Mathematical and Computational Applications, Vol. 20, No. 2, pp. 106-110, 2015
http://dx.doi.org/10.19029/mca-2015-009

A WEAKLY CONVERGENCE RESULT ON $L^{p(x)}$ SPACES

Yasin Kaya

Department of Mathematics Education
Dicle University, 21280
Diyarbakır TURKEY
ykaya@dicle.edu.tr

Abstract- In this paper with $1 < p^- \le p^+ < \infty$ condition we prove a weak convergence result under pointwise convergence and bounded of the sequence. Our theorem is an extension of classical result to variable exponent setting.

Key Words- Variable exponent Lebesgue spaces, Weak convergence

1. INTRODUCTION

Variable exponent spaces play an important role in the study of some nonlinear problems in natural science and engineering. More concretely the motivation to study such function spaces comes from applications to fluid dynamics [1], image processing [2], PDE and the calculus of variation [3,4]. The spaces can be traced back to W. Orlicz [5], but systematically studied in a survey article by Kováčik and Rákosník [6] and later on by Fan and Zhao [7].

Let $p:\Omega \to [1,\infty)$ be a measurable bounded function, called a variable exponet on Ω and denote $p^+ = \text{ess sup}_{x\in\Omega} p(x)$ and $p^- = \text{ ess inf}_{x\in\Omega}\ p(x)$. We define the variable exponent Lebesgue space $L^{p(.)}(\Omega)$ to consist of all measurable function $u:\Omega \to \mathbb{R}$ for which the modular $\varrho_{p(.)}(u) = \int_\Omega |u(x)|^{p(x)} dx$ is finite. We define a norm, the so-called Luxemburg norm, on this space by the formula $\|u\|_{p(.)} = \inf\left\{\lambda > 0 : \varrho_{p(.)}\left(\frac{u}{\lambda}\right) \le 1\right\}.$

With this norm variable exponent Lebesgue space is a Banach space. If p is a constant function, then the variable exponent Lebesgue spaces coincides with the classical Lebesgue space and so the notation can give rise to no confusion.

2. PRELIMINARIES

For fixed exponent spaces we have a very simple relationship between the norm and the modular. In the variable exponent case we have the following theorem.

Theorem 2.1. [7] Given Ω and $p(.)$, then

1) $\|u\|_{p(.)} < 1(=1;>1) \Leftrightarrow \varrho(u) < 1(=1;>1)$;

2) If $\|u\|_{p(.)}>1$, then $\|u\|_{p(.)}^{p^-}\le\varrho(u)\le\|u\|_{p(.)}^{p^+}$;

3) If $\|u\|_{p(.)}<1$, then $\|u\|_{p(.)}^{p^-}\le\varrho(u)\le\|u\|_{p(.)}^{p^+}$.

Given $p(.)$, we define the conjugate exponent function $p'(.)$ by the formula

$$\frac{1}{p(x)}+\frac{1}{p'(x)}=1. \tag{1}$$

Theorem 2.2. [6] Given Ω and $p(.)$, for all $u\in L^{p(.)}(\Omega)$ and $v\in L^{p'(.)}(\Omega)$, then $uv\in L^1(\Omega)$ and

$$\int_\Omega |u(x)v(x)|dx\le C\|u(x)\|_{p(.)}\|v(x)\|_{p'(.)}. \tag{2}$$

Theorem 2.3. [6] Given Ω and $p(.)$, the dual space to $L^{p(.)}(\Omega)$ is $L^{p'(.)}(\Omega)$ if and only if $p(.)$ is bounded.

3. FROM POINTWISE CONVERGENCE TO WEAK CONVERGENCE

Our following theorem is an extension of classical result to the variable exponent setting.

Theorem 3.1. Let (X,M,μ) be a σ-finite measure space. Consider $L^{p(.)}(X,\mathrm{M},\mu)$ where $1<p^-\le p^+<\infty$. Let (u_n) be a sequence in $L^{p(.)}(X,\mathrm{M},\mu)$ and let u be an element in $L^{p(.)}(X,\mathrm{M},\mu)$. Suppose (u_n) satisfies the following conditions:

1) $\lim_{n\to\infty}u_n=u$ a.e. on X,
2) $\varrho_{p(.)}(u_n)\le K$, $K>0$.

Then (u_n) converges weakly to u in $L^{p(.)}(X,\mathrm{M},\mu)$.

Proof. We first define

$$p_*=\begin{cases}p^-, \text{ if } \|u\|_{p(.)}>1\\ p^+, \text{ if } \|u\|_{p(.)}<1\\ 1, \text{ if } \|u\|_{p(.)}=1\end{cases}$$

Hence we get

$$\|u\|_{p(.)}\le\left(\varrho_{p(.)}(u)\right)^{\frac{1}{p_*}}. \tag{3}$$

We are to show that for every $v \in L^{p'(.)}(\Omega)$ we have

$$\lim_{n\to\infty} \int_X u_n(x) v(x)\, d\mu = \int_X u(x) v(x)\, d\mu.$$

First we show that $\varrho_{p(.)}(u_n) \le K$. By Fatou's lemma we have

$$\int_X |u(x)|^{p(x)} d\mu = \int_X |\liminf u_n(x)|^{p(x)} d\mu \le \liminf \int_X |u_n(x)|^{p(x)} d\mu \le K.$$

Thus $\varrho_{p(.)}(u) \le K$. Then for any $A \in \mathrm{M}$ we have

$$\int_A |u_n(x) - u(x)|^{p(x)} d\mu \le \int_X |u_n(x) - u(x)|^{p(x)} d\mu \tag{4}$$

$$\le 2^{p^+} \left(\int_X |u_n(x)|^{p(x)} d\mu + \int_X |u(x)|^{p(x)} d\mu \right)$$

$$\le 2^{p^+ + 1} K < \infty. \tag{5}$$

Let $v \in L^{p'(.)}(\Omega)$ be arbitrarily chosen and let $0 < \varepsilon \le 1$. Since $\int_X |v(x)|^{p'(x)} d\mu < \infty$ by absolute continuity of integral there exists $\delta > 0$ such that for every $E \in \mathrm{M}$ with $\mu(E) < \delta$ we have

$$\int_E |v(x)|^{p'(x)} d\mu < \varepsilon. \tag{6}$$

Since (X, M, μ) a σ-finite measure space there exists an increasing sequence (F_k) in M such that $\lim_{k\to\infty} F_k = \bigcup_{k\in\mathbb{N}} F_k = X$ and $\mu(F_k) < \infty$ for $k \in \mathbb{N}$. By the monotone convergence theorem we have

$$\lim_{k\to\infty} \int_{F_k} |v(x)|^{p'(x)} d\mu = \lim_{k\to\infty} \int_X \chi_{F_k}(x) |v(x)|^{p'(x)} d\mu = \int_X |v(x)|^{p'(x)} d\mu < \infty. \tag{7}$$

Thus for sufficiently large $k_0 \in \mathbb{N}$ we have

$$\int_X |v(x)|^{p'(x)} d\mu - \int_{F_{k_0}} |v(x)|^{p'(x)} d < \varepsilon. \tag{8}$$

Let $F = F_{k_0}$. Then $\mu(F) < \infty$ and

$$\int_{F^c} |v(x)|^{p'(x)} d\mu < \varepsilon. \tag{9}$$

Since $\mu(F)<\infty$ by Egorov's theorem there exists $G\in \mathrm{M}$ such that $G\subset F$, $\mu(F\backslash G)<\delta$ and $\lim_{n\to\infty} u_n = u$ uniformly on G. If $\varrho_{p'(.)}(v)=0$ then $v=0$ a.e. on X so that theorem holds. For $\varrho_{p'(.)}(v)>0$; since $\lim_{n\to\infty} u_n = u$ uniformly on G there exists $N\in\mathbb{N}$ such that for any $n\geq N$ we have

$$|u_n(x)-u(x)|<\varepsilon \text{ for all } x\in G. \tag{10}$$

Then for $n\geq N$ we have

$$\int_G |u_n(x)-u(x)|^{p(x)} d\mu < \varepsilon^{p^-}\mu(G). \tag{11}$$

Now $F\backslash G$, G, F^c are disjoint and their union is equal to X. Thus for $n\geq N$ we have

$$\left|\int_X u_n(x)v(x)\,d\mu - \int_X u(x)v(x)\,d\mu\right| \leq \int_X |u_n(x)-u(x)||v(x)|\,d\mu$$

$$= \int_{F\backslash G} |u_n(x)-u(x)||v(x)|\,d\mu + \int_G |u_n(x)-u(x)||v(x)|\,d\mu +$$

$$\int_{F^c} |u_n(x)-u(x)||v(x)|\,d\mu. \tag{12}$$

We estimate the last three integrals by applying Hölder's inequality and (3)

For the first of the three integrals, by (4), (5) and (6) we have

$$\int_{F\backslash G} |u_n(x)-u(x)||v(x)|\,d\mu \leq C\|u_n(x)-u(x)\|_{p(.)}\|v(x)\|_{p'(.)}$$

$$\leq C\left(\int_{F\backslash G}|u_n(x)-u(x)|^{p(x)}d\mu\right)^{\frac{1}{p_*}}\left(\int_{F\backslash G}|v(x)|^{p'(x)}d\mu\right)^{\frac{1}{p'_*}}$$

$$\leq C\left(2^{p^+ +1}K\right)^{\frac{1}{p_*}}\varepsilon^{\frac{1}{p'_*}}.$$

For the second integral, by (11) we have

$$\int_G |u_n(x)-u(x)||v(x)|\,d\mu \leq C\|u_n(x)-u(x)\|_{p(.)}\|v(x)\|_{p'(.)}$$

$$\leq C\left(\int_G |u_n(x)-u(x)|^{p(x)}d\mu\right)^{\frac{1}{p_*}}\|v(x)\|_{p'(.)}$$

$$\leq C\left(\varepsilon^{p^-}\mu(G)\right)^{\frac{1}{p_*}}\|v(x)\|_{p'(.)}.$$

And for the third integral, by (9) we have

$$\int_{F^c} |u_n(x)-u(x)||v(x)|d\mu \le C\|u_n(x)-u(x)\|_{p(.)}\|v(x)\|_{p'(.)}$$

$$\le C\left(\int_{F^c}|u_n(x)-u(x)|^{p(x)}d\mu\right)^{\frac{1}{p_*}}\left(\int_{F^c}|v(x)|^{p'(x)}d\mu\right)^{\frac{1}{p'_*}}$$

$$\le C\left(2^{p^+ +1}K\right)^{\frac{1}{p_*}}\varepsilon^{\frac{1}{p'_*}}.$$

Thus we have for $n \ge N$

$$\left|\int_X u_n(x)v(x)d\mu - \int_X u(x)v(x)d\mu\right|$$

$$\le C\left(2^{p^+ +1}K\right)^{\frac{1}{p_*}}\varepsilon^{\frac{1}{p'_*}} + C\left(\varepsilon^{p^-}\mu(G)\right)^{\frac{1}{p_*}}\|v(x)\|_{p'(.)} + C\left(2^{p^+ +1}K\right)^{\frac{1}{p_*}}\varepsilon^{\frac{1}{p'_*}}.$$

4. CONCLUDING REMARKS

We proved a weak convergence result which does not require any regularity of exponent functions. By inequalities relating the norm and modular in variable exponent Lebesque spaces it is also possible to use the norm boundedness of the sequence instead of the modular boundedness. But in this case the upper bound may be different from K.

5. REFERENCES

1. M. Růžička, *Elektrorheological Fluids: Modeling and mathematical Theory, Lecture notes in Mathematics*, vol.1748, Springer-Verlag, Berlin, 2000.
2. S. Chen, Y. Levine, and J. Stanich, Image restoration via nonstandard diffusion, Technical Report # 04-01, Department of Mathematics and Computer Sciences, Duquesne University, 2004.
3. V.V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, *Izvestiya Akademii Nauk SSSR Seriya Matematicheskaya* **50(4)**, 675-710, 887, 1986.
4. E. Acerbi and G. Mingione, Regularity results for a class of functionals with non-standard growth, *Archive for Rational Mechanics and Analysis* **156(2)**, 121-140, 2001.
5. W. Orlicz, Über konjugierte exponentenfolgen, *Studia Mathematica* **3**, 200-211 (German), 1931.
6. O. Kováčik and J. Rákosník, On spaces $L^{p(x)}$ and $W^{k,p(x)}$, *Czechoslovak Mathematical Journal* **41(116)**, 592-618, 1991.
7. X. Fan and D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *Journal of Mathematical Analysis and Applications* **263**, 424-446, 2001.