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# Integration of Bayesian Adaptive Exponentially Weighted Moving Average Control Chart and Paired Ranked-Based Sampling for Enhanced Semiconductor Manufacturing Process Monitoring

**Botao Liu [1], Muhammad Noor-ul-Amin [2], Imad Khan [3,*], Emad A. A. Ismail [4] and Fuad A. Awwad [4]**

[1] School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China; xjliubotao@sina.com
[2] Department of Statistics, COMSATS University Islamabad, Lahore Campus, Lahore 57000, Pakistan; nooramin.stats@gmail.com
[3] Department of Statistics, Abdul Wali Khan University Mardan, Mardan 23200, Pakistan
[4] Department of Quantitative Analysis, College of Business Administration, King Saud University, P.O. Box 71115, Riyadh 11587, Saudi Arabia; emadali@ksu.edu.sa (E.A.A.I.); fawwad@ksu.edu.sa (F.A.A.)
* Correspondence: imad_khan@awkum.edu.pk

**Abstract:** Exponentially weighted moving average (EWMA) and Shewhart control charts are commonly utilized to detect the small to moderate and large shifts in the process mean, respectively. This article introduces a novel Bayesian AEWMA control chart that employs various loss functions (LFs), including square error loss function (SELF) and LINEX loss function (LLF). The control chart incorporates an informative prior for posterior and posterior predictive distributions. Additionally, the control chart utilizes various paired ranked set sampling (PRSS) schemes to improve its accuracy and effectiveness. The average run length (ARL) and standard deviation of run length (SDRL) are used to evaluate the performance of the suggested control chart. Monte Carlo simulations are conducted to compare the performance of the proposed approach to other control charts. The results show that the proposed method outperforms in identifying out-of-control signals, particularly under PRSS schemes compared to simple random sampling (SRS). The proposed CCs effectiveness was validated using a real-life semiconductor manufacturing application, utilizing different PRSS schemes. The performance of the Bayesian AEWMA CC was evaluated, demonstrating its superiority in detecting out-of-control signs compared to existing CCs. This study introduces an innovative method incorporating various LFs and PRSS schemes, providing an enhanced and efficient approach for identifying shifts in the process mean.

**Keywords:** control charts; adaptive EWMA; Bayesian approach; loss function; statistical process control; semiconductor manufacturing

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## 1. Introduction

All processes, whether in manufacturing or non-manufacturing settings, inherently feature variations, which can be classified as either natural or abnormal. Natural variations are inescapable and do not pose any harm. However, abnormal variations can have detrimental effects on the process and compromise the quality of the final outcome. To ensure the efficiency of the process and promptly deal with these variations, it is vital to implement corrective measures. The term "shift" denotes the magnitude of irregular fluctuations in parameters, such as location and dispersion. In essence, the term "shift" denotes a systematic alteration of the location parameter of a probability distribution, such as a mean or median shift. The uneven fluctuation of parameters is not frequently used to characterize it. In statistical process control (SPC), quality control charts (CCs) are a crucial instrument for tracking and managing the caliber of a manufacturing process. Plotting process data across time and setting control limits based on past process performance is how quality CCs operate. The process is considered as "control" if the data lies within the control limits, meaning that it is operating within its expected range of variation. If the data deviates from the predetermined range, it may indicate a process issue that requires further detailed investigation. There are several kinds of quality CCs, each one intended to track a particular step in the process. Operators may immediately recognize changes and take appropriate action by using these charts to analyses trends, patterns, and shifts in the data produced by the process. Walter Shewhart [1] is the inventor of memoryless-type CCs, which are less susceptible to small or moderate shifts and effectively monitor significant shifts in the manufacturing process by using only the most recent sample information. In contrast, traditional memory-type control charts, such as the cumulative sum (CUSUM) and exponentially weighted moving average (EWMA) control charts, as suggested by [2] and [3], respectively, are commonly employed to effectively monitor small-to-moderate changes. The fundamental designs of traditional memory type CCs are continually being modified and enhanced (see for example, [4–8]). Haq and Woodall [9] highlight the close relationship between the modified EWMA CC and a moving average-based EWMA CC. Both exhibit unfavorable weighting functions that impact run length properties, leading to the suggestion that an ordinary EWMA CC may offer better overall performance. Yeganeh et al. [10] present adaptive GLR CCs for monitoring linear profiles using variable sampling intervals and sequential sampling, particularly focusing on scenarios where explanatory variables are uncontrollable. The SS approach exhibits superior performance with lower ATS values in simulations and real-life applications. Riaz et al. [11] introduces two adaptive CCs for process mean vector shift monitoring, using PCA dimensionality reduction and adaptive techniques like Huber and Bi-square functions. The PCA-based multivariate CUSUM CC outperforms classical EWMA CCs in simulations and real-life wind turbine manufacturing. Woodall et al. [12] assess CCs using RSS techniques, emphasizing their improved average run length performance due to reduced parameter estimation error. Nevertheless, they underscore the importance of caution when evaluating RSS benefits over time, especially in the context of monitoring process means. Haq et al. [13] has examined the capability of the EWMA CC in the existence of measurement inaccuracy. When shift size is already available or when it is needed to create a CC for a specific shift, CCs may be a beneficial tool. Before employing CCs, the amount of the shift is often already decided. To facilitate the identification of changes of various magnitudes, a quality investigator may focus on improving double and adaptive control charting techniques. The Shewhart and EWMA CC elements are combined in a user-friendly manner to create the adaptive exponentially weighted moving average (AEWMA) CC. Sabahno et al. [14] introduces adaptive schemes for simultaneous monitoring of the mean and variability in a multivariate normal quality characteristic, extending existing bivariate non-adaptive CCs and demonstrating the applicability of the adaptive scheme through a numerical example. Santorea et al. [15] introduce adaptive CCs utilizing RSS, featuring variable sample sizes and multiple dependent state sampling, demonstrating improved performance in comparison to non-adaptive RSS and adaptive SRS approaches, supported by comprehensive simulations and practical applications. For monitoring location parameters, the authors of [16] have established a comparison technique and have suggested using the AEWMA CC. The AEWMA CC was created with the Huber score function, which includes elements of both the EWMA and Shewhart CCs. The study has shown that while monitoring shifts of various magnitudes, the AEWMA CC outperformed the conventional Shewhart, optimal EWMA, Shewhart EWMA, and optimal CUSUM CCs. In the literature, various studies are conducted to explore the ACUSUM and AEWMA CCs for monitoring the process mean [17–21]. Abbas et al. [22] presents nonparametric DEWMA charts using the Wilcoxon signed-rank test for efficient process location monitoring, demonstrating superior performance compared to classical and nonparametric alternatives. A practical application involving piston ring diameter is also provided. Abbas et al. [23] introduces a PAEWMA chart for monitoring nonconformities per unit in industrial processes, demonstrating superior performance compared to existing schemes in detecting unknown shifts, with real-life applications from various datasets. The AEWMA CC is examined by Zaman et al. [24] utilizing the Tukey Bi-square function, which effectively tracks the location parameter of process. All of the aforementioned research has been conducted using the traditional methodology, which simply relies on sample data and leaves out previous knowledge. The Bayesian approach is an estimating technique that incorporates sample data and prior knowledge, which is updated to provide a posterior distribution. Girshick and Rubin [25] discuss SPC and continuous inspection methods optimized for a specific income function and a production model with four states, including known transition probabilities. The study utilizes Markov processes and integral equations to derive optimal procedures approaching a limiting distribution. Riaz et al. [26] analyze the Bayesian EWMA CC under three LFs with various informative and non-informative priors. Performance is assessed using ARL and SDRL via Monte Carlo simulations across different smoothing constant values, accompanied by a practical illustrative example. A Bayesian modified EWMA CC that includes posterior (P) and posterior predictive (PP) distribution is introduced in Asalam et al.'s [27] study, which has revealed the CC's greater capacity to detect out-of-control signals compared to other CCs by evaluating its performance using ARL and SDRL metrics. Noor et al. [28] have suggested the Bayesian AEWMA CC for monitoring process mean under various loss functions. Du et al. [29] propose a Bayesian-based lubricating oil replacement scheme using a hidden Markov chain model, enhancing machine health, lowering costs, and improving availability, surpassing age-based and failure-based methods in fault detection and average availability. Ali [30] presents Bayesian predictive monitoring using CUSUM and EWMA CCs, eliminating the need for large Phase-I datasets, enabling online monitoring, comparing Bayesian memory-type charts with frequentist ones, and assessing performance under practitioner-to-practitioner variability using AARL and SDARL. A brand-new Bayesian EWMA CC is developed by Lin et al. [31] for detecting changes in process variance without depending on presumptions on its underlying distribution and investigated the sampling properties of the proposed method to make its implementation simple for processes with time-varying distributions. They have also carried out a simulation study to show the value of the CC. In their unique Bayesian Hybrid EWMA CC, Imad et al. [32] have proposed a variety of RSS techniques and an instructive prior to monitor the process mean. The efficiency of the proposed method is assessed by using run length profiles, and it is then contrasted with that of other Bayesian CCs under SRS, such as Bayesian HEWMA and Bayesian AEWMA. Wang et al. [33] explore how measurement error affects the Bayesian EWMA CC, incorporating different RSS sampling designs and LFs, and find that the median ranked set sampling scheme performs best under these conditions.

PRSS is a technique that aims to reduce data collection costs and time while maintaining or improving accuracy. It involves collecting paired samples based on the ranked order of units, showing promise in various applications. Integrating Bayesian CCs with PRSS enhances its efficiency and effectiveness. Bayesian CCs enable the incorporation of prior information, aiding informed decisions and improved estimates. They offer more precise and reliable control limits compared to traditional charts by accommodating various data distributions. This integration also addresses the small sample size issue by utilizing paired sample information effectively. The Bayesian framework allows flexible modeling of complex relationships, useful in situations with nonlinear or non-monotonic dependencies. Overall, PRSS with Bayesian control charts improves sampling by incorporating prior knowledge, providing robust control limits, addressing sample size concerns, and allowing flexible modeling for better decision making and process improvement. The focus in the current study is to develop a novel Bayesian AEWMA CC that utilizes various paired RSS schemes, such as paired ranked set sampling (PRSS), quartiles paired ranked set sampling (QPRSS), and extreme paired ranked set sampling (EPRSS), including an informative prior for the P and PP distributions based on distinct LFs such as SELF and LLF. The study evaluates the capability of the suggested Bayesian CC using ARL and SDRL. The remaining work is organized as follows: Section 2 describes the Bayesian theory, Section 3 discusses several PRSS schemes, and Section 4 describes how to build a Bayesian AEWMA CC. Section 5 includes the simulation study. Results, debates, and findings are found in Section 6, and real-world applications are found in Section 7. Section 8 contains the conclusion.

## 2. Bayesian Approach

The Bayesian strategy, which combines data from the sample and the prior distribution, is a well-liked technique for estimating unknown population characteristics. Before any information is taken into consideration, the prior distribution provides initial assumptions about the population parameter (unknown). Informative and non-informative previous distribution are the two primary types. When there is knowledge about the prior distribution of the population parameter, an informative prior is utilized. When the prior and sample distributions are members of the same family of distributions, this is known as a conjugate prior. Consider the study variable $X$ for in-control process with $\theta$ (mean) and $\delta^2$ (variance), while taking normal prior with parameter $\theta_0$ and $\delta_0^2$ is defined as:

$$p(\theta) = \frac{1}{\sqrt{2\pi\delta_0^2}} \exp\left\{-\frac{1}{2\delta_0^2}(\theta - \theta_0)^2\right\}. \tag{1}$$

In situations where we lack substantial prior information about a population parameter, Bayesian analysis often resorts to employing a non-informative prior distribution. This non-informative prior is intentionally designed to have minimal impact on the resulting P distribution, thus allowing the data to predominantly dictate the posterior inference. To address this challenge, Jeffrey [34] proposed a specific form of non-informative prior, denoted as $p(\theta) \propto \sqrt{I(\theta)}$, which is proportional to the Fisher information matrix and captures any available information about the parameter. By incorporating this informative aspect into the analysis, the Bayesian framework becomes more adaptive and robust when dealing with situations where prior knowledge is limited or absent. This enables a more data driven and objective approach to parameter estimation and inference in such scenarios.

We obtain the P distribution by using prior and sampling distribution, $p(\theta|x)$ is given as $p(\theta|x) = \frac{p(x|\theta)p(\theta)}{\int p(x|\theta)p(\theta)d\theta}$, the PP distribution is a distribution used to predict future observations determined by the P distribution. It is often considered as a prior distribution for new data $Y$, as it incorporates the information obtained from the data and the prior distribution. By estimating PP distribution, we can make predictions about future observations with a measure of uncertainty. This distribution plays a significant role in Bayesian analysis, as it allows for updating the prior distribution based on new data. The PP distribution $p(y|x)$ is given as:

$$p(y|x) = \int p(y|\theta)p(\theta|x)d\theta. \tag{2}$$

### *2.1. Squared Error Loss Function*

In Bayesian decision theory, Gauss [35] suggested a symmetric type of squared error LF (SELF), which is commonly used and to measure the discrepancy between the true parameter and the estimated parameter. It is defined as the squared difference between the true parameter and the estimate and plays a crucial role in minimizing the expected posterior loss. If $\hat{\theta}$ is an estimator of $\theta$ of the variable under consideration $X$, then the SELF is given below:

$$L(\theta, \hat{\theta}) = (\theta - \hat{\theta})^2 \tag{3}$$

and applying SELF, the Bayes estimator is defined as:

$$\hat{\theta} = E_{\theta/x}(\theta). \tag{4}$$

### *2.2. Linex Loss Function*

The classic mean squared error (MSE) loss function utilized in regression issues is modified by the LINEX loss function. It is intended to punish either overly high or too low forecasts, but in a way that allows for some flexibility in the severity of the penalty. When overestimation is thought to be more important than underestimation, Varian [36] suggested an asymmetric loss function known as LINEX LF. The location parameter estimation method known as LINEX LF may be characterized as follows:

$$L(\theta,\hat{\theta}) = \left(e^{c(\theta-\hat{\theta})} - c(\theta-\hat{\theta}) - 1\right) \quad (5)$$

where $c$ is any constant and the Bayes estimator applying LLF is:

$$\hat{\theta} = -\frac{1}{c} InE_{\theta/x}\left(e^{-c\theta}\right) \quad (6)$$

## 3. Paired Ranked Set Sampling

Muttlak [37] proposed a new *RSS* scheme named paired *RSS* (*PRSS*) in which half units of the population are chosen for ordering and select two units from each set for measurement. *PRSS* entails choosing observational pairs from the population so that each pair's value difference is rated the same. In other words, for any pair of paired observations, the rank difference is the same. The whole method for selecting samples by using *PRSS* is given as follows: if the set size $l$ is odd then $(l(l+1)/2)th$ units are selected from the underlying population and distributed randomly into $((l+1)/2)th$ sets of similar size, arrange the units within each sets after ranking and select the first and last units from the first set, the second and $(l-1)th$ sampling units are selected from the second set and so on until the $((l+1)/2)th$ unit is selected from $((l+1)/2)th$ set. For even set size $l$, $(l^2/2)th$ units taken from the given population and distributed into $(l/2)th$ sets with the same set size $l$, ranked all the units within each set and choose the first and last units from the first set after ranking, select second and $(l-1)th$ from the second set and so on until the $(l/2)th$ and $(l/2+1)th$ units are chosen from the last set to finish off a cycle of *PRSS*. This procedure is replicated $r$ time to achieve $n = lr$ if required.

The mean estimator by using *PRSS* for one cycle is given by:

If $l$ is even, then:

$$\overline{Z}_{(PRSS)e} = \frac{1}{l}\left[\sum_{i=1}^{\frac{l}{2}} Z_{i(i)} + \sum_{i=1}^{\frac{l}{2}} Z_{i(l+1-i)}\right] \quad (7)$$

with variance:

$$\text{var}\left(\overline{Z}_{(PRSS)e}\right) = \text{var}\left(\overline{Z}_{(RSS)}\right) + \frac{2}{l^2}\sum_{i=1}^{\frac{l}{2}}\sum_{i<l+1-i}^{\frac{l}{2}} \text{cov}\left(Z_{(i)}, Z_{(l+1-i)}\right) \quad (8)$$

If $l$ is odd, then:

$$\overline{Z}_{(PRSS)o} = \frac{1}{l}\left[\sum_{i=1}^{\frac{(l+1)}{2}} Z_{i(i)} + \sum_{i=1}^{\frac{(l-1)}{2}} Z_{i(l+1-i)}\right] \quad (9)$$

with variance:

$$\text{var}\left(\overline{Z}_{(PRSS)o}\right) = \text{var}\left(\overline{Z}_{(RSS)}\right) + \frac{2}{l^2}\sum_{i=1}^{\frac{(l-1)}{2}}\sum_{i<l+1-i}^{\frac{(l-1)}{2}} \text{cov}\left(Z_{(i)}, Z_{(l+1-i)}\right). \quad (10)$$

### 3.1. Extreme Pair Ranked Set Sampling

Balci et al. [38] introduced a new modified design of PRSS known as extreme PRSS (EPRSS); this section provides a comprehensive explanation of the sample selection method using *EPRSS*. When the population has a heavy-tailed distribution, which occurs more frequently than in a normal distribution, *EPRSS* can be especially helpful. Standard sampling techniques might not be able to catch the extreme values in such circumstances, producing estimates that are skewed. *EPRSS* is narrated as: if $l$ is even, pick $(l^2/2)$ sampling units from the target population and distribute them into $(l/2)$ sets with similar size, rank the units in each set in ascending order and then take measurements on the first and last units In each ranked set. If $l$ is even, then pick $(l(l+1)/2)$ sampling units from the given population and randomly allocate into $(l-1/2)$ sets and rank all the units within each set. After ranking select the first and last units from the $(l-1/2)$ sets and the last unit is selected as a middle unit from the last set. If essential, the complete procedure of EPRSS is repeated $r$ time to obtain size $n = lr$.

The mean estimator for *EPRSS* along with variance for a single cycle is as follows:

If $l$ is even, then:

$$\overline{Z}_{(EPRSS)e} = \frac{1}{l}\sum_{i=1}^{\frac{l}{2}}\left[Z_{i(1)} + Z_{i(1)}\right] \tag{11}$$

with variance:

$$Var\left(\overline{Z}_{(EPRSS)e}\right) = \frac{1}{2l}\left[\begin{array}{c} Var\left(Z_{(1)}\right) + Var\left(Z_{(l)}\right) \\ +2Cov\left(Z_{(1)}, Z_{(l)}\right) \end{array}\right] \tag{12}$$

If $l$ is odd, then:

$$\overline{Z}_{(EPRSS)o} = \frac{1}{l}\left[\sum_{i=1}^{\frac{(l-1)}{2}}\left(Z_{i(1)} + Z_{i(l)}\right) + Z_{\frac{l+1}{2}\left(\frac{l+1}{2}\right)}\right] \tag{13}$$

and with variance:

$$Var\left(\overline{Z}_{(EPRSS)o}\right) = \frac{l-1}{2l^2}\left[\begin{array}{c} Var\left(Z_{(1)}\right) + Var\left(Z_{(l)}\right) \\ +2Cov\left(Z_{(1)}, Z_{(l)}\right) \end{array}\right] + \frac{1}{l^2}\left[Var\left(Z_{\left(\frac{l+1}{2}\right)}\right)\right] \tag{14}$$

### *3.2. Quartile Pair Ranked Set Sampling*

Tayyab et al. [39] suggested an efficient sampling scheme for estimating unknown population parameters known as quartile PRSS (*QPRSS*). Instead of measuring the values of the observations in each sample directly, it entails ranking them. The first, second, or third quartiles of the population are represented by the difference between paired observations in *QPRSS*. Pairs of observations are chosen from the population so that each pair has the same ranked difference in values. Compared to conventional random sample techniques, this enables a more accurate quartile estimate. The whole methodology of selecting sample by *QPRSS* is narrated as: if $l$ is even, $(l^2/2)$ sampling units are picked from the given population and distributed randomly into $(l/2)$ sets with similar set size $l$, ranked all units within each set and select $((l+1)/4)th$ and $(3(l+1)/4)th$ units from each ranked set. If $l$ is odd, then select $(l(l+1)/2)$ sampling units are selected from population and distributed them into $((l+1)/2)$ sets, order all the units within set in an ascending order. After ranking, choose $((l+1)/4)th$ and $(3(l+1)/4)th$ units out of $(l-1/2)th$ sets and the middle unit from the previous set is quantified for finishing a single *QPRSS* cycle. If essential replicate the whole procedure $r$ times to obtain $n = lr$.

The mean estimator for *QPRSS* for one cycle is defined as:

If $l$ is even, then:

$$\overline{Z}_{(QPRSS)e} = \frac{1}{l}\left[\sum_{i=1}^{\frac{l}{2}} Z_{i(q_1(l+1):l)} + \sum_{i=1}^{\frac{l}{2}} Z_{i(q_3(l+1):l)}\right] \tag{15}$$

and if $l$ is odd, then:

$$\overline{Z}_{(QPRSS)e} = \frac{1}{l}\left[\begin{array}{l} \sum_{i=1}^{\frac{l}{2}} Z_{i(q_1(l+1):l)} + \sum_{i=1}^{\frac{l}{2}} Z_{i(q_3(l+1):l)} \\ +Z_{\frac{l+1}{2}(q_2(l+1):l)} \end{array}\right] \tag{16}$$

with the corresponding variances are:

$$Var\left(\overline{Z}_{(QPRSS)e}\right) = \frac{1}{2l}\left[\begin{array}{l} \delta^2_{(q_1(l+1))} + \delta^2_{(q_3(l+1))} \\ +2\delta_{(q_1(l+1),q_3(l+1))} \end{array}\right] \tag{17}$$

and

$$Var\left(\overline{Z}_{(QPRSS)o}\right) = \frac{l-1}{2l^2}\left[\begin{array}{l} \delta^2_{(q_1(l+1))} + \delta^2_{(q_3(l+1))} \\ +2\delta_{(q_1(l+1),q_3(l+1))} \end{array}\right] + \frac{1}{l^2}\delta^2_{(q_1(l+1))}. \tag{18}$$

The last set is quantified for obtaining a single cycle of *QPRSS*. If required, the overall method is replicated $r$ times to obtain $n = lr$.

## 4. Proposed AEWMA CC Utilizing Bayesian Approach with Different *PRSS* Schemes under LF

The proposed Bayesian AEMWA CC based on different *PRSS* schemes for monitoring location parameter of normally distributed process has been discussed in this section. Let $X_1, X_2, X_3, \ldots X_n$ the independently and normally distributed random with mean $\theta$ and variance $\sigma^2$ thus the probability density function is defined as:

$$f\left(x_t : \theta, \sigma^2\right) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x_t - \theta)^2\right) \tag{19}$$

Let $\hat{\delta}_t^*$ estimate for mean shift be the sequence of EMWA statistic based on $\{X_t\}$ is given by:

$$\hat{\delta}_t^* = \psi X_t + (1-\psi)\hat{\delta}_{t-1}^* \tag{20}$$

where $\hat{\delta}_0^* = 0$ and $\psi$ is smoothing constant, the estimator $\hat{\delta}_t^*$ is unbiased for under control process and biased for the out-of-control process. The unbiased estimator of $\delta$ for both the case of in-control and the out-of-control process is suggested by Haq et al. [13], which is defined as:

$$\hat{\delta}_t^{**} = \frac{\hat{\delta}_t^*}{1-(1-\psi)^t}. \tag{21}$$

It is suggested to use $\hat{\delta}_t = \left|\hat{\delta}_t^{**}\right|$ for $\delta$ estimation.

The suggested Bayesian AEWMA control chart under different *PRSS* schemes and with different LFs for the location parameter of process using the sequence $\{X_t\}$ is given by:

$$W_t = \theta(\widetilde{\delta}_t)\hat{\theta}_{(PRSS_i)LF} + (1-\theta(\widetilde{\delta}_t))W_{t-1}, \tag{22}$$

where $i = 1, 2, 3,\ \begin{array}{l} RSS_1 = RSS \\ RSS_2 = MRSS \\ RSS_3 = ERSS \end{array}, \theta\left(\hat{\delta}_t\right) \in (0, 1]$ and $W_0 = 0$ such that:

$$\left(\hat{\delta}_t\right) = \begin{cases} 0.015\ if\ 0.00 < \hat{\delta}_t \leq 0.25 \\ 0.10\ if\ 0.25 < \hat{\delta}_t \leq 0.75 \\ 0.20\ if\ 0.75 < \hat{\delta}_t \leq 1.00 \\ 0.25\ if\ 1.00 < \hat{\delta}_t \leq 1.50 \\ 0.50\ if\ 1.50 < \hat{\delta}_t \leq 2.50 \\ 0.80\ if\ 2.50 < \hat{\delta}_t \leq 3.50 \\ 1\ if\ \hat{\delta}_t > 3.50 \end{cases} \tag{23}$$

When the computed plotting statistic surpasses the predefined threshold, value denoted as $h$, it signifies that the process is identified as out of control. Conversely, if the statistic falls below or equals the threshold value, it indicates that the process is deemed in control. This threshold-based classification is fundamental in statistical process control, enabling the timely detection of deviations from normal process behavior.

In cases where both the likelihood function and prior distribution are normally distributed, the resulting P distribution also follows a normal distribution with a mean of $\theta_n$ and a variance of $\delta_n^2$. The probability density function (pdf) can be expressed as follows:

$$P(\theta/x) = \frac{1}{\sqrt{2\pi}\sqrt{\frac{\delta^2\delta_0^2}{\delta^2+n\delta_0^2}}} * \exp\left[-\frac{1}{2}\left(\frac{\theta - \sum_{i=1}^{n}\frac{x_i\delta_0^2+\theta_0\delta_0^2}{\delta^2+n\delta_0^2}}{\sqrt{\frac{\delta^2\delta_0^2}{\delta^2+n\delta_0^2}}}\right)^2\right] \tag{24}$$

where, $\theta_n = \frac{n\bar{x}\delta_0^2+\delta^2\theta_0}{\delta^2+n\delta_0^2}$ and $\delta_n^2 = \frac{\delta^2\delta_0^2}{\delta^2+n\delta_0^2}$, respectively.

The estimator for the proposed method by employing Bayesian technique under *PRSS* designs for P and PP distribution under SELF is defined as:

$$\hat{\theta}_{(SELF)} = \frac{n\overline{x}_{(PRSS_i)}\delta_0^2 + \delta^2\theta_0}{\delta^2 + n\delta_0^2}. \tag{25}$$

The properties of the $\hat{\theta}_{(SELF)}$ is given as $E\left(\hat{\theta}_{(SELF)}\right) = \frac{n\theta_1\delta_0^2+\delta^2\theta_0}{\delta^2+n\delta_0^2}$ and $sd\left(\hat{\theta}_{(SELF)}\right) = \sqrt{\frac{n\delta_{(PRSS_i)}^2\delta_0^4}{\delta^2+n\delta_0^2}}$, respectively. The Bayes estimator for the suggested CC using LLF based on *PRSS* designs is derived as:

$$\hat{\theta}_{(LLF)} = \frac{n\overline{x}_{(PRSS_i)}\delta_0^2 + \delta^2\theta_0}{\delta^2 + n\delta_0^2} - \frac{C'}{2}\delta_n^2. \tag{26}$$

The mean of $\hat{\theta}_{(LLF)}$ is given by $E\left(\hat{\theta}_{LLF}\right) = \frac{n\theta_1\delta_0^2+\delta^2\theta_0}{\delta^2+n\delta_0^2} - \frac{C'}{2}$.

Let the feature observations of size $h$, i.e., $y_1, y_2, \ldots, y_h$ then the structure of the Bayesian AEWMA CC using distinct PRSS designs for PP distribution, the probability density function of $Y/X$ is shown as:

$$p(y/x) = \frac{1}{\sqrt{2\pi\delta_1^2}}\exp\left\{-\frac{1}{2\delta_1^2}(Y - \theta_n)^2\right\} \tag{27}$$

which is normally distributed with the mean $\theta_n$ and the standard deviation $\delta_1$, derived as $\delta_1 = \sqrt{\delta^2 + \frac{\delta^2\delta_0^2}{\delta^2+n\delta_0^2}}$. Then $\theta$ is estimated for PP distribution using LLF based on various *PRSS* designs by:

$$\hat{\theta}_{LLF} = \frac{n\overline{x}_{(PRSS_i)}\delta_0^2 + \delta^2\theta_0}{\delta^2 + n\delta_0^2} - \frac{C'}{2}\tilde{\delta}_1^2 \tag{28}$$

where, $\tilde{\delta}_1^2 = \frac{\delta^2}{k} + \frac{\delta^2\delta_0^2}{\delta^2+n\delta_0^2}$, $E\left(\hat{\theta}_{LLF}\right) = \frac{n\theta_1\delta_0^2+\delta^2\theta_0}{\delta^2+n\delta_0^2} - \frac{C'}{2}\tilde{\delta}_1^2$, and $sd\left(\hat{\theta}_{LLF}\right) = \sqrt{\frac{n\delta_{(PRSS_i)}^2\delta_0^4}{\left(\delta^2+n\delta_0^2\right)^2}}$ are the first two moments of $\hat{\theta}_{LLF}$.

## 5. Simulation Study

In order to assess the effectiveness of the proposed AEWMA CC when used in conjunction with various PRSS schemes, we apply the Bayesian methodology, which incorporates an informative prior distribution, through the Monte Carlo simulation method. This simulation allows us to replicate real-world scenarios and assess the effectiveness of the CC applying various conditions. Specifically, we examine the impact of two distinct smoothing constants, ψ = 0.10 and ψ = 0.25, to gain insights into how they influence the behavior and effectiveness of the Bayesian AEWMA CC. Below, we provide a comprehensive description of the step-by-step simulation process, which covers all the details and procedures involved in this evaluation.

**Estimating the threshold for an in-control ARL:**

i. While determining the mean and variance of both the P distribution and PP distribution applying various LFs, we utilized the standard normal distribution as both the sampling and prior distribution. This entailed determining values such as $E\left(\hat{\theta}_{(PRSS_i)LF}\right)$ and $\delta_{(PRSS_i)LF}$ for various LFs;
ii. When employing a fixed smoothing constant (ψ) in a mathematical or statistical context, it is crucial to select a suitable value for another parameter referred to as "$h$". This choice of '$h$' can significantly impact the performance or behavior of the system or model being studied;
iii. Select a paired ranked set sample with a size of "$n$" from a population that follows a normal distribution, as a representation of an in-control process, i.e., $X \sim N\left(E\left(\hat{\theta}\right), \delta^2\right)$;
iv. Determine the plotting statistic as specified in Equation (22) and proceed with the process evaluation.
v. If it is confirmed that the process is under control, proceed with the described steps iteratively until an out-of-control signal is identified, while maintaining a record of the consecutive in-control run lengths.

**Setting the threshold for out-of-control ARL:**

i. Generate a random sample obtained from a normal distribution, where the mean has been intentionally adjusted or shifted from its usual position, i.e., $X \sim N\left(E\left(\hat{\theta}_{LF}\right) + \delta\frac{\sigma}{\sqrt{n}}, \delta\right)$;
ii. Compute the value of Wt and evaluate the procedure by applying the proposed AEWMA CC within the Bayesian framework, utilizing different PRSS designs.

### *5.1. ARL Methods*

In the context of CCs and SPC, Zero State ARL and Steady State ARL are important performance metrics used to assess the effectiveness of a CC in detecting out-of-control conditions in a manufacturing or industrial process.

#### 5.1.1. Zero State ARL

This refers to the initial state of the CC, where the process is assumed to be operating under normal or in-control conditions from the very beginning. ARL is a measure of how long, on average, it takes for a CC to signal an out-of-control condition when the process is indeed operating under normal conditions (in the zero state). ARL is essentially the expected or average number of samples (or data points) needed to detect a problem or shift in the process.

#### 5.1.2. Steady State ARL

After an initial period of time (transient period) where the CC adapts to the process and stabilizes, the process enters a steady state. In this state, the CC is expected to perform optimally and consistently detect out-of-control conditions. Steady State ARL, like Zero State ARL, is a measure of how long, on average, it takes for the CC to signal an out-of-control condition, but specifically in the steady state. It provides an indication of the CCs performance when the process has reached a stable and predictable condition. Both Zero State ARL and Steady State ARL are crucial for evaluating the efficiency and effectiveness of a CC. A CC should ideally have a low Steady State ARL to quickly detect process deviations while having a relatively high Zero State ARL to avoid frequent false alarms when the process is initially in control. The balance between these two metrics is essential for a CC to perform optimally in quality control and process monitoring. In the current study we have utilized the Zero state ARL as a performance metrics.

## 6. Results, Discussions, and Findings

In Tables 1–6, we compare the proposed AEWMA CC under Bayesian analysis using PRSS and with SRS using the same smoothing constant for both the P and PP distributions. Both CCs are created under various PRSS schemes using two different LFs. The suggested CC is more significantly monitoring the out-of-control signs than the existing CC using SRS, according to the performance measures run length results of the proposed CC using PRSS strategies applying SELF under informative prior and Bayesian AEWMA CC utilizing SRS. For instance, outcomes of the AEWMA CC utilizing the Bayesian method with SRS using SELF for P and PP distribution with a particular value of the smoothing constant $\psi = 0.10$ and different shifts, i.e., $\sigma$ = 0.0, 0.30, 0.50, 0.80, 1.50, 4 are 370.86, 35.40, 13.55, 5.62, 2.25, and 1.02. In a similar circumstance, ARL value of the suggested Bayesian AEWMA CC using PRSS are 370.50, 23.75, 9.94, 3.58, 1.43, and 1, and using QPRSS are 370.04, 23.05, 9.01, 3.45, 1.38, and 1. The ARL results of the proposed CC under EPRSS are 370.80, 23.13, 9.45, 3.92, 1.37, and 1. The comparison clearly demonstrates the efficacy of the recommended Bayesian AEWMA CC under PRSS strategies. Additionally, the performance of the proposed Bayesian AEWMA CC under PRSS design with an informative prior and distinct LFs is compared to the Bayesian AEWMA CC using SRS at $\psi = 0.25$ and different shifts $\sigma$ = 0.0, 0.30, 0.50, 0.80, 1.50, 4 are 369.25, 55.67, 27.50, 12.91, 4.08, and 1.08. The outcomes show how well the suggested Bayesian AEWMA CC performs under PRSS schemes. Also, the proposed Bayesian AEMWA CC utilizing PRSS schemes with an informative prior and two distinct LFs and the Bayesian AEWMA CC using SRS with a set smoothing constant are compared at $\psi = 0.25$, and different shifts $\sigma$ = 0.0, 0.30, 0.50, 0.80, 1.50, 4 indicates ARLs of 369.25, 55.67, 27.50, 12.91, 4.08, and 1.08, respectively. The ARL results for the proposed method using PRSS are 370.74, 31.72, 11.98, 4.86, 2.12, and 1, while for QPRSS, the ARL values are 370.56, 31.82, 14.12, 5.83, 1.93, and 1. The EPRSS has ARL values of 369.62, 30.03, 10.89, 4.47, 2.02, and 1. In comparison to the existing Bayesian CC under SRS, the results show that the offered approach under PRSS schemes demonstrates a quick decline in values at bigger shifts, proving its superior capacity to detect out-of-control signals. These results can be summarized in the following points:

- The run length results of the provided Bayesian CC using the SELF across different PRSS schemes exhibit a rapid decrease as the mean shift increases. This observation suggests that the proposed method is unbiased, as evidenced in Tables 1 and 2. For example, from Table 1 at $ARL_0 = 370$ and smoothing constant $\psi = 0.10$ at various shifts, i.e., $\delta$ = 0.20 and 0.70. The ARL outcomes are 42.72 and 4.03 for PRSS, 42.35 and 4.55 for QPRSS, and the values of ARL for EPRSS are 41.79 and 4.48;
- From Tables 3 and 4, it can be observed that the proposed technique can be affected with changes in the value of the smoothing constant, i.e., $\psi = 0.10$ and 0.25. Under LLF, the results of ARL and SDRL for the proposed method with posterior distribution are displayed in Tables 3 and 4, which indicate that the efficiency decreases with increase in the value of smoothing constant for the proposed method. For example, at $ARL_0 = 370$, $\psi = 0.10$ and shift $\delta$ = 0.20; however, the respective ARL values for the proposed method under PRSS, QPRSS, and EPRSS are 40.78, 38.42, and 40.47. For the same shift $\delta$ = 0.20 and $\psi = 0.25$, the value of ARL for PRSS is 66.76, under QPRSS is 64.23, and under EPRSS is 60.82;
- For the proposed method the results of ARL under PRSS are shown in Tables 5 and 6. They show that the offered CC uses PRSS schemes under LLF for P and PP distribution at $ARL_0 = 370$, $\delta$ = 0.50 and smoothing constant $\psi = 0.10$ is 9.35 and the ARL value at $\psi = 0.25$ is 11.28; in a similar case the ARL for QPRSS are 9.41 and 10.79. The ARL values using EPRSS are 9.32 and 10.91;

- From Tables 1–6, we can see that the recommended Bayesian AEWMA CC is quite vulnerable in identifying out-of-control signals as compared with existing AEWMA CC applying SRS, according to the data (see Figures 1–7).

**Table 1.** The ARL and SDRL results by using SELF in the proposed AEWMA CC, for $\psi$ = 0.10 and $n$ = 5.

| Shift | Bayes SRS | | Bayes PRSS | | Bayes QPRSS | | Bayes EPRSS | |
|---|---|---|---|---|---|---|---|---|
| | *ARL* | *SDRL* | *ARL* | *SDRL* | *ARL* | *SDRL* | *ARL* | *SDRL* |
| | **$h$ = 0.0856** | | **$h$ = 0.0475** | | **$h$ = 0.0456** | | **$h$ = 0.0448** | |
| 0.00 | 370.86 | 537.77 | 370.50 | 473.98 | 370.04 | 453.73 | 370.80 | 482.75 |
| 0.10 | 179.91 | 247.52 | 105.26 | 109.93 | 103.11 | 105.26 | 101.91 | 104.05 |
| 0.20 | 70.61 | 91.12 | 42.72 | 39.83 | 42.35 | 39.31 | 41.79 | 36.90 |
| 0.30 | 35.40 | 44.53 | 23.75 | 22.10 | 23.05 | 21.16 | 23.13 | 20.55 |
| 0.40 | 21.15 | 26.36 | 14.77 | 16.77 | 14.38 | 13.80 | 14.73 | 13.86 |
| 0.50 | 13.55 | 16.69 | 9.94 | 10.51 | 9.01 | 10.02 | 9.45 | 9.69 |
| 0.60 | 9.46 | 11.23 | 6.75 | 7.38 | 6.37 | 6.98 | 6.36 | 6.93 |
| 0.70 | 7.08 | 7.70 | 4.71 | 5.08 | 4.55 | 4.92 | 4.48 | 4.84 |
| 0.75 | 6.15 | 6.43 | 4.03 | 4.24 | 3.85 | 4.02 | 3.92 | 4.06 |
| 0.80 | 5.62 | 5.82 | 3.58 | 3.61 | 3.45 | 3.50 | 3.34 | 3.23 |
| 0.90 | 4.51 | 4.18 | 2.83 | 2.48 | 2.75 | 2.44 | 2.71 | 2.44 |
| 1.00 | 3.85 | 3.20 | 2.37 | 1.87 | 2.27 | 1.77 | 2.26 | 1.70 |
| 1.50 | 2.25 | 1.29 | 1.43 | 0.64 | 1.38 | 0.62 | 1.37 | 0.61 |
| 2.00 | 1.66 | 0.78 | 1.14 | 0.36 | 1.11 | 0.33 | 1.11 | 0.32 |
| 2.50 | 1.36 | 0.56 | 1.03 | 0.19 | 1.02 | 0.16 | 1.02 | 0.15 |
| 3.00 | 1.17 | 0.39 | 1 | 0 | 1 | 0 | 1 | 0 |
| 4.00 | 1.02 | 0.14 | 1 | 0 | 1 | 0 | 1 | 0 |

**Table 2.** ARL and SDRL outcomes by applying SESL in the suggested CC, for $\psi$ = 0.25 $n$ = 5.

| Shift | Bayes SRS | | Bayes PRSS | | Bayes QPRSS | | Bayes EPRSS | |
|---|---|---|---|---|---|---|---|---|
| | *ARL* | *SDRL* | *ARL* | *SDRL* | *ARL* | *SDRL* | *ARL* | *SDRL* |
| | **$h$ = 0.241** | | **$h$ = 0.0869** | | **$h$ = 0.0812** | | **$h$ = 0.0734** | |
| 0.00 | 369.00 | 367.39 | 371.60 | 419.90 | 369.85 | 397.69 | 369.29 | 416.73 |
| 0.10 | 210.23 | 195.27 | 168.65 | 164.37 | 160.33 | 155.21 | 163.25 | 162.23 |
| 0.20 | 97.04 | 80.91 | 67.14 | 63.35 | 62.09 | 58.98 | 60.79 | 7.69 |
| 0.30 | 55.71 | 42.80 | 32.85 | 29.77 | 31.94 | 28.95 | 29.61 | 27.38 |
| 0.40 | 36.15 | 25.09 | 18.64 | 16.81 | 17.93 | 16.22 | 16.88 | 15.69 |
| 0.50 | 25.95 | 17.04 | 12.10 | 10.67 | 11.50 | 10.07 | 10.55 | 9.51 |
| 0.60 | 19.80 | 12.20 | 8.33 | 6.78 | 8.14 | 6.74 | 7.42 | 6.36 |
| 0.70 | 15.41 | 9.09 | 6.16 | 4.63 | 5.97 | 4.51 | 5.50 | 4.31 |
| 0.75 | 14.11 | 8.17 | 5.54 | 3.92 | 5.27 | 3.79 | 4.81 | 3.52 |
| 0.80 | 12.87 | 7.26 | 4.92 | 3.27 | 4.73 | 3.21 | 4.38 | 3.08 |
| 0.90 | 10.76 | 5.97 | 4.11 | 2.47 | 3.85 | 2.32 | 3.63 | 2.24 |
| 1.00 | 9.17 | 4.96 | 3.49 | 1.90 | 3.34 | 1.80 | 3.08 | 1.69 |
| 1.50 | 4.90 | 2.77 | 2.14 | 0.87 | 2.06 | 0.79 | 1.97 | 0.71 |
| 2.00 | 2.98 | 1.83 | 1.56 | 0.60 | 1.54 | 0.56 | 1.49 | 0.54 |
| 2.50 | 1.98 | 1.15 | 1.23 | 0.43 | 1.13 | 0.33 | 1.21 | 0.41 |
| 3.00 | 1.48 | 0.72 | 1.06 | 0.24 | 1 | 0 | 1 | 0 |
| 4.00 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 |

**Table 3.** ARLs and SDRLs results for the P distribution, by using LLF with $\psi = 0.10$ $n = 5$.

| Shift | Bayes SRS | | Bayes PRSS | | Bayes QPRSS | | Bayes EPRSS | |
|---|---|---|---|---|---|---|---|---|
| | *ARL* | *SDRL* | *ARL* | *SDRL* | *ARL* | *SDRL* | *ARL* | *SDRL* |
| | $h = 0.086$ | | $h = 0.0462$ | | $h = 0.0479$ | | $h = 0.0448$ | |
| 0.00 | 370.98 | 539.06 | 370.94 | 416.25 | 370.09 | 451.21 | 369.57 | 489.53 |
| 0.10 | 184.38 | 254.91 | 100.40 | 107.66 | 98.76 | 105.55 | 100.94 | 107.63 |
| 0.20 | 71.98 | 92.48 | 40.78 | 38.36 | 38.42 | 35.78 | 40.47 | 36.40 |
| 0.30 | 36.26 | 45.49 | 23.14 | 21.42 | 22.30 | 20.25 | 22.78 | 20.45 |
| 0.40 | 21.09 | 26.30 | 14.33 | 14.21 | 13.50 | 14.05 | 14.55 | 13.83 |
| 0.50 | 13.71 | 16.73 | 9.48 | 10.15 | 9.56 | 9.99 | 9.35 | 9.73 |
| 0.60 | 9.53 | 11.25 | 6.53 | 7.19 | 6.48 | 7.05 | 6.27 | 6.82 |
| 0.70 | 7.09 | 7.86 | 4.59 | 5.05 | 4.56 | 4.92 | 4.52 | 4.93 |
| 0.75 | 6.20 | 6.50 | 4.01 | 4.18 | 3.88 | 4.02 | 3.88 | 4.09 |
| 0.80 | 5.54 | 5.54 | 3.45 | 3.46 | 3.92 | 4.11 | 3.39 | 3.40 |
| 0.90 | 4.52 | 4.17 | 2.76 | 2.51 | 3.43 | 3.44 | 2.72 | 2.46 |
| 1.00 | 3.83 | 3.20 | 2.33 | 1.81 | 2.28 | 1.77 | 2.25 | 1.71 |
| 1.50 | 2.26 | 1.27 | 1.41 | 0.65 | 1.40 | 0.62 | 1.37 | 0.62 |
| 2.00 | 1.66 | 0.78 | 1.13 | 0.35 | 1.11 | 0.33 | 1.10 | 0.32 |
| 2.50 | 1.34 | 0.55 | 1.02 | 0.16 | 1 | 0 | 1.02 | 0.15 |
| 3.00 | 1.16 | 0.39 | 1 | 0 | 1 | 0 | 1 | 0 |
| 4.00 | 1.02 | 0.15 | 1 | 0 | 1 | 0 | 1 | 0 |

**Table 4.** ARLs and SDRLs outcomes for suggested CC by applying Bayesian approach utilizing LLF, for $\psi = 0.25$ $n = 5$.

| Shift | Bayes SRS | | Bayes PRSS | | Bayes QPRSS | | Bayes EPRSS | |
|---|---|---|---|---|---|---|---|---|
| | *ARL* | *SDRL* | *ARL* | *SDRL* | *ARL* | *SDRL* | *ARL* | *SDRL* |
| | $h = 0.242$ | | $h = 0.0845$ | | $h = 0.0789$ | | $h = 0.0735$ | |
| 0.00 | 370.14 | 434.88 | 371.13 | 456.32 | 371.39 | 394.59 | 370.44 | 447.97 |
| 0.10 | 212.09 | 198.42 | 167.63 | 165.47 | 161.83 | 158.98 | 157.34 | 162.61 |
| 0.20 | 86.77 | 83.25 | 66.32 | 61.82 | 64.73 | 61.05 | 60.20 | 57.76 |
| 0.30 | 55.44 | 42.26 | 31.80 | 29.41 | 28.51 | 26.37 | 28.48 | 26.35 |
| 0.40 | 36.76 | 25.98 | 18.68 | 16.80 | 17.30 | 15.61 | 16.85 | 15.50 |
| 0.50 | 25.86 | 16.88 | 12.04 | 10.38 | 11.55 | 10.18 | 10.68 | 9.59 |
| 0.60 | 19.65 | 12.16 | 8.27 | 6.84 | 7.81 | 6.50 | 7.30 | 6.19 |
| 0.70 | 15.62 | 9.17 | 6.15 | 4.60 | 5.79 | 4.38 | 5.48 | 4.26 |
| 0.75 | 14.23 | 8.29 | 5.54 | 3.97 | 5.20 | 3.72 | 4.87 | 3.61 |
| 0.80 | 12.83 | 7.30 | 4.93 | 3.36 | 4.53 | 3.01 | 4.30 | 3.03 |
| 0.90 | 10.79 | 5.90 | 4.07 | 2.48 | 3.80 | 2.33 | 3.66 | 2.24 |
| 1.00 | 9.25 | 5.00 | 3.49 | 1.93 | 3.25 | 1.80 | 3.11 | 1.70 |
| 1.50 | 4.95 | 2.80 | 2.12 | 0.85 | 2.04 | 0.77 | 1.96 | 0.71 |
| 2.00 | 2.97 | 1.81 | 1.55 | 0.59 | 1.55 | 0.56 | 1.49 | 0.54 |
| 2.50 | 1.97 | 1.13 | 1.23 | 0.43 | 1.23 | 0.42 | 1.20 | 0.40 |
| 3.00 | 1.48 | 0.73 | 1.06 | 0.25 | 1.03 | 0.17 | 1.05 | 0.23 |
| 4.00 | 1.09 | 0.30 | 1 | 0 | 1 | 0 | 1 | 0 |

**Table 5.** Run length outcomes for proposed CC using PP distribution with LLF, for $\psi$ = 0.10 and $n$ = 5.

| Shift | Bayes SRS | | Bayes PRSS | | Bayes QPRSS | | Bayes EPRSS | |
|---|---|---|---|---|---|---|---|---|
| | *ARL* | *SDRL* | *ARL* | *SDRL* | *ARL* | *SDRL* | *ARL* | *SDRL* |
| | $h$ = 0.0856 | | $h$ = 0.0464 | | $h$ = 0.0445 | | $h$ = 0.0449 | |
| 0.00 | 369.58 | 524.70 | 370.78 | 451.12 | 370.60 | 472.13 | 369.90 | 402.14 |
| 0.10 | 178.57 | 250.19 | 100.99 | 104.36 | 108.49 | 109.96 | 101.24 | 103.87 |
| 0.20 | 70.53 | 91.22 | 45.11 | 38.28 | 43.14 | 38.81 | 40.35 | 36.06 |
| 0.30 | 35.71 | 45.25 | 24.96 | 21.42 | 23.53 | 21.29 | 22.73 | 20.37 |
| 0.40 | 21.24 | 26.29 | 14.75 | 14.37 | 14.53 | 14.00 | 14.37 | 13.62 |
| 0.50 | 13.66 | 16.90 | 9.36 | 9.98 | 9.41 | 9.93 | 9.32 | 9.63 |
| 0.60 | 9.46 | 11.08 | 6.44 | 7.05 | 6.38 | 7.00 | 6.32 | 6.99 |
| 0.70 | 6.94 | 7.70 | 4.60 | 4.97 | 4.69 | 5.07 | 4.46 | 4.82 |
| 0.75 | 6.22 | 6.53 | 4.03 | 4.24 | 3.97 | 4.17 | 3.85 | 4.05 |
| 0.80 | 5.50 | 5.58 | 3.44 | 4.09 | 3.45 | 3.38 | 3.35 | 3.39 |
| 0.90 | 4.52 | 4.15 | 2.75 | 2.49 | 2.70 | 2.33 | 2.67 | 2.39 |
| 1.00 | 3.77 | 3.17 | 2.31 | 1.79 | 2.28 | 1.76 | 2.21 | 1.72 |
| 1.50 | 2.26 | 1.29 | 1.41 | 0.64 | 1.39 | 0.63 | 1.36 | 0.60 |
| 2.00 | 1.66 | 0.78 | 1.13 | 0.35 | 1.13 | 0.34 | 1.10 | 0.32 |
| 2.50 | 1.35 | 0.55 | 1.03 | 0.17 | 1.03 | 0.16 | 1.02 | 0.15 |
| 3.00 | 1.16 | 0.39 | 1 | 0 | 1 | 0 | 1 | 0 |
| 4.00 | 1.02 | 0.15 | 1 | 0 | 1 | 0 | 1 | 0 |

**Table 6.** The run length outcomes of the recommended CC applying Bayesian approach for PP distribution by using LLF, for $\psi$ = 0.25 and $n$ = 5.

| Shift | Bayes SRS | | Bayes PRSS | | Bayes QPRSS | | Bayes EPRSS | |
|---|---|---|---|---|---|---|---|---|
| | *ARL* | *SDRL* | *ARL* | *SDRL* | *ARL* | *SDRL* | *ARL* | *SDRL* |
| | $h$ = 0.2414 | | $h$ = 0.0547 | | $h$ = 0.0765 | | $h$ = 0.0764 | |
| 0.00 | 369.67 | 359.45 | 370.74 | 396.18 | 369.62 | 381.94 | 370.84 | 377.82 |
| 0.10 | 210.29 | 197.72 | 169.22 | 165.09 | 163.28 | 168.74 | 172.88 | 177.32 |
| 0.20 | 98.16 | 83.24 | 66.27 | 62.70 | 65.07 | 60.06 | 62.59 | 59.57 |
| 0.30 | 54.92 | 41.45 | 31.72 | 29.03 | 30.03 | 27.89 | 29.91 | 27.79 |
| 0.40 | 36.19 | 25.48 | 18.51 | 16.52 | 16.86 | 15.75 | 17.16 | 15.75 |
| 0.50 | 25.97 | 17.13 | 11.98 | 10.47 | 10.89 | 9.74 | 10.91 | 9.75 |
| 0.60 | 19.68 | 12.21 | 8.15 | 6.68 | 7.73 | 6.40 | 7.55 | 6.40 |
| 0.70 | 15.56 | 9.19 | 6.12 | 4.64 | 5.63 | 4.37 | 5.54 | 4.18 |
| 0.75 | 14.18 | 8.26 | 5.45 | 3.99 | 5.03 | 3.65 | 4.95 | 3.66 |
| 0.80 | 12.79 | 7.24 | 4.86 | 3.34 | 4.47 | 3.10 | 4.40 | 3.04 |
| 0.90 | 10.74 | 5.93 | 4.03 | 2.46 | 3.69 | 2.22 | 3.63 | 2.20 |
| 1.00 | 9.20 | 4.98 | 3.44 | 1.88 | 3.18 | 1.78 | 3.15 | 1.71 |
| 1.50 | 4.94 | 2.79 | 2.12 | 0.84 | 2.02 | 0.74 | 1.99 | 0.72 |
| 2.00 | 2.95 | 1.81 | 1.55 | 0.60 | 1.52 | 0.54 | 1.52 | 0.54 |
| 2.50 | 1.98 | 1.14 | 1.24 | 0.43 | 1.23 | 0.42 | 1.21 | 0.41 |
| 3.00 | 1.48 | 0.72 | 1.09 | 0.28 | 1.07 | 0.26 | 1.06 | 0.24 |
| 4.00 | 1.09 | 0.30 | 1 | 0 | 1 | 0 | 1 | 0 |

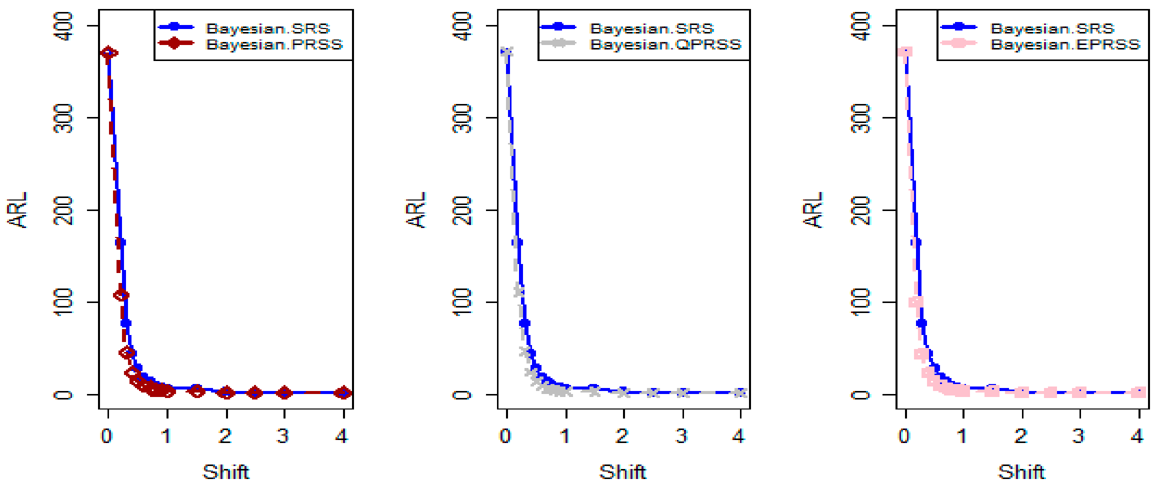


**Figure 1.** ARL plots the utilizing SELF for suggested CC under PRSS, QPRSS, and EPRSS for P and PP distribution.

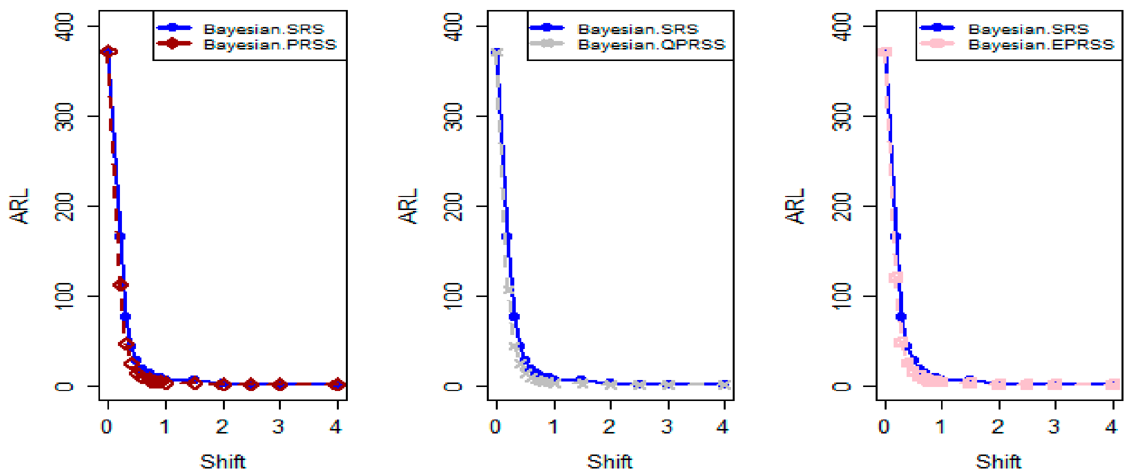


**Figure 2.** ARL plots of the P distribution using LLF for the Bayesian AEWMA CC applying PRSS, QPRSS, and EPRSS.

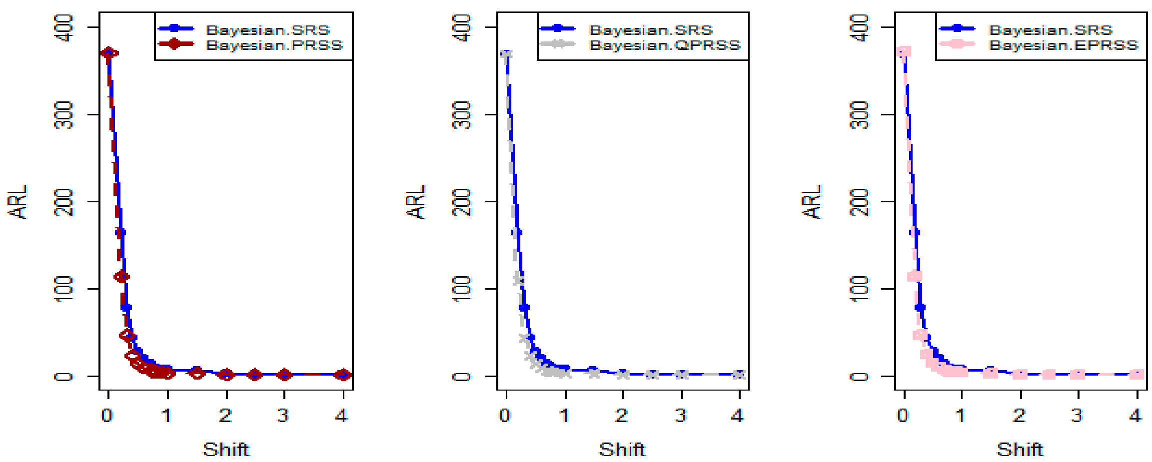


**Figure 3.** ARL plots showing the PP distribution using LLF for the offered CC under PRSS, QPRSS, and EPRSS.

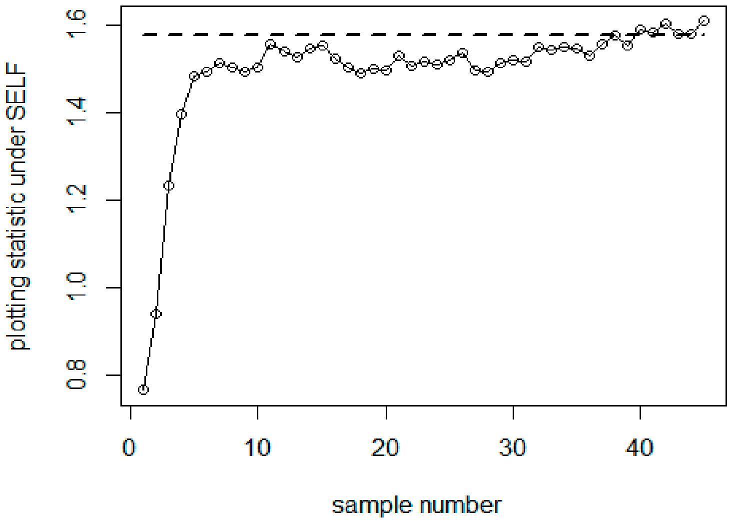


**Figure 4.** Applying SRS, the plot shows Bayesian AEWMA CC utilizing SELF.

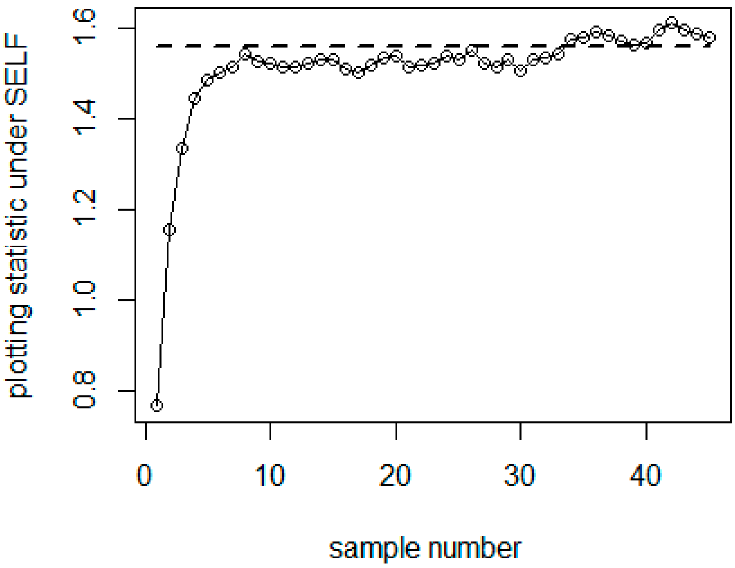


**Figure 5.** Utilizing PRSS, the plot shows suggested CC with SELF.

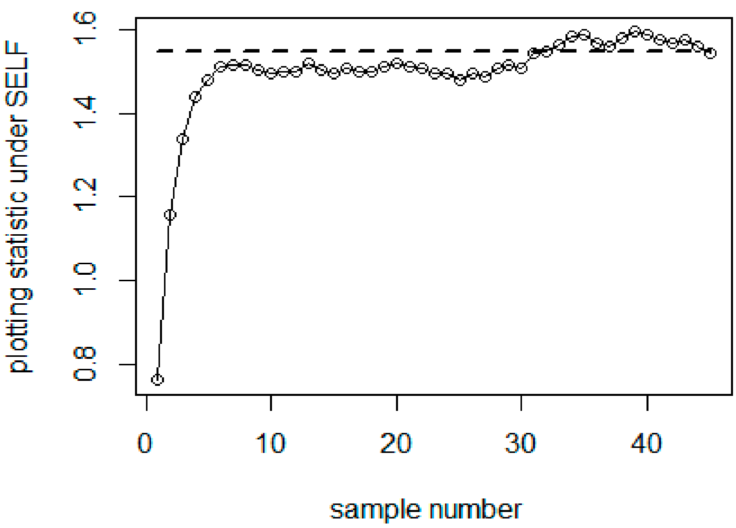


**Figure 6.** Under SELF, the plot shows ARL of suggested CC using QPRSS.

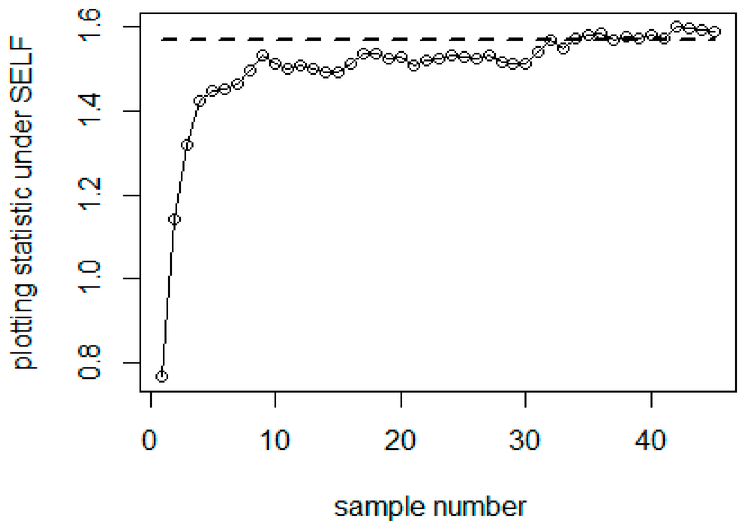


**Figure 7.** Under SELF, the plot shows ARL results of CC utilizing EPRSS.

## 7. Real Data Applications

The numerical illustration of the recommended method based on Bayesian approach using PRSS strategies with distinct LFs under informative prior is evaluated through data given by Montgomery [40] related to hard-bake processes in semiconductor production. The photoresist material used in the photolithography process is heated to a high temperature during this process, which is a crucial stage in the manufacture of semiconductors. The photoresist must be cleaned of any leftover solvent in this stage in order to produce a homogenous, stable surface that is ready to be exposed to light. The information was gathered at a semiconductor production plant and consisted of critical dimension measurements for 90 wafers that underwent various hard-bake processes. We consider 45 samples, each of the size 5 wafers. In this process, flow width is measured by taking the sample time interval is one hour and the microns are utilized. The first 30 samples are considered as in-control and the last 15 are considered as out of control.

Figure 4 depicts the Bayesian AEWMA CC's detection of an out-of-control signal on the 40th sample utilizing SELF for P and PP distribution with SRS. Figures 5–7, on the other hand, show the proposed CC employing PP distribution based on SELF and PRSS schemes for posterior. The graphs show that, for PRSS, QPRSS, and EPRSS, respectively, the recommended PRSS-based Bayesian AEWMA CC identifies out-of-control signals on the 36th, 33rd, and 35th samples. According to Figures 1–7, the PRSS-based Bayesian AEWMA CC is very subtle in spotting out-of-control signs.

## 8. Conclusions

The current study proposes the use of a Bayesian AEWMA CC with a range of LFs and utilizing PRSS schemes for P and PP distribution to monitor the location parameter of a process. This proposed method is compared to the current CC under SRS with an informative prior, and the comparison is presented in Tables 1–6. The results of the recommended method show better performance than the existing CC.

To demonstrate implementation of the proposed method, the study uses a real-world dataset as an illustrative example. The example shows the proposed technique can be utilized to track the location parameter of a process and detect any deviations from the desired target. The study suggests future research directions to enhance the Bayesian AEWMA CC. It proposes exploring non-normal distributions to assess the method's robustness and applicability. Additionally, investigating alternative sampling strategies, such as consecutive sampling, could refine the control chart's accuracy. By considering these directions, the proposed method can be tailored to different contexts, leading to more effective process monitoring and quality control. The study emphasizes the importance of these advancements in handling different types of data and offers valuable guidance for future research, contributing to improved process monitoring and quality management practices.

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