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DTM-BF METHOD FOR THE FLOW AND HEAT TRANSFER OVER A NONLINEARLY STRETCHING SHEET WITH NANOFLUIDS

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Abstract- This paper investigates an analytical analysis for the flow and heat transfer in a viscous fluid over a nonlinear stretching sheet. The governing partial differential equations are transformed into coupled nonlinear differential equations by introducing a similarity transformation. The asymptotic analytical solutions are obtained by using differential transform method-basic functions (DTM-BF). Four types of nanofluids, namely Cu-water, Ag-water, Al_2O_3-water and TiO_2-water were studied. The influence of the nanoparticle volume fraction ϕ, the nonlinear stretching parameter n, Prandtl number Pr, Eckert number Ec and different nanoparticles on the velocity and temperature are discussed and shown graphically. The comparison with the numerical results is presented and it is found to be in excellent agreement.

Key Words- heat and mass transfer, asymptotic expansion, volume fraction influence, nanoparticle reaction

1. INTRODUCTION

In the past few years the flow over stretching surface has received considerable attention because of its many engineering applications. Crane [1] considered the steady two-dimensional flow of a Newtonian fluid driven by a sheet moving in its own plane with a velocity varying linearly with the distance from a fixed point. Ho et al. [2] identified the effects due to uncertainties in effective dynamic viscosity and thermal conductivity of nanofluid on laminar natural convection heat transfer in a square enclosure. Khan and Pop [3] investigated numerically the problem of laminar fluid flow resulting from the stretching of a flat surface in a nanofluid. Mahdy and Sameh [4] reported numerical analysis for laminar free convection over a vertical wavy surface embedded in a porous medium saturated with a nanofluid.

Due to many applications of nanofluids in technical process, the heat and mass transfer of nanofluids with a chemical reaction also caused more attention. The term 'nanofluid' was first proposed by Choi [5] to indicate a liquid suspension containing ultra-fine particles. Eastman [6] obtained an excellent assessment of nanofluid physics and developments. Kuznetsov and Nield [7] studied the influence of nanoparticles on natural convection boundary layer flow past a vertical plate by taking Brownian motion and thermophoresis into account. Besides, Mokmeli [8] and Xuan [9] explained that nanofluids clearly exhibit enhanced thermal conductivity, which gone up with increasing volumetric fraction of nanoparticles.

Motivated by the above works, in this paper we present similarity solutions for the nonlinear problem of flow and heat transfer of nanofluid past a nonlinearly stretching sheet, which are then solved analytically using DTM-BF. This method was first proposed by Xiaohong Su et al.[10]. In this paper, we can find that the approximate solution agrees very well with the numerical solution, which shows the reliability and validity of the present work. The effects of the governing parameters and different nanoparticles on the velocity and temperature are discussed by graph in detail.

2. MATHEMATICAL FORMULATION

Consider the two-dimensional steady laminar flow of viscous and incompressible nanofluid past a flat sheet. The fluid is a water based nanofluid containing different types of nanoparticles such as copper Cu, silver Ag, alumina Al_2O_3 and titanate TiO_2. It is assumed that the base fluid and the nanoparticles are in thermal equilibrium and no slip occurs between them. The thermophysical properties of the nanofluid are listed in Table 1. We choose the slit, from which the sheet is drawn, as the origin of the system. In this coordinate frame the x-axis is taken along the direction of the continuous stretching surface and the y-axis is measured normal to the surface of the sheet.

Under above assumptions, the steady, two-dimensional boundary layer equations for this fluid can be written as follow [11]:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{1}$$

$$\rho_{nf}(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y})=\mu_{nf}\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$(\rho C_p)_{nf}(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y})=k_{nf}\frac{\partial^2 T}{\partial y^2}+\mu_{nf}(\frac{\partial u}{\partial y})^2. \tag{3}$$

Subject to the following boundary conditions

$$u=U_w=ax^n,\ v=0,\ T=T_w,\ \text{at}\ y=0, \tag{4}$$

$$u\to 0,\ T\to T_\infty,\ \text{as}\ y\to\infty, \tag{5}$$

where u,v are the velocity components in the x and y directions, respectively. T is the temperature of the nanofluid, T_∞ is the temperature of the fluid far from the sheet. a and n are parameters related to the surface stretching speed. The effective dynamic viscosity of the nanofluid μ_{nf}, the effective density ρ_{nf} and the heat capacitance of the nanofluid are given by [12]

$$\mu_{nf} = \frac{\mu_f}{(1-\phi)^{2.5}},\ \rho_{nf} = (1-\phi)\rho_f + \phi\rho_s,\ \ (\rho Cp)_{nf} = (1-\phi)(\rho Cp)_f + \phi(\rho Cp)_s. \tag{6}$$

The thermal conductivity of nanofluids restricted to spherical nanoparticles is approximated [13]:

$$k_{nf} = k_f\left[\frac{k_s + 2k_f - 2\phi(k_f - k_s)}{k_s + 2k_f + \phi(k_f - k_s)}\right], \tag{7}$$

where ϕ is the solid volume fraction of nanoparticles, the subscripts nf, f and s denote the thermophysical properties of the nanofluid, base fluid and nano-solid particles, respectively.

Introducing the following similarity variables:

$$u = ax^n f'(\eta),\ v = -\sqrt{\frac{a\nu_f(n+1)}{2}}x^{\frac{n-1}{2}}\left[f(\eta) + \frac{n-1}{n+1}\eta f'(\eta)\right], \tag{8}$$

$$\eta = \sqrt{\frac{a(n+1)}{2\nu_f}}yx^{\frac{n-1}{2}},\quad T = T_\infty + (T_w - T_\infty)g(\eta). \tag{9}$$

The transformed momentum and energy equations together with the boundary conditions given by Eqs. (2)-(5) can be written as

$$f''' + \phi_1\left[ff'' - \frac{2n}{n+1}(f')^2\right] = 0, \tag{10}$$

$$g'' + \Pr\frac{k_f}{k_{nf}}(\phi_2 fg' + \phi_3 E_c f''^2) = 0. \tag{11}$$

$$f(0) = 0,\ f'(0) = 1,\ f'(\infty) \to 0, \tag{12}$$

$$g(0) = 1, g(\infty) \to 0, \tag{13}$$

where $\Pr = \frac{\nu_f(\rho C_p)_f}{k_f}$ is the Prandtl number, $E_c = \frac{U_w^2}{(C_p)_f(T_w - T_\infty)}$ is the Eckert number.

Besides, $\phi_1 = (1-\phi)^{2.5}(1-\phi+\phi\frac{\rho_s}{\rho_f})$, $\phi_2 = 1-\phi+\phi\frac{(\rho C_p)_s}{(\rho C_p)_f}$, $\phi_3 = \frac{1}{(1-\phi)^{2.5}}$.

Table 1. Thermophysical properties of water and nanoparticles.

	Pure water	Ag	Cu	Al2O3	TiO2
C_p $[J/KgK]$	4179	235	385	765	686.2
ρ $[kg/m^3]$	997.1	10500	8933	3970	4250
k $[W/mK]$	0.613	429	400	40	8.9538

3. DTM-BF METHOD FOR THE SIMILARITY SOLUTION

Differential transformation of the function $f(\eta)$ is defined as follows

$$F(k)=\frac{1}{k!}[\frac{d^k f(\eta)}{d\eta^k}]_{\eta=\eta_0} \quad . \tag{14}$$

In equation (14), $f(\eta)$ is the original function and $F(k)$ is the transformed function which is called the T-function. The differential inverse transformation of $F(k)$ is defined as

$$f(\eta)=\sum_{k=0}^{\infty}F(k)(\eta-\eta_0)^k \approx \sum_{k=0}^{N}F(k)(\eta-\eta_0)^k \quad . \tag{15}$$

Combining equations (14) and (15), we obtain

$$f(\eta)\approx \sum_{k=0}^{N}[\frac{d^k f(\eta)}{d\eta^k}]_{\eta=\eta_0}\frac{(\eta-\eta_0)^k}{k!} \quad . \tag{16}$$

Theorems proposed by Rashidi, Mohimanian and Laraqito[14] in the transformation procedure are given below

Theorem 1. If $f(\eta)=g(\eta)\pm h(\eta)$, then $F(k)=G(k)\pm H(k)$.
Theorem 2. If $f(\eta)=cg(\eta)$, then $F(k)=cG(k)$, where c is a constant.
Theorem 3. If $f(\eta)=\frac{d^n g(\eta)}{d\eta^n}$, then $F(k)=\frac{(k+n)!}{k!}G(k+n)$.
Theorem 4. If $f(\eta)=g(\eta)h(\eta)$, then $F(k)=\sum_{l=0}^{k}G(l)H(k-l)$.

Taking differential transform of equation (10), the following equation can be obtained

$$F(k+3)=\frac{\phi_1}{(k+1)(k+2)(k+3)}\{\sum_{i=0}^{k}[\frac{2n}{n+1}(i+1)(k-i+1)F(i+1)F(k-i+1) -(k-i+1)(k-i+2)F(i)F(k-i+2)]\} \quad , \tag{17}$$

where $F(k)$ is the differential transforms of f.

The transform of the boundary conditions are

$$F(0)=0, F(1)=1, F(2)=\beta . \tag{18}$$

where β is a constant.

Using equation (18) and the iterative formula (17), $F(k)$ can be calculated. Then the following series solutions of the initial value problem is

$$f(\eta)=\sum_{k=0}^{\infty}F(k)\eta^k \approx \sum_{k=0}^{m}F(k)\eta^k \quad . \tag{19}$$

Using equation (10) and the boundary condition (12) and selecting the basic functions { $f_{0,0}(\eta)$, $f_{i,j}(\eta)_{(i,j=2,3,\cdots)}$ } where $f_{i,j}(\eta)=\eta^j e^{ia\eta}$, then one can regarded the expression of $f(\eta)$ as a linear combination of the basic functions

$$f(\eta)\approx f_{N_1,N_2}(\eta)= f_{0,0}(\eta)+\sum_{j=2}^{N_1}\sum_{i=2}^{N_2} b_{i,j} f_{i,j}(\eta)= f_{0,0}(\eta)+\sum_{j=2}^{N_1}\sum_{i=2}^{N_2} b_{i,j}\eta^j e^{ia\eta} \quad . \tag{20}$$

where $f_{00}=a_0-a_0e^{a\eta}+(aa_0+1)\eta e^{a\eta}+a_1\eta^2 e^{a\eta}$ satisfies the boundary conditions (12) and $f_{i,j}(\eta)=\eta^j e^{ia\eta}$ satisfy the following boundary conditions

$$f(0)=0,\ f'(0)=0,\ f'(\infty)\to 0 \quad . \tag{21}$$

where $a<0$ is an attenuation parameter which to be determined.

In practical applications, we will get a good precision when N_i less than 5.

For the case $\phi=0, n=1, \Pr=6.2, Ec=0.3$, let $N_1=2, N_2=4$, then expending equation (20) in the following power series

$$f(\eta)=\eta+[-\frac{a_0a^2}{2!}+(aa_0+1)a+a_1+\sum_{i=2}^{4}b_{i,2}]\eta^2+[-\frac{a_0a^3}{3!}+(aa_0+1)\frac{a^2}{2!}+a_1a+\sum_{i=2}^{4}b_{i,2}ai]\eta^3$$
$$+[-\frac{a_0a^4}{4!}+(aa_0+1)\frac{a^3}{3!}+a_1\frac{a^2}{2!}+\sum_{i=2}^{4}b_{i,2}\frac{(ai)^2}{2!}]\eta^4+[-\frac{a_0a^5}{5!}+(aa_0+1)\frac{a^4}{4!}+a_1\frac{a^3}{3!}+\sum_{i=2}^{4}b_{i,2}\frac{(ai)^3}{3!}]\eta^5+\cdots \tag{22}$$

The following equations can be obtained from the equations (19)(20) and (22) by comparing the coefficient of the same order of η^k

$$-\frac{a_0a^j}{j!}+(aa_0+1)\frac{a^{(j-1)}}{(j-1)!}+a_1\frac{a^{(j-2)}}{(j-2)!}+\sum_{i=2}^{4}b_{i,2}\frac{(ai)^{(j-2)}}{(j-2)!}=F(j), \qquad j=2\cdots 8. \tag{23}$$

It is easy to solve these equations successively, and then the series solution of the equation can be obtained

$$f=0.9999999952-0.9999999952e^{a\eta}-0.001604696\eta e^{a\eta}-0.1285555685\times10^{-5}\eta^2e^{a\eta}$$
$$+0.1790266428\times10^{-9}\eta^2e^{2a\eta}-0.2174269615\times10^{-10}\eta^2e^{3a\eta}+0.1738231524\times10^{-11}\eta^2e^{4a\eta}+\cdots \tag{24}$$

where $a=-1.00160470\ 1$.

Similarly, we can get the corresponding analytical solution for $g(\eta)$

$$g=e^{b\eta}+2.81515272\ 3\eta e^{b\eta}+6.64405373\ 3\eta^2e^{b\eta}-4.90531936\ 4\eta^2e^{2b\eta}$$
$$+0.624574852\ 6\eta^2e^{3b\eta}-8.77976883\ 7\eta^3e^{b\eta}-7.07210493\ 1\eta^3e^{2b\eta}$$
$$+0.768981512\ 8\eta^3e^{3b\eta}+4.07829186\ 9\eta^4e^{b\eta}-8.82337901\ 9\eta^4e^{2b\eta}+0.266249655\ 0\eta^4e^{3b\eta} \tag{25}$$

where $b=-3.97207381\ 2$.

4. DISCUSSION OF THE SOLUTIONS

The boundary layer flow and heat transfer due to stretching vertical sheet have been investigated analytically. All the results obtained by the DTM-BF method are compared with the numerical results obtained by bvp4c with Matlab. Fig.1 shows a comparison between analytical and numerical solutions for f, f' and g. Moreover, the results are also illustrated in Table 2. It is obvious that excellent agreement exists for all values considered.

The effects of the nonlinear stretching parameter n, Prandtl number Pr and Eckert number Ec on the velocity f, f' and g are plotted in Fig. 2. It is observed that as the nonlinear stretching parameter increases, f and f' decreases. Besides, Ec tends to increase the temperature profiles while Pr tends to decrease it. In addition, both of Eckert number and Prandtl number doesn't affect the velocity profile.

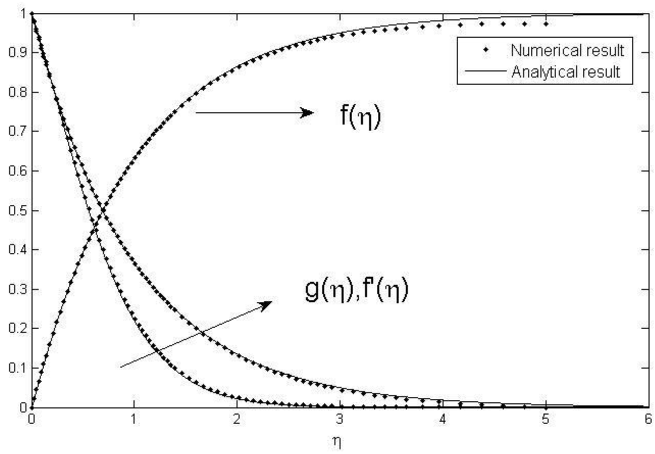

Figure 1. Comparison of numerical and analytical solutions for f, f' and g as $\phi = 0, n = 1, \mathrm{Pr} = 3, Ec = 0.3$.

Table 2. Comparison of $f'(\eta)$ as $\phi = 0, n = 1, \mathrm{Pr} = 6.2, Ec = 0.3$.

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
DTM-BF	1.0000	0.9048	0.8187	0.7408	0.6703	0.6065	0.5488	0.4966	0.4493	0.4066	0.3679
Bvp 4c	1.0000	0.9009	0.8116	0.7397	0.6741	0.6002	0.5469	0.4983	0.4436	0.4041	0.3638

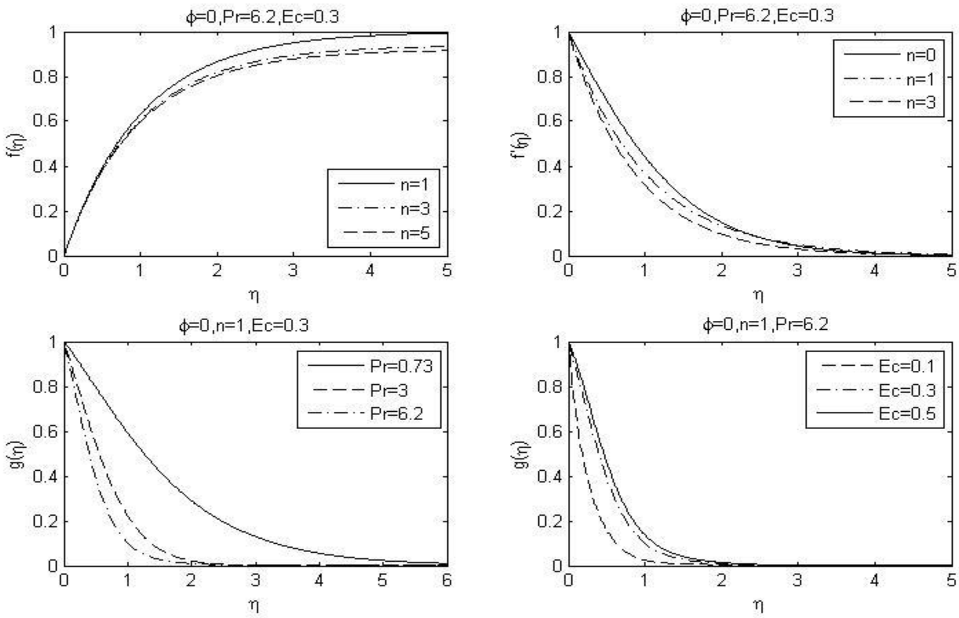

Figure 2. Effects of the parameters n, Pr and Ec for f, f' and g as $\phi = 0$.

Variations of f, f' and g against the volume fraction are plotted for different nanofluids in Figs. 3 and 4. Fig. 3 displays the effects of solid volume fraction on f and f' as $n = 1, \Pr = 6.2, Ec = 0.3$. It is found that as the solid volume fraction of nanoparticles increases, f and f' decreases. This refers that a high velocity can be obtained for lower solid volume concentration of nanoparticles. Fig. 4 depicts the effect of solid volume fraction on temperature related with Cu-water, Ag-water, Al_2O_3-water and TiO_2-water nanofluid, respectively. It is observed that increasing the volume fraction results in an increase in temperature. Besides, adding nanoparticle to the pure water decreases velocity profile while increases temperature profiles.

Fig.5 exhibits the velocity and temperature distributions for different nanoparticles respectively when $n = 1, \Pr = 6.2, Ec = 0.3$. It can be observed that the velocity and temperature distributions decrease gradually far away from the surface of the stretching sheet. Moreover, Al_2O_3-water nanofluid and TiO_2-water nanofluid exhibits higher velocity and lower temperature than that of the other nanofluid concerned while Ag-water nanofluid exhibits lower velocity and higer temperature instead. Besides, Al_2O_3-water nanofluid and TiO_2-water nanofluid have the similar profiles. Therefor, in the cooling intensification process, it is more suitable for selecting metal oxides such as aluminum oxide and titanium oxide than copper and silver.

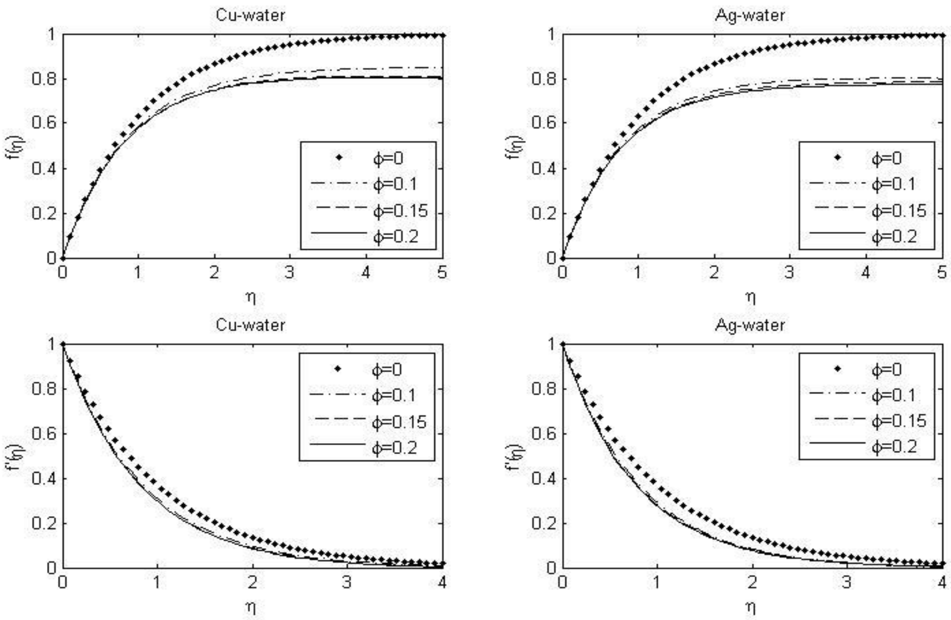

Figure 3. Effects of the solid volume fraction ϕ on f and f' for Cu and Ag as $n=1, \Pr=6.2, Ec=0.3$.

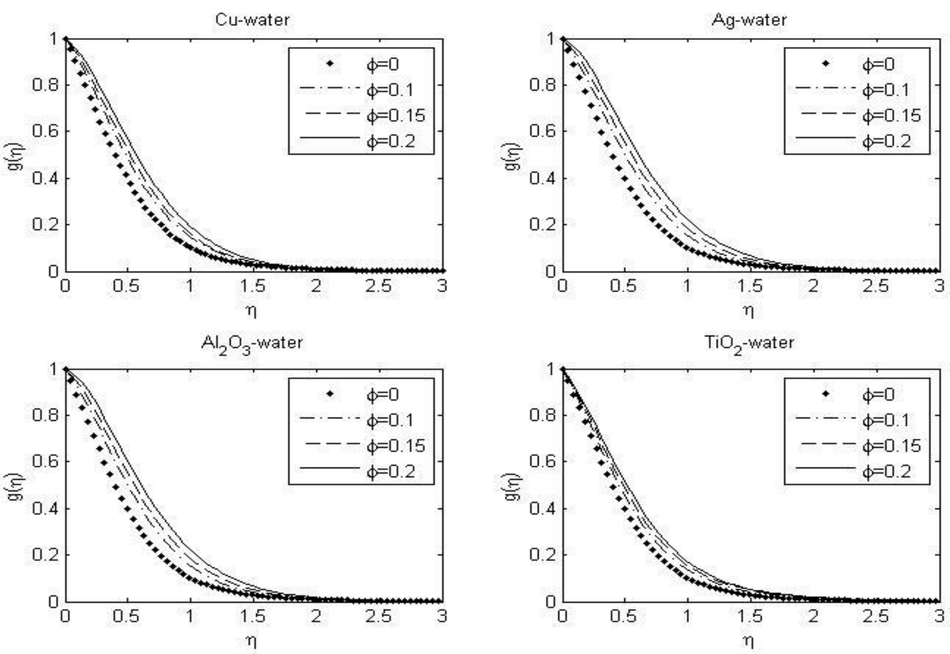

Figure 4. Effects of the solid volume fraction ϕ on temperature g for Cu, Ag, Al_2O_3 and TiO_2 as $n=1, \Pr=6.2, Ec=0.3$.

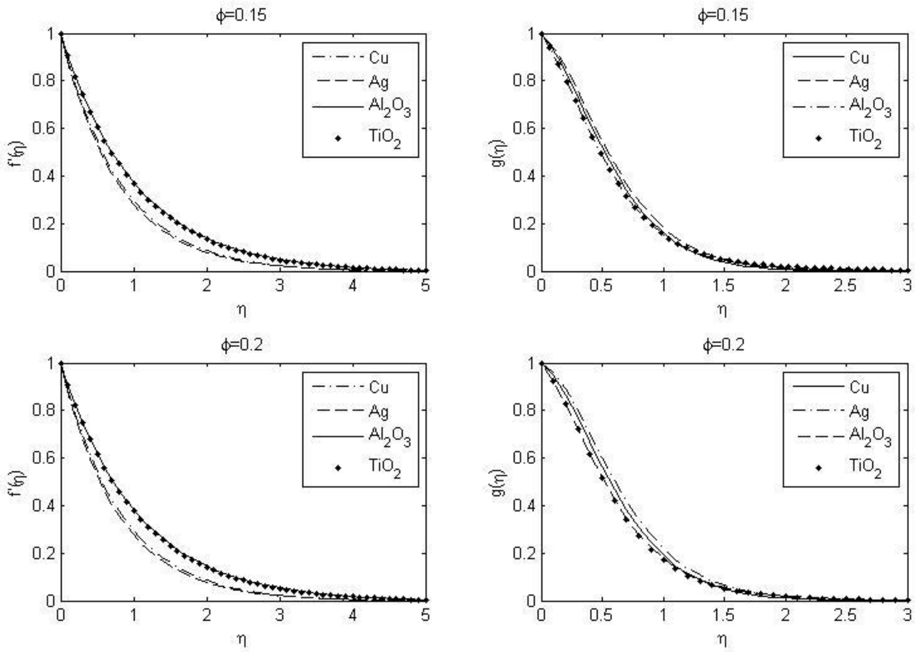

Figure 5. Comparison of different elements on velocity f' and temperature g for $\phi = 0.15$, $\phi = 0.2$ as $n = 1, \Pr = 6.2, Ec = 0.3$.

5. CONCLUSION

In the present paper, we have investigated the problem of flow and heat transfer in a viscous nanofluid over a nonlinear stretching sheet. The momentum and energy boundary layer transfer characteristics for different parameters are discussed. Some important conclusion can be drawn.

(i) Variations of the dimensionless velocity and temperature are affected with the solid volume fraction of nanoparticles, the nonlinear stretching parameter, Eckert number and Prandtl number. f and f' decreases with an increase in the nonlinear stretching parameter, Ec tends to increase the temperature profiles while Pr tends to decrease it.

(ii) A high velocity can be obtained for lower solid volume concentration of nanoparticles since f' decrease with the increasing parameter ϕ. Besides, the increasing volume fraction results in an increase in temperature. In addition, adding nanoparticle to the pure water decreases velocity profile while increases temperature profiles.

(iii) In the cooling intensification process, It is more suitable for selecting metal oxides nanofluids than the metal nanofluids since Al_2O_3-water nanofluid and TiO_2-water nanofluid exhibits higher velocity and lower temperature while Ag-water nanofluid exhibits lower velocity and higer temperature instead.

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