H}_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t})'\Big](x_t - \mathbf{E}x_t) dt \\
&+ \mathbf{E}\int_0^T [(u_t - \mathbf{E}u_t) - (u^*_t - \mathbf{E}u^*_t)]' \mathcal{M}^\xi_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t}) [(u_t - \mathbf{E}u_t) - (u^*_t - \mathbf{E}u^*_t)] dt \\
&+ \int_0^T (\mathbf{E}x_t)' \Big[\dot{\widetilde{\mathcal{P}}}^\xi_{t,\zeta_t} + \widehat{\mathcal{L}}_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t}, \widetilde{\mathcal{P}}^\xi_{t,\zeta_t}) - \widehat{\mathcal{H}}_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t}, \widetilde{\mathcal{P}}^\xi_{t,\zeta_t}) \\
&\quad \times \widehat{\mathcal{M}}^\xi_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t}, \widetilde{\mathcal{P}}^\xi_{t,\zeta_t})^{-1} \widehat{\mathcal{H}}_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t}, \widetilde{\mathcal{P}}^\xi_{t,\zeta_t})'\Big](\mathbf{E}x_t) dt \\
&+ \int_0^T [\mathbf{E}u_t - \mathbf{E}u^*_t]' \widehat{\mathcal{M}}^\xi_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t}, \widetilde{\mathcal{P}}^\xi_{t,\zeta_t}) [\mathbf{E}u_t - \mathbf{E}u^*_t] dt. \\
&= \mathbf{E}\int_0^T [(u_t - \mathbf{E}u_t) - (u^*_t - \mathbf{E}u^*_t)]' \mathcal{M}^\xi_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t}) [(u_t - \mathbf{E}u_t) - (u^*_t - \mathbf{E}u^*_t)] dt \\
&+ \int_0^T [\mathbf{E}u_t - \mathbf{E}u^*_t]' \widehat{\mathcal{M}}^\xi_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t}, \widetilde{\mathcal{P}}^\xi_{t,\zeta_t}) [\mathbf{E}u_t - \mathbf{E}u^*_t] dt.
\end{aligned} \tag{12}$$

where

$u^*_t - \mathbf{E}u^*_t = -\mathcal{M}^\xi_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t})^{-1} \mathcal{H}_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t})'(x_t - \mathbf{E}x_t)$,

$\mathbf{E}u^*_t = -\widehat{\mathcal{M}}^\xi_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t}, \widetilde{\mathcal{P}}^\xi_{t,\zeta_t})^{-1} \widehat{\mathcal{H}}_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t}, \widetilde{\mathcal{P}}^\xi_{t,\zeta_t})' \mathbf{E}x_t$.

Since $\mathcal{M}^\xi_i(\widetilde{\mathcal{Q}}^\xi_{t,i}) > 0$ and $\widehat{\mathcal{M}}^\xi_i(\widetilde{\mathcal{Q}}^\xi_{t,i}, \widetilde{\mathcal{P}}^\xi_{t,i}) > 0$, (12) provides $\widehat{J}^T_\xi(0,u) \geq 0$, which means $\|\Gamma\|^{[0,T]}_{-} \geq \xi$.

To prove $\widehat{J}^T_\xi(0,u) > 0$, we define the operator $\widetilde{\mathcal{L}}_1 : \widetilde{\mathcal{L}}_1(u_t - \mathbf{E}u_t) = (u_t - \mathbf{E}u_t) - (u^*_t - \mathbf{E}u^*_t)$ with its realization

$$\begin{cases} d(x_t - \mathbf{E}x_t) = [A_{t,\zeta_t}(x_t - \mathbf{E}x_t) + B_{t,\zeta_t}(u_t - \mathbf{E}u_t)]dt \\ \qquad + [C_{t,\zeta_t}(x_t - \mathbf{E}x_t) + \overline{C}_{t,\zeta_t}\mathbf{E}x_t + D_{t,\zeta_t}(u_t - \mathbf{E}u_t) + \overline{\mathcal{D}}_{t,\zeta_t}\mathbf{E}u_t]dW_t, \\ (u_t - \mathbf{E}u_t) - (u^*_t - \mathbf{E}u^*_t) = (u_t - \mathbf{E}u_t) + \mathcal{M}^\xi_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t})^{-1} \mathcal{H}_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t})'(x_t - \mathbf{E}x_t). \end{cases}$$

and define the operator $\widetilde{\mathcal{L}}_2 : \widetilde{\mathcal{L}}_2(\mathbf{E}u_t) = \mathbf{E}u_t - \mathbf{E}u^*_t$ with its realization

$$\begin{cases} d\mathbf{E}x_t = (\overline{\mathcal{A}}_{t,\zeta_t}\mathbf{E}x_t + \overline{\mathcal{B}}_{t,\zeta_t}\mathbf{E}u_t)dt, \\ \mathbf{E}u_t - \mathbf{E}u^*_t = \mathbf{E}u_t + \widehat{\mathcal{M}}^\xi_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t}, \widetilde{\mathcal{P}}^\xi_{t,\zeta_t})^{-1} \widehat{\mathcal{H}}_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t}, \widetilde{\mathcal{P}}^\xi_{t,\zeta_t})' \mathbf{E}x_t. \end{cases}$$

Then $\widetilde{\mathcal{L}}_1^{-1}$ and $\widetilde{\mathcal{L}}_2^{-1}$ exist, which are determined by

$$\begin{cases} d(x_t - \mathbf{E}x_t) = \{B_{t,\zeta_t}[(u_t - \mathbf{E}u_t) - (u^*_t - \mathbf{E}u^*_t)] \\ \quad + [A_{t,\zeta_t} - B_{t,\zeta_t}\mathcal{M}^\xi_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t})^{-1} \mathcal{H}_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t})'](x_t - \mathbf{E}x_t)\}dt \\ \quad + \{[C_{t,\zeta_t} - D_{t,\zeta_t}\mathcal{M}^\xi_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t})^{-1} \mathcal{H}_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t})'](x_t - \mathbf{E}x_t) + C_{t,\zeta_t}[(u_t - \mathbf{E}u_t) - (u^*_t - \mathbf{E}u^*_t)] \\ \quad + [\overline{C}_{t,\zeta_t} - \overline{\mathcal{D}}_{t,\zeta_t}\widehat{\mathcal{M}}^\xi_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t}, \widetilde{\mathcal{P}}^\xi_{t,\zeta_t})^{-1} \widehat{\mathcal{H}}_{\zeta_t}(\widetilde{\mathcal{Q}}^\xi_{t,\zeta_t}, \widetilde{\mathcal{P}}^\xi_{t,\zeta_t})']\mathbf{E}x_t + \overline{\mathcal{D}}_{t,\zeta_t}(\mathbf{E}u_t - \mathbf{E}u^*_t)\}dW_t, \\ x_0 - \mathbf{E}x_0 = 0 \end{cases}$$

and

$$\begin{cases} d\mathbf{E}x_t = \{\overline{\mathcal{B}}_{t,\zeta_t}(\mathbf{E}u_t - \mathbf{E}u_t^*) + [\overline{\mathcal{A}}_{t,\zeta_t} - \overline{\mathcal{B}}_{t,\zeta_t}\widehat{\mathcal{M}}^{\xi}_{\zeta_t}(\widetilde{\mathcal{Q}}^{\xi}_{t,\zeta_t}, \widetilde{\mathcal{P}}^{\xi}_{t,\zeta_t})^{-1}\widehat{\mathcal{H}}_{\zeta_t}(\widetilde{\mathcal{Q}}^{\xi}_{t,\zeta_t}, \widetilde{\mathcal{P}}^{\xi}_{t,\zeta_t})']\mathbf{E}x_t\}dt, \\ \mathbf{E}x_0 = 0, \end{cases}$$

respectively, where

$$\begin{aligned} u_t - \mathbf{E}u_t =& -\mathcal{M}^{\xi}_{\zeta_t}(\widetilde{\mathcal{Q}}^{\xi}_{t,\zeta_t})^{-1}\mathcal{H}_{\zeta_t}(\widetilde{\mathcal{Q}}^{\xi}_{t,\zeta_t})'(x_t - \mathbf{E}x_t) \\ &+ [(u_t - \mathbf{E}u_t) - (u_t^* - \mathbf{E}u_t^*)], \\ \mathbf{E}u_t =& -\widehat{\mathcal{M}}^{\xi}_{\zeta_t}(\widetilde{\mathcal{Q}}^{\xi}_{t,\zeta_t}, \widetilde{\mathcal{P}}^{\xi}_{t,\zeta_t})^{-1}\widehat{\mathcal{H}}_{\zeta_t}(\widetilde{\mathcal{Q}}^{\xi}_{t,\zeta_t}, \widetilde{\mathcal{P}}^{\xi}_{t,\zeta_t})'\mathbf{E}x_t \\ &+ (\mathbf{E}u_t - \mathbf{E}u_t^*). \end{aligned}$$

Since $\mathcal{M}^{\xi}_i(\widetilde{\mathcal{Q}}^{\xi}_{t,i}) > 0$ and $\widehat{\mathcal{M}}^{\xi}_i(\widetilde{\mathcal{Q}}^{\xi}_{t,i}, \widetilde{\mathcal{P}}^{\xi}_{t,i}) > 0, i = 1, \ldots, s$ are continuous functions on $[0, T]$, there exist $\lambda_i > 0$ and $\widehat{\lambda}_i > 0$ for $i = 1, \ldots, s$, such that $\mathcal{M}^{\xi}_i(\widetilde{\mathcal{Q}}^{\xi}_{t,i}) > \lambda_i I_{n_1}$ and $\widehat{\mathcal{M}}^{\xi}_i(\widetilde{\mathcal{Q}}^{\xi}_{t,i}, \widetilde{\mathcal{P}}^{\xi}_{t,i}) > \widehat{\lambda}_i I_{n_1}$ on $[0, T]$. Let $\lambda = min\{\lambda_1, \lambda_2, \ldots, \lambda_s, \widehat{\lambda}_1, \widehat{\lambda}_2, \ldots, \widehat{\lambda}_s\}$, it follows that $\mathcal{M}^{\xi}_{\zeta_t}(\widetilde{\mathcal{Q}}^{\xi}_{t,\zeta_t}) > \lambda I_{n_1}$ and $\widehat{\mathcal{M}}^{\xi}_{\zeta_t}(\widetilde{\mathcal{Q}}^{\xi}_{t,\zeta_t}, \widetilde{\mathcal{P}}^{\xi}_{t,\zeta_t}) > \lambda I_{n_1}$ for $t \in [0, T], i \in \widetilde{S}$. So, there exist constants $\theta > 0$ and $\varepsilon > 0$, such that

$$\begin{aligned} \widetilde{J}^T_{\xi}(0, u) =& \mathbf{E}\int_0^T [\mathbf{E}u_t - \mathbf{E}u_t^*]'\widehat{\mathcal{M}}^{\xi}_{\zeta_t}(\widetilde{\mathcal{Q}}^{\xi}_{t,\zeta_t}, \widetilde{\mathcal{P}}^{\xi}_{t,\zeta_t})[\mathbf{E}u_t - \mathbf{E}u_t^*]dt \\ &+ \mathbf{E}\int_0^T [(u_t - \mathbf{E}u_t) - (u_t^* - \mathbf{E}u_t^*)]'\mathcal{M}^{\xi}_{\zeta_t}(\widetilde{\mathcal{Q}}^{\xi}_{t,\zeta_t})[(u_t - \mathbf{E}u_t) - (u_t^* - \mathbf{E}u_t^*)]dt \\ &\geq \lambda[\|\widetilde{\mathcal{L}}_2(u_t - \mathbf{E}u_t)\|^2_{[0,T]} + \|\widetilde{\mathcal{L}}_1(\mathbf{E}u_t)\|^2_{[0,T]}] \\ &\geq \lambda\delta[\|u_t - \mathbf{E}u_t\|^2_{[0,T]} + \|\mathbf{E}u_t\|^2_{[0,T]}] = \varepsilon\|u_t\|^2_{[0,T]} > 0 \end{aligned}$$

for $u_t \neq 0$, which yields $\|y_t\|^2_{[0,T]} - \xi^2\|u_t\|^2_{[0,T]} \geq \varepsilon\|u_t\|^2_{[0,T]}$. So, $\|y_t\|^2_{[0,T]} \geq (\xi^2 + \varepsilon)\|u_t\|^2_{[0,T]}$, it is concluded that $\|\Gamma\|^{[0,T]}_{-} \geq (\xi^2 + \varepsilon) > \xi$, which ends the proof. □

In what follows, the necessary condition of $\|\Gamma\|^{[0,T]}_{-} > \xi$ will be given for the following square and time-invariant system:

$$\begin{cases} dx_t = (A_{\zeta_t}x_t + A^0_{\zeta_t}\mathbf{E}x_t + B_{\zeta_t}u_t + B^0_{\zeta_t}\mathbf{E}u_t)dt + (C_{\zeta_t}x_t + C^0_{\zeta_t}\mathbf{E}x_t)dW_t, \\ y_t = K_{\zeta_t}x_t + K^0_{\zeta_t}\mathbf{E}x_t + F_{\zeta_t}u_t + F^0_{\zeta_t}\mathbf{E}u_t, \\ x_0 \in \mathbf{R}^n, t \in [0, T]. \end{cases} \tag{13}$$

Theorem 2. *For system (13) and some given $\xi > 0$, which satisfies $F_i'F_i - \xi^2 I_{n_1} > 0$ and $\overline{\mathcal{F}}_i'\overline{\mathcal{F}}_i - \xi^2 I_{n_1} > 0$, if $\|\Gamma\|^{[0,T]}_{-} > \xi$, there exist unique $\widetilde{\mathcal{Q}}^{\xi}_t = [\widetilde{\mathcal{Q}}^{\xi}_{t,1}, \widetilde{\mathcal{Q}}^{\xi}_{t,2}, \ldots, \widetilde{\mathcal{Q}}^{\xi}_{t,s}]$ and $\widetilde{\mathcal{P}}^{\xi}_t = [\widetilde{\mathcal{P}}^{\xi}_{t,1}, \widetilde{\mathcal{P}}^{\xi}_{t,2}, \ldots, \widetilde{\mathcal{P}}^{\xi}_{t,s}]$, $t \in [0, T]$ satisfying the following GDRE*

$$\begin{cases} A_i'\widetilde{\mathcal{Q}}_{t,i} + \widetilde{\mathcal{Q}}_{t,i}A_i + C_i'\widetilde{\mathcal{Q}}_{t,i}C_i + K_i'K_i + \dot{\widetilde{\mathcal{Q}}}_{t,i} + \sum_{j=1}^s g_{ij}\widetilde{\mathcal{Q}}_{t,j} \\ \quad = (\widetilde{\mathcal{Q}}_{t,i}B_i + K_i'F_i)(F_i'F_i - \xi^2 I_{n_1})^{-1}(\widetilde{\mathcal{Q}}_{t,i}B_i + K_i'F_i)', \\ \overline{\mathcal{A}}_i'\widetilde{\mathcal{P}}_{t,i} + \widetilde{\mathcal{P}}_{t,i}\overline{\mathcal{A}}_i + \overline{\mathcal{C}}_i'\widetilde{\mathcal{Q}}_{t,i}\overline{\mathcal{C}}_i + \overline{\mathcal{K}}_i'\overline{\mathcal{K}}_i + \dot{\widetilde{\mathcal{P}}}_{t,i} + \sum_{j=1}^s g_{ij}\widetilde{\mathcal{P}}_{t,j} \\ \quad = (\widetilde{\mathcal{P}}_{t,i}\overline{\mathcal{B}}_i + \overline{\mathcal{K}}_i'\overline{\mathcal{F}}_i)(\overline{\mathcal{F}}_i'\overline{\mathcal{F}}_i - \xi^2 I_{n_1})^{-1}(\widetilde{\mathcal{P}}_{t,i}\overline{\mathcal{B}}_i + \overline{\mathcal{K}}_i'\overline{\mathcal{F}}_i)', \\ \widetilde{\mathcal{Q}}_{T,i} = \widetilde{\mathcal{P}}_{T,i} = 0, i = 1, \ldots, s \end{cases} \tag{14}$$

Moreover, $\widetilde{J}_{\xi}^{T}(x_0,i,u)$ and $\widetilde{J}_{\xi}^{T}(x_0,u)$ are minimized by $u_t^* = U_{\zeta_t}^*(x_t^{u^*} - \mathbf{E}x_t^{u^*}) + \overline{U}_{\zeta_t}^* \mathbf{E}x_t^{u^*}$, with $U_{\zeta_t}^* = -(F_{\zeta_t}' F_{\zeta_t} - \xi^2 I_{n_1})^{-1}(B_{\zeta_t}' \widetilde{\mathcal{Q}}_{t,\zeta_t}^{\xi} + F_{\zeta_t}' K_{\zeta_t})$ and $\overline{U}_{\zeta_t}^* = (\overline{\mathcal{F}}_{\zeta_t}' \overline{\mathcal{F}}_{\zeta_t} - \xi^2 I_{n_1})^{-1}(\overline{\mathcal{B}}_{\zeta_t}' \widetilde{\mathcal{P}}_{t,\zeta_t}^{\xi} + \overline{\mathcal{F}}_{\zeta_t}' \overline{\mathcal{K}}_{\zeta_t})$, where $x_t^{u^*}$ is the state trajectory of system (13) when $u_t = u_t^*$, and

$$\begin{cases} \min\limits_{u_t \in M_{\mathcal{F}}^2([0,T],\mathbf{R}^{n_1}), i \in \widetilde{S}} \widetilde{J}_{\xi}^{T}(x_0,i,u) = \widetilde{J}_{\xi}^{T}(x_0,i,u^*) = \mathbf{E}x_0' \widetilde{\mathcal{P}}_{0,i}^{\xi} \mathbf{E}x_0 \\ \min\limits_{u_t \in M_{\mathcal{F}}^2([0,T],\mathbf{R}^{n_1}), \zeta_0 \in \widetilde{S}} \widetilde{J}_{\xi}^{T}(x_0,u) = \widetilde{J}_{\xi}^{T}(x_0,u^*) = \mathbf{E}x_0' \widetilde{\mathcal{P}}_{0,\zeta_0}^{\xi} \mathbf{E}x_0 = \mathbf{E}x_0' \sum_{i=1}^{s} \widetilde{\mathcal{P}}_{0,i}^{\xi} \mathcal{P}(\zeta_0 = i) \mathbf{E}x_0, \end{cases} \tag{15}$$

where $\overline{\mathcal{A}}_{\zeta_t} = A_{\zeta_t} + A_{\zeta_t}^0$, $\overline{\mathcal{B}}_{\zeta_t} = B_{\zeta_t} + B_{\zeta_t}^0$, $\overline{\mathcal{C}}_{\zeta_t} = C_{\zeta_t} + C_{\zeta_t}^0$, $\overline{\mathcal{K}}_{\zeta_t} = K_{\zeta_t} + K_{\zeta_t}^0$, $\overline{\mathcal{F}}_{\zeta_t} = F_{\zeta_t} + F_{\zeta_t}^0$.

Proof. It will be proved that $\|\Gamma\|_{-}^{[0,T]} > \xi$ can imply that there is a unique solution $(\widetilde{\mathcal{Q}}_t^{\xi}, \widetilde{\mathcal{P}}_t^{\xi})$ of (14) on $[0,T]$. Otherwise, according to standard theory of differential equations, there is a finite escape time for (14), i.e., (14) has a unique solution $\widetilde{\mathcal{P}}_t^{\xi}$ on a maximal interval $(t_1, T]$ with $t_1 \geq 0$, and $\widetilde{\mathcal{P}}_t^{\xi}$ becomes unbounded when $t \to t_1$. Next, a contradiction will be derived.

For $0 < \rho < T - t_1$, $x_{t_1+\rho} = x_{t_1,\rho} \in \mathbf{R}^n$, similar to the method of Theorem 3.1, it can be shown that

$$\begin{aligned} \widetilde{J}_{\xi}^{T}(x_{t_1,\rho},i,u) =& \mathbf{E}\left\{\int_{t_1+\rho}^{T} [\|y_t\|^2 - \xi^2\|u_t\|^2] dt | \zeta_{t_1+\rho} = i\right\} \\ =& \mathbf{E}x_{t_1,\rho}' \widetilde{\mathcal{P}}_{t_1+\rho,i}^{\xi} \mathbf{E}x_{t_1,\rho} + \left\{\int_{t_1+\rho}^{T} [\mathbf{E}u_t - \mathbf{E}u_t^*]' (\overline{\mathcal{F}}_{\zeta_t}' \overline{\mathcal{F}}_{\zeta_t} - \xi^2 I_{n_1}) [\mathbf{E}u_t - \mathbf{E}u_t^*] dt | \zeta_{t_1+\rho} = i\right\} \\ &+ \mathbf{E}\left\{\int_{t_1+\rho}^{T} [(u_t - \mathbf{E}u_t) - (u_t^* - \mathbf{E}u_t^*)]' (F_{\zeta_t}' F_{\zeta_t} - \xi^2 I_{n_1}) [(u_t - \mathbf{E}u_t) - (u_t^* - \mathbf{E}u_t^*)] dt | \zeta_{t_1+\rho} = i\right\}, \end{aligned} \tag{16}$$

where

$$\begin{aligned} \mathbf{E}u_t^* &= -(\overline{\mathcal{F}}_{\zeta_t}' \overline{\mathcal{F}}_{\zeta_t} - \xi^2 I_q)^{-1} (\overline{\mathcal{B}}_{\zeta_t}' \widetilde{\mathcal{P}}_t^T + \overline{\mathcal{F}}_{\zeta_t}' \overline{\mathcal{K}}_{\zeta_t}) \mathbf{E}x_t, \\ u_t^* - \mathbf{E}u_t^* &= -(F_{\zeta_t}' F_{\zeta_t} - \xi^2 I_q)^{-1} (B_{\zeta_t}' \widetilde{\mathcal{Q}}_t^T + F_{\zeta_t}' K_{\zeta_t})(x_t - \mathbf{E}x_t). \end{aligned}$$

In addition, it is obviously that there exists $\varpi_1 > 0$ such that

$$\begin{aligned} \widetilde{J}_{\xi}^{T}(x_{t_1,\rho},i,0) =& \mathbf{E}\left\{\int_{t_1+\rho}^{T} y_t' y_t dt | \zeta_{t_1+\rho} = i\right\} \\ =& \mathbf{E}\left\{\int_{t_1+\rho}^{T} (\|K_{\zeta_t}(x_{(t,0;x_{t_1,\rho},\zeta_{t_1+\rho})} - \mathbf{E}x_{(t,0;x_{t_1,\rho},\zeta_{t_1+\rho})})\|^2 dt | \zeta_{t_1+\rho} = i\right\} \\ &+ \left\{\int_{t_1+\rho}^{T} \|\overline{\mathcal{K}}_{\zeta_t} \mathbf{E}x_{(t,0;x_{t_1,\rho},\zeta_{t_1+\rho})}\|^2) dt | \zeta_{t_1+\rho} = i\right\} \\ \leq& \varpi_1 [\|x_{t_1,\rho} - \mathbf{E}x_{t_1,\rho}\|^2 + \|\mathbf{E}x_{t_1,\rho}\|^2] = \varpi_1 \|x_{t_1,\rho}\|^2. \end{aligned} \tag{17}$$

Combining (16) and (17), it yields immediately

$$\begin{aligned} \min_{u_t \in M_{\mathcal{F}}^2([t_1+\rho,T],\mathbf{R}^{n_1})} \widetilde{J}_{\xi}^{T}(x_{t_1,\rho},i,u) = \widetilde{J}_{\xi}^{T}(x_{t_1,\rho},i,u^*) = \mathbf{E}x_{t_1,\rho}' \widetilde{\mathcal{P}}_{t_1+\rho,i}^{\xi} \mathbf{E}x_{t_1,\rho} \\ \leq \widetilde{J}_{\xi}^{T}(x_{t_1,\rho},i,0) \leq \varpi_1 \|x_{t_1,\rho}\|^2. \end{aligned} \tag{18}$$

Let $x_{(t,u;x_{t_1,\rho},\zeta_{t_1+\rho})}$ be the solution of (13) with initial state $\zeta_{t_1+\rho}$ and $x_{t_1,\rho}$, by linearity

$$x_{(t,u;x_{t_1,\rho},\zeta_{t_1+\rho})} = x_{(t,0;x_{t_1,\rho},\zeta_{t_1+\rho})} + x_{(t,u;0,\zeta_{t_1+\rho})}.$$

Suppose $\Omega_t = (\Omega_{t,1}, \dots, \Omega_{t,l})$ and $\Xi_t = (\Xi_{t,1}, \dots \Xi_{t,l})$ satisfy the following equation

$$\begin{cases} \dot{\Omega}_{t,i} + \sum_{j=1}^{s} g_{ij}\Omega_{t,j} + K_i'K_i \\ +A_i'\Omega_{t,i} + \Omega_{t,i}A_i + C_i'\Omega_{t,i}C_i = 0, \\ \dot{\Xi}_{t,i} + \sum_{j=1}^{s} g_{ij}\Xi_{t,j} + \overline{\mathcal{K}}_i'\overline{\mathcal{K}}_i \\ +\overline{\mathcal{A}}_i'\Xi_{t,i} + \Xi_{t,i}\overline{\mathcal{A}}_i + \overline{\mathcal{C}}_i'\Omega_{t,i}\overline{\mathcal{C}}_i = 0, \\ \Omega_{T,i} = \Xi_{T,i} = 0, i = 1, \dots, s, \end{cases} \tag{19}$$

one has

$$\begin{aligned} \widetilde{J}_\xi^T(x_{t_1,\rho}, i, u) = &\widetilde{J}_\xi^T(0, i, u) + \mathbf{E}x_{t_1,\rho}'\Xi_{t_1+\rho,i}\mathbf{E}x_{t_1,\rho} \\ &+ [x_{t_1,\rho} - \mathbf{E}x_{t_1,\rho}]'\Omega_{t_1+\rho,i}[x_{t_1,\rho} - \mathbf{E}x_{t_1,\rho}] \\ &+ \mathbf{E}\left\{\int_{t_1+\rho}^{T} 2V_{\zeta_t}[u_t - \mathbf{E}u_t]dt|\zeta_{t_1+\rho} = i\right\} \\ &+ \left\{\int_{t_1+\rho}^{T} 2\overline{V}_{\zeta_t}\mathbf{E}u_t dt|\zeta_{t_1+\rho} = i\right\}, \end{aligned} \tag{20}$$

where $V_{\zeta_t} = [x_{(t,0;x_{t_1,\rho},\zeta_{t_1+\rho})} - \mathbf{E}x_{(t,0;x_{t_1,\rho},\zeta_{t_1+\rho})}]'(\Omega_{t,\zeta_t}B_{\zeta_t} + K_{\zeta_t}'F_{\zeta_t})$, $\overline{V}_{\zeta_t} = \mathbf{E}x_{(t,0;x_{t_1,\rho},\zeta_{t_1+\rho})}'(\Xi_{t,\zeta_t}\overline{\mathcal{B}}_{\zeta_t} + \overline{\mathcal{K}}_{\zeta_t}'\overline{\mathcal{F}}_{\zeta_t})$.

According to (16), there exists $\alpha > 0$ such that

$$\begin{aligned} \widetilde{J}_\xi^T(0, i, u) = &\left\{\int_{t_1+\rho}^{T} [\mathbf{E}u_t - \mathbf{E}u_t^*]'(\overline{\mathcal{F}}_{\zeta_t}'\overline{\mathcal{F}}_{\zeta_t} - \xi^2 I_{n_1})[\mathbf{E}u_t - \mathbf{E}u_t^*]dt|\zeta_{t_1+\rho} = i\right\} \\ &+ \mathbf{E}\left\{\int_{t_1+\rho}^{T} [(u_t - \mathbf{E}u_t) - (u_t^* - \mathbf{E}u_t^*)]'(F_{\zeta_t}'F_{\zeta_t} - \xi^2 I_{n_1})[(u_t - \mathbf{E}u_t) - (u_t^* - \mathbf{E}u_t^*)]dt|\zeta_{t_1+\rho} = i\right\} \\ &\geq \alpha(\|u_t - \mathbf{E}u_t\|_{[t_1+\rho,T]}^2 + \|u_t\|_{[t_1+\rho,T]}^2). \end{aligned} \tag{21}$$

Combining (20) and (21), it follows that

$$\begin{aligned} \widetilde{J}_\xi^T(x_{t_1,\rho}, i, u) \geq &[x_{t_1,\rho} - \mathbf{E}x_{t_1,\rho}]'\Omega_{t_1,\rho}[x_{t_1,\rho} - \mathbf{E}x_{t_1,\rho}] + \mathbf{E}x_{t_1,\rho}'\Xi_{t_1+\rho}\mathbf{E}x_{t_1,\rho} \\ &+ \mathbf{E}\left\{\int_{t_1+\rho}^{T} (2V_{\zeta_t}[u_t - \mathbf{E}u_t] + \alpha\|u_t - \mathbf{E}u_t\|_{[t_1+\rho,T]}^2)dt|\zeta_{t_1+\rho} = i\right\} \\ &+ \left\{\int_{t_1+\rho}^{T} (2\overline{V}_{\zeta_t}\mathbf{E}u_t + \alpha\|u_t\|_{[t_1+\rho,T]}^2)dt|\zeta_{t_1+\rho} = i\right\}, \\ \geq &[x_{t_1,\rho} - \mathbf{E}x_{t_1,\rho}]'\Omega_{t_1+\rho,i}[x_{t_1,\rho} - \mathbf{E}x_{t_1,\rho}] + \mathbf{E}x_{t_1,\rho}'\Xi_{t_1+\rho,i}\mathbf{E}x_{t_1,\rho} \\ &+ \mathbf{E}\left\{\int_{t_1+\rho}^{T} \|\alpha[(u_t - \mathbf{E}u_t) + \alpha^{-2}V_{\zeta_t}']\|^2 dt|\zeta_{t_1+\rho} = i\right\} \\ &- \mathbf{E}\left\{\int_{t_1+\rho}^{T} \|\alpha^{-1}V_{\zeta_t}'\|^2 dt|\zeta_{t_1+\rho} = i\right\} \\ &+ \left\{\int_{t_1+\rho}^{T} \|\alpha[\mathbf{E}u_t + \alpha^{-2}\overline{V}_{\zeta_t}'\|^2 dt|\zeta_{t_1+\rho} = i\right\} \\ &- \mathbf{E}\left\{\int_{t_1+\rho}^{T} \|\alpha^{-1}\overline{V}_{\zeta_t}'\|^2 dt|\zeta_{t_1+\rho} = i\right\} \\ &+ \mathbf{E}x_{t_1,\rho}'\Xi_{t_1+\rho,i}\mathbf{E}x_{t_1,\rho} \\ \geq &[x_{t_1,\rho} - \mathbf{E}x_{t_1,\rho}]'\Omega_{t_1+\rho,i}[x_{t_1,\rho} - \mathbf{E}x_{t_1,\rho}] + \mathbf{E}x_{t_1,\rho}'\Xi_{t_1+\rho,i}\mathbf{E}x_{t_1,\rho} \\ &- \mathbf{E}\left\{\int_{t_1+\rho}^{T} \|\alpha^{-1}V_{\zeta_t}'\|^2 dt|\zeta_{t_1+\rho} = i\right\} \\ &- \mathbf{E}\left\{\int_{t_1+\rho}^{T} \|\alpha^{-1}\overline{V}_{\zeta_t}'\|^2 dt|\zeta_{t_1+\rho} = i\right\}. \end{aligned} \tag{22}$$

Obviously, there exists $\eta > 0$ such that

$$\eta(\|x_{t_1,\rho}-\mathbf{E}x_{t_1,\rho}\|^2+\|\mathbf{E}x_{t_1,\rho}\|^2)\geq \mathbf{E}\int_{t_1+\rho}^{T}\|[x_{(t,0;x_{t_1,\rho},\zeta_{t_1+\rho})}-\mathbf{E}x_{(t,0;x_{t_1,\rho},\zeta_{t_1+\rho})}]\|^2dt + \int_{t_1+\rho}^{T}\|\mathbf{E}x_{(t,0;x_{t_1,\rho},\zeta_{t_1+\rho})}]\|^2dt. \tag{23}$$

Therefore, there is some constant $\omega_0>0$ satisfying the following inequality

$$\mathbf{E}\Big\{\int_{t_1+\rho}^{T}\|\alpha^{-1}V'_{\zeta_t}\|^2dt|\zeta_{t_1+\rho}=i\Big\}+\Big\{\int_{t_1+\rho}^{T}\|\alpha^{-1}\overline{V}'_{\zeta_t}\|^2dt|\zeta_{t_1+\rho}=i\Big\} \leq\omega_0\|x_{t_1,\rho}-\mathbf{E}x_{t_1,\rho}\|^2+\omega_0\|\mathbf{E}x_{t_1,\rho}\|^2. \tag{24}$$

According to Lemma 1,

$$\mathbf{E}\{\Upsilon_{T,x_T,\zeta_T}-\Upsilon_{t_1+\rho,x_{t_1,\rho},\zeta_{t_1+\rho}}|\zeta_{t_1+\rho}=i\}=\mathbf{E}\Big\{\int_{t_1+\rho}^{T}\Delta\Upsilon_{t,x_t,\zeta_t}dt|\zeta_{t_1+\rho}=i\Big\}.$$

Let $\Upsilon_{t,x_t,\zeta_t}=\mathbf{E}x'_t\Xi_{t,\zeta_t}\mathbf{E}x_t$ and $\Upsilon_{t,x_t,\zeta_t}=(x_t-\mathbf{E}x_t)'\Omega_{t,\zeta_t}(x_t-\mathbf{E}x_t)$ respectively, there is $\omega_1>0$ such that

$$\begin{aligned}&[x_{t_1,\rho}-\mathbf{E}x_{t_1,\rho}]'\Omega_{t_1+\rho}[x_{t_1,\rho}-\mathbf{E}x_{t_1,\rho}]+\mathbf{E}x'_{t_1,\rho}\Xi_{t_1+\rho}\mathbf{E}x_{t_1,\rho}\\ =&-\mathbf{E}\Big\{\int_{t_1+\rho}^{T}\|K_{\zeta_t}[x_{(t,0;x_{t_1,\rho},\zeta_{t_1+\rho})}-\mathbf{E}x_{(t,0;x_{t_1,\rho},\zeta_{t_1+\rho})}]\|^2dt|\zeta_{t_1+\rho}=i\Big\}\\ &-\Big\{\int_{t_1+\rho}^{T}\|\overline{\mathcal{K}}_{\zeta_t}\mathbf{E}x_{(t,0;x_{t_1,\rho},\zeta_{t_1+\rho})}\|^2dt|\zeta_{t_1+\rho}=i\Big\}\\ \geq&-\omega_1\|x_{t_1,\rho}\|^2.\end{aligned} \tag{25}$$

From (22), (24), and (25), it can be obtained that

$$\widetilde{J}^T_\xi(x_{t_1,\rho},i,u)\geq-(\omega_0+\omega_1)\|x_{t_1,\rho}\|^2. \tag{26}$$

Combining (18) and (26), it yields $-(\omega_0+\omega_1)I_{n_2}\leq\widetilde{\mathcal{P}}^\xi_{t_1+\rho,i}\leq\varpi_1 I_{n_2}$. So $\widetilde{\mathcal{P}}^\xi_{t_1+\rho,i}$ can not be unbounded when $\rho\to0$, which lead to a contradiction. Therefore, there is unique $\widetilde{\mathcal{Q}}^\xi_t=[\widetilde{\mathcal{Q}}^\xi_{t,1},\widetilde{\mathcal{Q}}^\xi_{t,2},\ldots,\widetilde{\mathcal{Q}}^\xi_{t,s}]$ and $\widetilde{\mathcal{P}}^\xi_t=[\widetilde{\mathcal{P}}^\xi_{t,1},\widetilde{\mathcal{P}}^\xi_{t,2},\ldots,\widetilde{\mathcal{P}}^\xi_{t,s}]$ satisfying (14) on $t\in[0,T]$.

Moreover, similar to Equation (18), we have

$$\begin{cases}\min\limits_{u_t\in M^2_{\mathcal{F}}([0,T],\mathbf{R}^{n_1}),i\in\widetilde{S}}\widetilde{J}^T_\xi(x_0,i,u)=\widetilde{J}^T_\xi(x_0,i,u^*)=\mathbf{E}x'_0\widetilde{\mathcal{P}}^\xi_{0,i}\mathbf{E}x_0\\ \min\limits_{u_t\in M^2_{\mathcal{F}}([0,T],\mathbf{R}^{n_1}),\zeta_0\in\widetilde{S}}\widetilde{J}^T_\xi(x_0,u)=\widetilde{J}^T_\xi(x_0,u^*)=\mathbf{E}x'_0\widetilde{\mathcal{P}}^\xi_{0,\zeta_0}\mathbf{E}x_0=\mathbf{E}x'_0\sum_{i=1}^s\widetilde{\mathcal{P}}^\xi_{0,i}\mathcal{P}(\zeta_0=i)\mathbf{E}x_0.\end{cases}$$

□

For system (13), Theorem 1 and Theorem 2 can yield the following equivalence relationships immediately:

Theorem 3. *For system (13) and some given $\xi>0$, which satisfies $F'_iF_i-\xi^2I_{n_1}>0$ and $\overline{\mathcal{F}}'_i\overline{\mathcal{F}}_i-\xi^2I_{n_1}>0$, the following are equivalent:*

(1) $\|\Gamma\|^{[0,T]}_{-}>\xi$,

(2) there exist unique $\widetilde{\mathcal{Q}}^\xi_t=[\widetilde{\mathcal{Q}}^\xi_{t,1},\widetilde{\mathcal{Q}}^\xi_{t,2},\ldots,\widetilde{\mathcal{Q}}^\xi_{t,s}]$ and $\widetilde{\mathcal{P}}^\xi_t=[\widetilde{\mathcal{P}}^\xi_{t,1},\widetilde{\mathcal{P}}^\xi_{t,2},\ldots,\widetilde{\mathcal{P}}^\xi_{t,s}]$, $t\in[0,T]$ satisfying GDRE (14)

Moreover

$$\begin{cases}\min\limits_{u_t\in M^2_{\mathcal{F}}([0,T],\mathbf{R}^{n_1})}\widetilde{J}^T_\xi(x_0,i,u)=\widetilde{J}^T_\xi(x_0,i,u^*)=\mathbf{E}x'_0\widetilde{\mathcal{P}}^\xi_{0,i}\mathbf{E}x_0\\ \min\limits_{u_t\in M^2_{\mathcal{F}}([0,T],\mathbf{R}^{n_1})}\widetilde{J}^T_\xi(x_0,u)=\widetilde{J}^T_\xi(x_0,u^*)=\mathbf{E}x'_0\widetilde{\mathcal{P}}^\xi_{0,\zeta_0}\mathbf{E}x_0,\end{cases}$$

Theorem 4. *For system (13) and some given $\xi > 0$, which satisfies $F_i'F_i - \xi^2 I_{n_1} > 0$ and $\overline{\mathcal{F}}_i'\overline{\mathcal{F}}_i - \xi^2 I_{n_1} > 0$, $\|\Gamma\|_-^{[0,T]} > \xi$, then the $\widetilde{\mathcal{P}}_{t,i}^T$ satisfying (14) decreases as T increases for $t \in [0,T]$.*

Proof. Suppose $\widehat{T} > \overline{T}$, $\widetilde{J}_\xi^{\overline{T}-t}(x_0, i, u)$ is optimal when $u = u^{\overline{T}-t,*}$, set

$$\mathbf{E}u_\kappa = \begin{cases} \mathbf{E}u_\kappa^{\overline{T}-t,*}, & \kappa \in [0, \overline{T}-t], \\ -(\overline{\mathcal{F}}_{\zeta_t}'\overline{\mathcal{F}}_{\zeta_t} - \xi^2 I_{n_1})^{-1}\overline{\mathcal{F}}_{\zeta_t}'\overline{\mathcal{K}}_{\zeta_t}\mathbf{E}x_\kappa, & \kappa \in (\overline{T}-t, \widehat{T}-t]. \end{cases}$$

$$u_\kappa - \mathbf{E}u_\kappa = \begin{cases} u_\kappa^{\overline{T}-t,*} - \mathbf{E}u_\kappa^{\overline{T}-t,*}, & \kappa \in [0, \overline{T}-t], \\ -(F_{\zeta_t}'F_{\zeta_t} - \xi^2 I_{n_1})^{-1}F_{\zeta_t}'K_{\zeta_t}(x_\kappa - \mathbf{E}x_\kappa), & \kappa \in (\overline{T}-t, \widehat{T}-t], \end{cases}$$

The time-invariance of $\widetilde{\mathcal{P}}_{t,i}^T$ yields $\widetilde{\mathcal{P}}_{0,i}^{T-t} = \widetilde{\mathcal{P}}_{t,i}^T$. Therefore, for $\forall x_0 \in \mathbf{R}^n$, we can obtain

$$\begin{aligned}
&\mathbf{E}x_0'\widetilde{\mathcal{P}}_{t,i}^{\widehat{T}}\mathbf{E}x_0 = \mathbf{E}x_0'\widetilde{\mathcal{P}}_{0,i}^{\widehat{T}-t}\mathbf{E}x_0 \\
\le& \widetilde{J}_\xi^{\overline{T}-t}(x_0, i, u^{\overline{T}-t,*}) + \mathbf{E}\left\{\int_{\overline{T}-t}^{\widehat{T}-t}[y_\kappa'y_\kappa - \xi^2 u_\kappa'u_\kappa]d\kappa|\zeta_0 = i\right\} \\
=& \widetilde{J}_\xi^{\overline{T}-t}(x_0, i, u^{\overline{T}-t,*}) + \left\{\mathbf{E}\int_{\overline{T}-t}^{\widehat{T}-t}\mathbf{E}x_\kappa'\overline{\mathcal{K}}_{\zeta_\kappa}'[I - \overline{\mathcal{F}}_{\zeta_\kappa}(\overline{\mathcal{F}}_{\zeta_\kappa}'\overline{\mathcal{F}}_{\zeta_\kappa} - \xi^2 I_{n_1})^{-1}\overline{\mathcal{F}}_{\zeta_\kappa}']\overline{\mathcal{K}}_{\zeta_\kappa}\mathbf{E}x_\kappa d\kappa|\zeta_0 = i\right\} \\
&+\mathbf{E}\left\{\int_{\overline{T}-t}^{\widehat{T}-t}[K_{\zeta_t}(x_\kappa - \mathbf{E}x_\kappa)]'[I - F_{\zeta_\kappa}(F_{\zeta_\kappa}'F_{\zeta_\kappa} - \xi^2 I_{n_1})^{-1}F_{\zeta_\kappa}'][K_{\zeta_t}(x_\kappa - \mathbf{E}x_\kappa)]d\kappa|\zeta_0 = i\right\} \\
\le& \widetilde{J}_\xi^{\overline{T}-t}(x_0, i, u^{\overline{T}-t,*}) = \mathbf{E}x_0'\widetilde{\mathcal{P}}_{t,i}^{\overline{T}}\mathbf{E}x_0,
\end{aligned}$$

which indicates that $\widetilde{\mathcal{P}}_{t,i}^T$ decreases as T increases for $t \in [0,T]$. □

4. Numerical Example

A simple example of $\mathcal{H}_-$ index is given to demonstrate the effectiveness of our results.

Example 1. *We consider mean-field stochastic Markov system (13), corresponding parameters are given as following:*

$$A_1 = \begin{bmatrix} 0.3 & 0.2 \\ 0.1 & 0.2 \end{bmatrix}, A_1^0 = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}, A_2 = \begin{bmatrix} 0.3 & 0.1 \\ 0.2 & 0.2 \end{bmatrix}, A_2^0 = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.2 & 0.4 \end{bmatrix}, B_1^0 = \begin{bmatrix} 0.1 & 0.2 \\ 0.1 & 0.1 \end{bmatrix}, B_2 = \begin{bmatrix} 0.2 & 0.1 \\ 0.2 & 0.4 \end{bmatrix}, B_2^0 = \begin{bmatrix} 0.2 & 0.2 \\ 0.1 & 0.2 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.3 & 0.1 \\ 0.2 & 0.1 \end{bmatrix}, C_1^0 = \begin{bmatrix} 0.1 & 0.3 \\ 0.1 & 0.1 \end{bmatrix}, C_2 = \begin{bmatrix} 0.2 & 0.2 \\ 0.1 & 0.2 \end{bmatrix}, C_2^0 = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.1 \end{bmatrix},$$

$$K_1 = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.8 \end{bmatrix}, K_1^0 = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.5 \end{bmatrix}, K_2 = \begin{bmatrix} 0.7 & 0 \\ 0 & 0.7 \end{bmatrix}, K_2^0 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix},$$

$$F_1 = \begin{bmatrix} 0.8 & 0 \\ 0 & 0.7 \end{bmatrix}, F_1^0 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, F_2 = \begin{bmatrix} 0.7 & 0 \\ 0 & 0.8 \end{bmatrix}, F_2^0 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$\mathcal{G} = \begin{bmatrix} -0.8 & 0.8 \\ 0.6 & -0.6 \end{bmatrix}, \xi = 0.6.$$

By solving the GDRE (14), we can obtain

$$\widetilde{\mathcal{Q}}_{t,1} = \begin{bmatrix} \widetilde{\mathcal{Q}}_{t,1}^{11} & \widetilde{\mathcal{Q}}_{t,1}^{12} \\ \widetilde{\mathcal{Q}}_{t,1}^{12} & \widetilde{\mathcal{Q}}_{t,1}^{22} \end{bmatrix}, \widetilde{\mathcal{P}}_{t,1} = \begin{bmatrix} \widetilde{\mathcal{P}}_{t,1}^{11} & \widetilde{\mathcal{P}}_{t,1}^{12} \\ \widetilde{\mathcal{P}}_{t,1}^{12} & \widetilde{\mathcal{P}}_{t,1}^{22} \end{bmatrix},$$

$$\widetilde{\mathcal{Q}}_{t,2} = \begin{bmatrix} \widetilde{\mathcal{Q}}_{t,2}^{11} & \widetilde{\mathcal{Q}}_{t,2}^{12} \\ \widetilde{\mathcal{Q}}_{t,2}^{12} & \widetilde{\mathcal{Q}}_{t,2}^{22} \end{bmatrix}, \ \widetilde{\mathcal{P}}_{t,2} = \begin{bmatrix} \widetilde{\mathcal{P}}_{t,2}^{11} & \widetilde{\mathcal{P}}_{t,2}^{12} \\ \widetilde{\mathcal{P}}_{t,2}^{12} & \widetilde{\mathcal{P}}_{t,2}^{22} \end{bmatrix}.$$

Figures 1–3 present their trajectories.

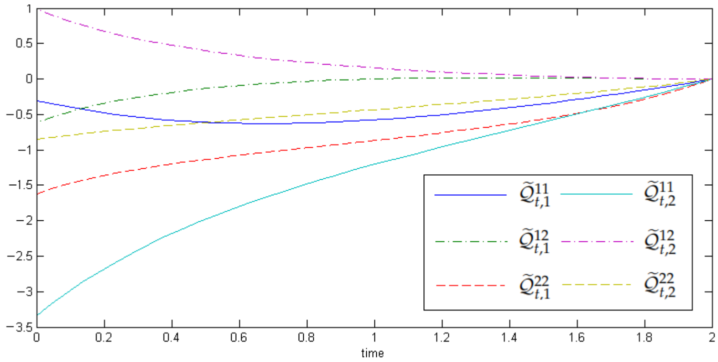

Figure 1. The trajectories of $\widetilde{\mathcal{Q}}_{t,1}$ and $\widetilde{\mathcal{Q}}_{t,2}$.

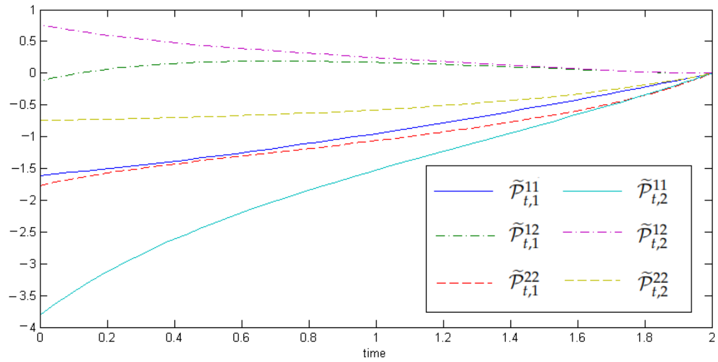

Figure 2. The trajectories of $\widetilde{\mathcal{P}}_{t,1}$ and $\widetilde{\mathcal{P}}_{t,2}$.

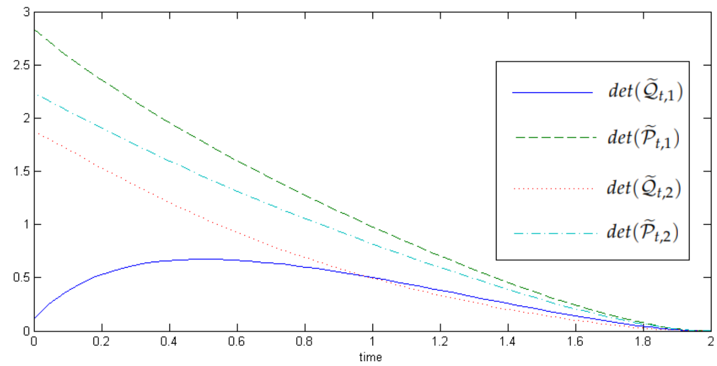

Figure 3. The trajectories of $det(\widetilde{\mathcal{Q}}_{t,1})$, $det(\widetilde{\mathcal{Q}}_{t,2})$, $det(\widetilde{\mathcal{P}}_{t,1})$, and $det(\widetilde{\mathcal{P}}_{t,2})$.

Figures 1–3 show that $\widetilde{\mathcal{P}}_{t,1} \leq 0$ and $\widetilde{\mathcal{P}}_{t,2} \leq 0$, which means $\|\Gamma\|_{-}^{[0,T]} > \xi$.

5. Conclusions

This paper investigates the problem of $\mathcal{H}_{-}$ index for stochastic mean-field type Markov jump systems with multiplicative noise. It is shown that when corresponding generalized differential Riccati equations is solvable, $\mathcal{H}_{-}$ index is greater than a given positive number for mean-field stochastic differential equation with state and input-dependent noise. Particularly, under some appropriate conditions, we obtain a sufficient and necessary condition for mean-field stochastic system with only x-dependent noise, which illustrate that $\mathcal{H}_{-}$ index greater than a given positive number is equivalent to the solvability of GDREs. A numerical example is given to shed light on obtained theoretical results.

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