Mathematical and Computational Applications, Vol. 20, No. 2, pp. 137-150, 2015
http://dx.doi.org/10.19029/mca-2015-012

SOLUTION OF QUADRATIC NONLINEAR PROBLEMS WITH MULTIPLE SCALES LINDSTEDT-POINCARE METHOD

Mehmet Pakdemirli and Gözde Sarı

Applied Mathematics and Computation Center, Celal Bayar University, Muradiye, 45140 Manisa, Turkey.
mpak@cbu.edu.tr, gozde.deger@cbu.edu.tr

Abstract- A recently developed perturbation algorithm namely the multiple scales Lindstedt-Poincare method (MSLP) is employed to solve the mathematical models. Three different models with quadratic nonlinearities are considered. Approximate solutions are obtained with classical multiple scales method (MS) and the MSLP method and they are compared with the numerical solutions. It is shown that MSLP solutions are better than the MS solutions for the strongly nonlinear case of the considered models.

Keywords- perturbation methods; numerical solutions; systems with quadratic nonlinearities

1. INTRODUCTION

Perturbation theories have been widely used to obtain approximate analytical solutions of linear and nonlinear physical problems. Although the methods provide acceptable solutions for weakly nonlinear problems, the solutions do not represent the physics for the strongly nonlinear cases. Recently, for solution of the strongly nonlinear problems, a new perturbation method was developed by Pakdemirli *et al.* [1]. This method named multiple scales Lindstedt-Poincare method (MSLP) combines the classical multiple scales method and the Lindstedt-Poincare method. Pakdemirli and Karahan [2] and Pakdemirli *et al*. [3] applied the method to many strongly nonlinear problems and obtained good results compatible with the numerical solutions.

This new method has not been tested for problems with strong quadratic nonlinearities. In this study, three different quadratic nonlinear problems are solved by MSLP and MS method. The approximate solutions are contrasted with the numerical solutions. For weak nonlinearities, all three methods yield similar solutions. As the nonlinearity is increased, the solutions deviate from each other, with MSLP yielding better approximate solutions in contrast to the numerical solutions.

2. QUADRATIC NONLINEAR MODEL I

Consider the below problem with a quadratic nonlinearity

$$\ddot{u} + \omega_0^2 u + \varepsilon u \dot{u} = 0 \tag{1}$$

with initial conditions

$$u(0) = a_0 \qquad \dot{u}(0) = 0 \tag{2}$$

2. 1. Multiple Scales Method (MS)

First, the problem is solved with the classical method. Solutions are assumed to be of the form;

$$u = u_0(T_0,T_1,T_2) + \varepsilon u_1(T_0,T_1,T_2) + \varepsilon^2 u_2(T_0,T_1,T_2) \tag{3}$$

where T_0=t is the usual fast time scale, T_1=εt and T_2=ε^{2}t are the slow time scales. Time derivatives are defined as

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2, \quad \frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_1^2 + 2D_0 D_2) \tag{4}$$

where $D_n = \partial / \partial T_n$.

If (3) and (4) are substituted into the original equation, the following equations are obtained at each order of ε;

$$O(1) \qquad D_0^2 u_0 + \omega_0^2 u_0 = 0 \qquad u_0(0) = a_0 \quad D_0 u_0(0) = 0 \tag{5}$$

$$\mathrm{O}(\varepsilon) \qquad D_0^2 u_1 + \omega_0^2 u_1 = -2D_0 D_1 u_0 - u_0 D_0 u_0 \qquad u_1(0) = 0 \quad (D_0 u_1 + D_1 u_0)(0) = 0 \tag{6}$$

$$\mathrm{O}(\varepsilon^2) \qquad D_0^2 u_2 + \omega_0^2 u_2 = -2D_0 D_1 u_1 - (D_1^2 + 2D_0 D_2) u_0 - u_0 (D_0 u_1 + D_1 u_0) - u_1 D_0 u_0 \tag{7}$$

At order 1, the solution may be expressed as (8)

$$u_0 = A e^{i\omega_0 T_0} + cc$$

where cc stands for the complex conjugates of the preceding terms and

$$A = \frac{1}{2} a e^{i\beta} \tag{9}$$

The first order solution is obtained in terms of real amplitude and phase

$$u_0 = a(T_1,T_2)\cos(\omega_0 t + \beta(T_1,T_2)) \tag{10}$$

Applying the initial conditions yields

$$\beta(0) = 0 \quad a(0) = a_0 \tag{11}$$

Equation (8) is substituted into (6) and secular terms are eliminated

$$D_1 A = 0 \Rightarrow A = A(T_2) \tag{12}$$

The solution at order ε is

$$u_1 = B e^{i\omega_0 T_0} + \frac{i}{3\omega_0} A^2 e^{2i\omega_0 T_0} + cc \tag{13}$$

This solution may be represented in terms of real amplitude and phase

$$u_1 = -b\sin(\omega_0 T_0 + \gamma) - \frac{a^2}{6\omega_0}\sin(2\omega_0 T_0 + 2\beta) \tag{14}$$

where

$$B = \frac{1}{2} i b e^{i\gamma} \tag{15}$$

Applying the initial conditions yields

$$\gamma(0) = 0 \quad b(0) = -\frac{a_0^2}{3\omega_0} \tag{16}$$

Equation (8) and (13) are inserted into (7) and secular terms are eliminated,

$$-2i\omega_0 D_1 B - 2i\omega_0 D_2 A + \frac{1}{3} A^2 \bar{A} = 0 \tag{17}$$

If (5), (15), (16) are inserted into (17), one finally has

$$a = a_0, \quad b = -\frac{a_0^2}{3\omega_0}, \quad \gamma = 0, \quad \beta = -\frac{a_0^2}{24\omega_0} T_2 \tag{18}$$

The final solution is

$$u = a_0 \cos((\omega_0 - \frac{\varepsilon^2 a_0^{\ 2}}{24\omega_0})t) + \frac{\varepsilon a_0^{\ 2}}{6\omega_0}(2\sin(\omega_0 t) - \sin(2(\omega_0 - \frac{\varepsilon^2 a_0^{\ 2}}{24\omega_0})t)) \tag{19}$$

For valid solutions, the correction term should be much smaller than the leading term. For the problem, this criterion yields

$$\frac{\varepsilon a_0}{6\omega_0} << 1 \tag{20}$$

2.2. Multiple Scales Lindstedt-Poincare Method (MSLP)

Details of the method was presented in the previous literature [1-3]. The time transformation $\tau = \omega t$ is applied to the model

$$\omega^2 u'' + \omega_0^{\ 2} u + \varepsilon u \omega u' = 0 \tag{21}$$

where prime denotes derivative with respect to the new variable τ . The time scales in this method are slightly different from classical multiple scales

$$T_0 = \tau,\ \ T_1 = \varepsilon\tau,\ \ T_2 = \varepsilon^2\tau \tag{22}$$

Expressing the time derivatives

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 \qquad \frac{d^2}{dt^2} = D_0^{\ 2} + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_1^{\ 2} + 2D_0 D_2) \tag{23}$$

with $D_n = \partial / \partial T_n$ and substituting the expansions

$$u = u_0(T_0, T_1, T_2) + \varepsilon u_1(T_0, T_1, T_2) + \varepsilon^2 u_2(T_0, T_1, T_2) \tag{24}$$

$$\omega_0^{\ 2} = \omega^2 - \varepsilon\omega_1 - \varepsilon^2\omega_2 \tag{25}$$

into the original equation, the following equations are obtained at each order of approximation;

$$O(1) \quad \omega^2 D_0^{\ 2} u_0 + \omega^2 u_0 = 0,\ \ u_0(0) = a_0 \quad D_0 u_0(0) = 0 \tag{26}$$

$$O(\varepsilon) \quad \omega^2 D_0^{\ 2} u_1 + \omega^2 u_1 = -2\omega^2 D_0 D_1 u_0 + \omega_1 u_0 - \omega u_0 D_0 u_0 \qquad u_1(0) = 0 \quad (D_0 u_1 + D_1 u_0)(0) = 0 \tag{27}$$

$$O(\varepsilon^2) \quad \omega^2 D_0^{\ 2} u_2 + \omega^2 u_2 = -2\omega^2 D_0 D_1 u_1 - \omega^2 (D_1^{\ 2} + 2D_0 D_2) u_0 + \omega_1 u_1 + \omega_2 u_0 - \omega(u_0(D_0 u_1 + D_1 u_0) + u_1 D_0 u_0) \tag{28}$$

The first order solution is

$$u_0 = Ae^{iT_0} + cc = a\cos(T_0 + \beta) \tag{29}$$

Applying the initial conditions yields

$$\beta(0) = 0,\ \ a(0) = a_0 \tag{30}$$

Equation (29) is substituted into (27) and secular terms are eliminated

$$-2i\omega^2 D_1 A + \omega_1 A = 0 \tag{31}$$

If $D_1A=0$ is selected, $a=a(T_2)$, $\beta= \beta\ (T_2)$ and $\omega_1=0$. Since ω_1 is not complex, this choice is admissible. The solution at order ε is

$$u_1 = Be^{iT_0} + \frac{i}{3\omega} A^2 e^{2iT_0} + cc \qquad B = \frac{1}{2} ibe^{i\gamma} \tag{32}$$

In terms of real amplitude and phase, the solution is

$$u_1 = -b\sin(T_0 + \gamma) - \frac{a^2}{6\omega}\sin(2T_0 + 2\beta) \tag{33}$$

Applying the initial conditions yields

$$\gamma(0) = 0 \quad b(0) = -\frac{a_0^{\ 2}}{3\omega} \tag{34}$$

Equations (29) and (32) are inserted into (28) and secular terms are eliminated

$$-2i\omega^2 D_1 B - 2i\omega^2 D_2 A + \omega_2 A + \frac{1}{3} A^2 \bar{A} = 0 \tag{35}$$

$D_1B=0$ can be assumed. If $D_2A=0$ is selected, ω_2 comes out to be real and this is again an admissible choice. After algebraic calculations, Equation (35) yields

$$a = a_0, \quad b = -\frac{a_0^{\ 2}}{3\omega}, \quad \gamma = \beta = 0, \quad \omega_2 = -\frac{1}{12} a^2 \tag{36}$$

The frequency is

$$\omega = \sqrt{\omega_0^{\ 2} - \frac{\varepsilon^2 a_0^{\ 2}}{12}} \tag{37}$$

The final solution in terms of this frequency is

$$u = a_0 \cos(\omega t) + \frac{\varepsilon a_0^{\ 2}}{6\omega}(2\sin(\omega t) - \sin(2\omega t)) + O(\varepsilon^2) \tag{38}$$

For valid solutions, the perturbation criteria is

$$\frac{\varepsilon a_0}{6\omega_0 \sqrt{1 - \frac{\varepsilon^2}{12\omega_0^{\ 2}} a_0^{\ 2}}} << 1 \tag{39}$$

2.3. Comparisons with the Numerical Solutions

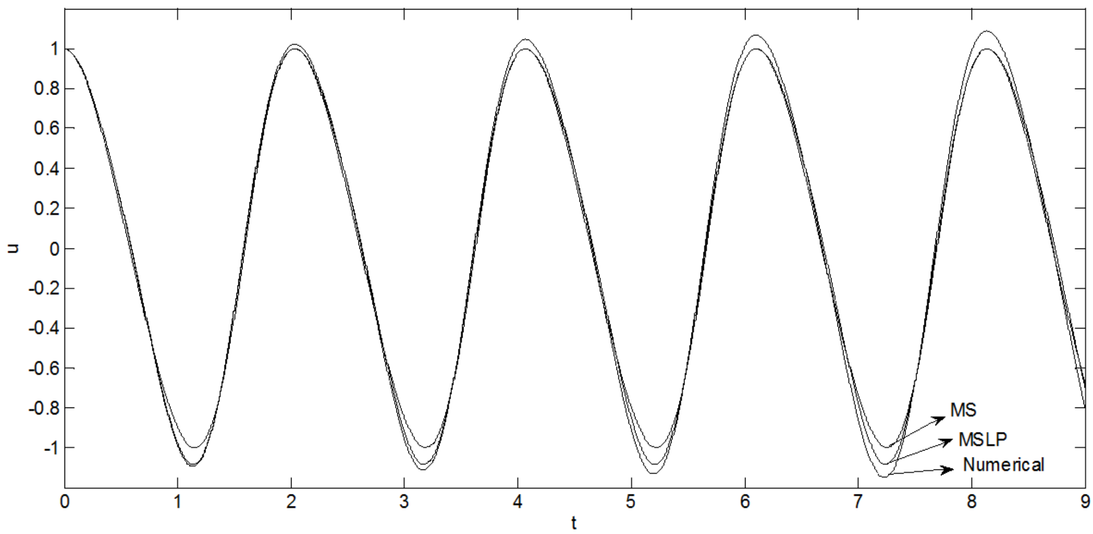

Figure 1. Comparison of numerical solutions and approximate analytical solutions (MS and MSLP) for $\varepsilon = 2,\ a_0 = 1,\ \omega_0 = \pi$

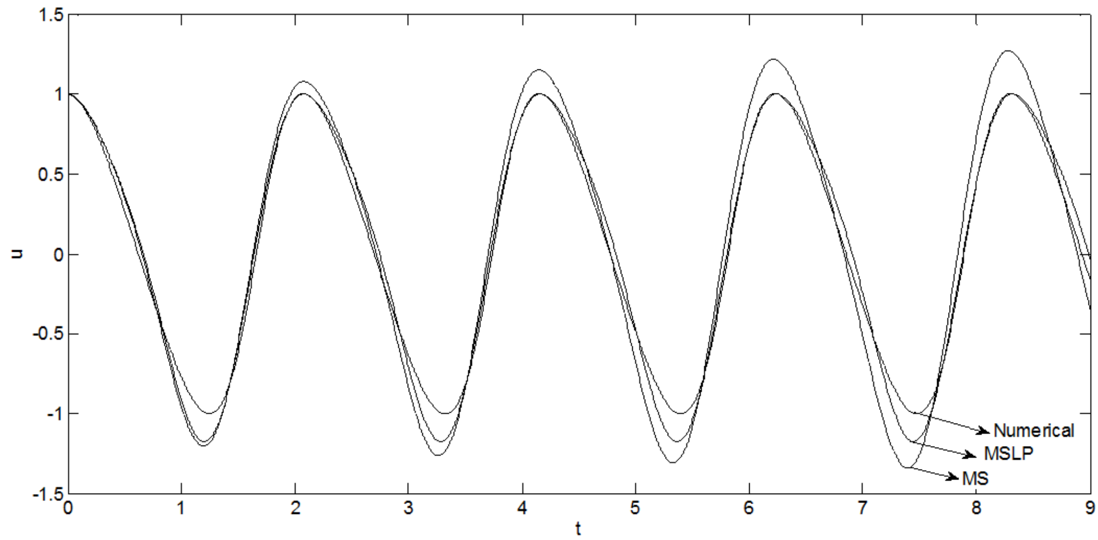

Figure 2. Comparison of numerical solutions and approximate analytical solutions (MS and MSLP) for $\varepsilon = 3,\ a_0 = 1,\ \omega_0 = \pi$

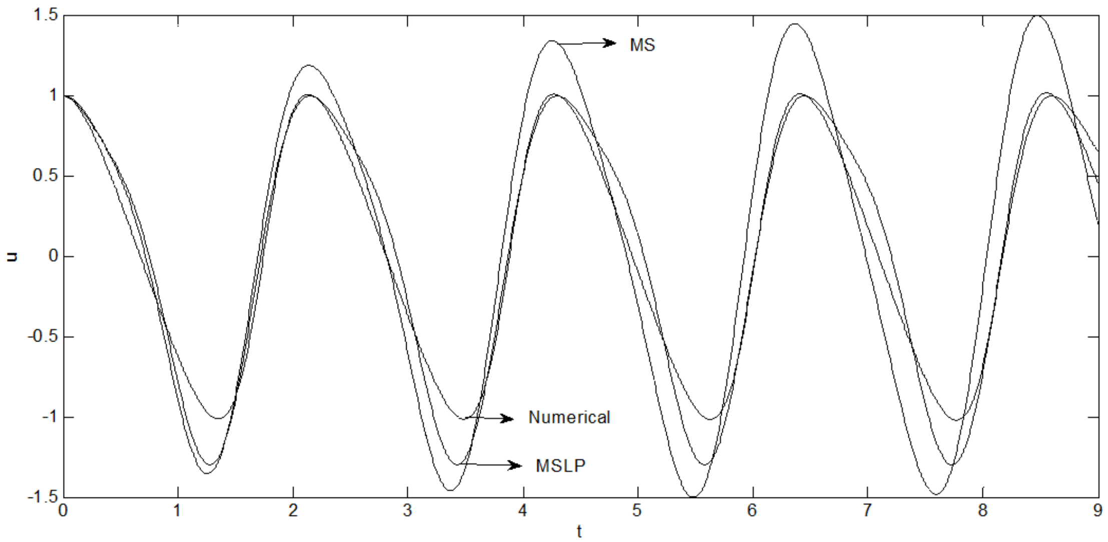

Figure 3. Comparison of numerical solutions and approximate analytical solutions (MS and MSLP) for $\varepsilon = 4,\ a_0 = 1,\ \omega_0 = \pi$

In this section, the approximate solutions are contrasted with the numerical solutions for the quadratic nonlinear model considered. In Figure 1, results are compared for ε=2. The agreement between MSLP and numerical solutions is better than MS solution and the amplitude values of the MS solution yield higher errors. The positive amplitudes agree with numerical and MSLP cases whereas the positive amplitudes introduce errors in case of MS solutions. For negative amplitude values, the error is less for MSLP. ε=3 is selected in Figure 2 and MSLP and numerical solutions agree well for positive values of amplitudes whereas, the error is smaller for negative amplitudes for MSLP solutions. For ε=4, in Figure 3, the same trend is observed with more amplification.

3. QUADRATIC NONLINEAR MODEL II

Consider the below problem with a quadratic nonlinearity

$$\ddot{u}+\omega_0^{2}u+\varepsilon\alpha u^2=0 \tag{40}$$

with initial conditions

$$u(0)=a_0 \qquad \dot{u}(0)=0 \tag{41}$$

3.1. Multiple Scales Method (MS)

First, the problem is solved with the classical method. Solutions are assumed to be of the form;

$$u=u_0(T_0,T_1,T_2)+\varepsilon u_1(T_0,T_1,T_2)+\varepsilon^2 u_2(T_0,T_1,T_2) \tag{42}$$

where $T_0=t$ is the usual fast time scale, $T_1=\varepsilon t$ and $T_2=\varepsilon^2 t$ are the slow time scales. Time derivatives are defined as

$$\frac{d}{dt}=D_0+\varepsilon D_1+\varepsilon^2 D_2, \qquad \frac{d^2}{dt^2}=D_0^{2}+2\varepsilon D_0 D_1+\varepsilon^2(D_1^2+2D_0D_2) \tag{43}$$

where $D_n=\partial/\partial T_n$.

If (42) and (43) are substituted into the original equation, the following equations are obtained at each order of ε;

$$O(1) \qquad D_0^{2}u_0+\omega_0^{2}u_0=0 \qquad u_0(0)=a_0 \quad D_0u_0(0)=0 \tag{44}$$

$$O(\varepsilon) \qquad D_0^{2}u_1+\omega_0^{2}u_1=-2D_0D_1u_0-\alpha u_0^{2} \qquad u_1(0)=0 \quad (D_0u_1+D_1u_0)(0)=0 \tag{45}$$

$$O(\varepsilon^2) \qquad D_0^{2}u_2+\omega_0^{2}u_2=-2D_0D_1u_1-(D_1^2+2D_0D_2)u_0-2\alpha u_0u_1 \tag{46}$$

At order 1, the solution may be expressed as (47)

$$u_0=Ae^{i\omega_0T_0}+cc$$

where cc stands for the complex conjugates of the preceding terms and

$$A=\frac{1}{2}ae^{i\beta} \tag{48}$$

The first order solution is obtained in terms of real amplitude and phase

$$u_0=a(T_1,T_2)\cos(\omega_0T_0+\beta(T_1,T_2)) \tag{49}$$

Applying the initial conditions yields

$$\beta(0)=0 \quad a(0)=a_0 \tag{50}$$

Equation (47) is substituted into (45) and secular terms are eliminated

$$D_1A=0 \Rightarrow A=A(T_2) \tag{51}$$

The solution at order ε is

$$u_1=Be^{i\omega_0T_0}+cc+\frac{\alpha}{3\omega_0^{2}}(A^2e^{2i\omega_0T_0}+cc)-\frac{\alpha}{\omega_0^{2}}2A\bar{A} \tag{52}$$

This solution may be represented in terms of real amplitude and phase

$$u_1=b\cos(\omega_0T_0+\gamma)+\frac{\alpha}{6\omega_0^{2}}a^2\cos(2\omega_0T_0+2\beta)-\frac{\alpha}{2\omega_0^{2}}a^2 \tag{53}$$

where

$$B=\frac{1}{2}ibe^{i\gamma} \tag{54}$$

Applying the initial conditions yields

$$\gamma(0)=0 \quad b(0)=\frac{\alpha}{3\omega_0^{2}}a_0^{2} \tag{55}$$

Equation (47) and (52) are inserted into (46) and secular terms are eliminated,

$$-2i\omega_0 D_1 B - 2i\omega_0 D_2 A + \frac{10}{3}\frac{\alpha^2}{\omega_0^{\,2}} A^2 \bar{A} = 0 \tag{56}$$

If (45), (54), (55) are inserted into (56), one finally has

$$a = a_0, \quad b = \frac{\alpha}{3\omega_0^{\,2}} a_0^{\,2}, \quad \gamma = 0, \quad \beta = -\frac{5}{12}\frac{\alpha^2}{\omega_0^{\,3}} a_0^{\,2} T_2 \tag{57}$$

The final solution is

$$u = a_0 \cos((\omega_0 - \varepsilon^2 \frac{5\alpha^2}{12\omega_0^{\,3}} a_0^{\,2})t) + \varepsilon(\frac{\alpha}{3\omega_0^{\,2}} a_0^{\,2} \cos(\omega_0 t) \tag{58}$$

$$+\frac{\alpha}{6\omega_0^{\,2}} a_0^{\,2} (\cos(2(\omega_0 - \varepsilon^2 \frac{5\alpha^2}{12\omega_0^{\,3}} a_0^{\,2})t) - 3) + O(\varepsilon^2)$$

3.2. Multiple Scales Lindstedt-Poincare Method (MSLP)

The time transformation $\tau = \omega t$ is applied to the model

$$\omega^2 u'' + \omega_0^{\,2} u + \varepsilon \alpha u^2 = 0 \tag{59}$$

where prime denotes derivative with respect to the new variable τ. The time scales in this method are slightly different from classical multiple scales

$$T_0 = \tau, \;\; T_1 = \varepsilon \tau, \;\; T_2 = \varepsilon^2 \tau \tag{60}$$

Expressing the time derivatives

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 \qquad \frac{d^2}{dt^2} = D_0^{\,2} + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_1^2 + 2D_0 D_2) \tag{61}$$

with $D_n = \partial / \partial T_n$ and substituting the expansions

$$u = u_0(T_0, T_1, T_2) + \varepsilon u_1(T_0, T_1, T_2) + \varepsilon^2 u_2(T_0, T_1, T_2) \tag{62}$$

$$\omega_0^{\,2} = \omega^2 - \varepsilon \omega_1 - \varepsilon^2 \omega_2 \tag{63}$$

into the original equation, the following equations are obtained at each order of approximation;

$$O(1) \quad \omega^2 D_0^{\,2} u_0 + \omega^2 u_0 = 0, \quad u_0(0) = a_0 \quad D_0 u_0(0) = 0 \tag{64}$$

$$\mathrm{O}(\varepsilon) \quad \omega^2 D_0^{\,2} u_1 + \omega^2 u_1 = -2\omega^2 D_0 D_1 u_0 + \omega_1 u_0 - \alpha u_0^{\,2} \qquad u_1(0) = 0 \quad (D_0 u_1 + D_1 u_0)(0) = 0 \tag{65}$$

$$\mathrm{O}(\varepsilon^2) \quad \omega^2 D_0^{\,2} u_2 + \omega^2 u_2 = -2\omega^2 D_0 D_1 u_1 - \omega^2 (D_1^2 + 2D_0 D_2) u_0 + \omega_1 u_1 + \omega_2 u_0 - 2\alpha u_0 u_1 \tag{66}$$

The first order solution is

$$u_0 = A e^{iT_0} + cc = a \cos(T_0 + \beta) \tag{67}$$

Applying the initial conditions yields

$$\beta(0) = 0, \;\; a(0) = a_0 \tag{68}$$

Equation (67) is substituted into (65) and secular terms are eliminated

$$-2i\omega^2 D_1 A + \omega_1 A = 0 \tag{69}$$

If $D_1 A = 0$ is selected, $a = a(T_2)$, $\beta = \beta\,(T_2)$ and $\omega_1 = 0$. Since ω_1 is not complex, this choice is admissible. The solution at order ε is

$$u_1 = B e^{iT_0} + cc + \frac{\alpha}{3\omega^2}(A^2 e^{2iT_0} + cc) - \frac{\alpha}{\omega^2} 2A\bar{A} \qquad B = \frac{1}{2} i b e^{i\gamma} \tag{70}$$

In terms of real amplitude and phase, the solution is

$$u_1 = b \cos(T_0 + \gamma) + \frac{\alpha}{6\omega^2} a^2 \cos(2T_0 + 2\beta) - \frac{\alpha}{2\omega^2} a^2 \tag{71}$$

Applying the initial conditions yields

$$\gamma(0)=0 \quad b(0)=\frac{\alpha}{3\omega^2}a_0^{\,2} \tag{72}$$

Equations (67) and (70) are inserted into (66) and secular terms are eliminated

$$-2i\omega^2 D_1 B - 2i\omega^2 D_2 A + \omega_2 A + (4-\frac{2}{3})\frac{\alpha^2}{\omega^2}A^2\bar{A}=0 \tag{73}$$

$D_1B=0$ can be assumed. If $D_2A=0$ is selected, ω_2 comes out to be real and this is again an admissible choice. After algebraic calculations, Equation (73) yields

$$a=a_0, \quad b=\frac{\alpha}{3\omega^2}a_0^{\,2}, \quad \gamma=\beta=0, \quad \omega_2=-\frac{5}{6}\frac{\alpha^2}{\omega^2}a^2 \tag{74}$$

The frequency is

$$\omega=\sqrt{\frac{1}{2}\omega_0^{\,2}+\frac{1}{2}\sqrt{\omega_0^{\,4}-\frac{10}{3}\varepsilon^2\alpha^2 a_0^{\,2}}} \tag{75}$$

The final solution in terms of this frequency is

$$u=a_0\cos(\omega t)+\varepsilon\left(\frac{\alpha}{3\omega^2}a_0^{\,2}\cos(\omega t)+\frac{\alpha a_0^{\,2}}{6\omega^2}(\cos(2\omega t)-3)\right)+O(\varepsilon^2) \tag{76}$$

For valid solutions, the perturbation criteria is

$$\frac{\varepsilon\alpha a_0}{3\omega^2} << 1 \tag{77}$$

3.3. Comparisons with the Numerical Solutions

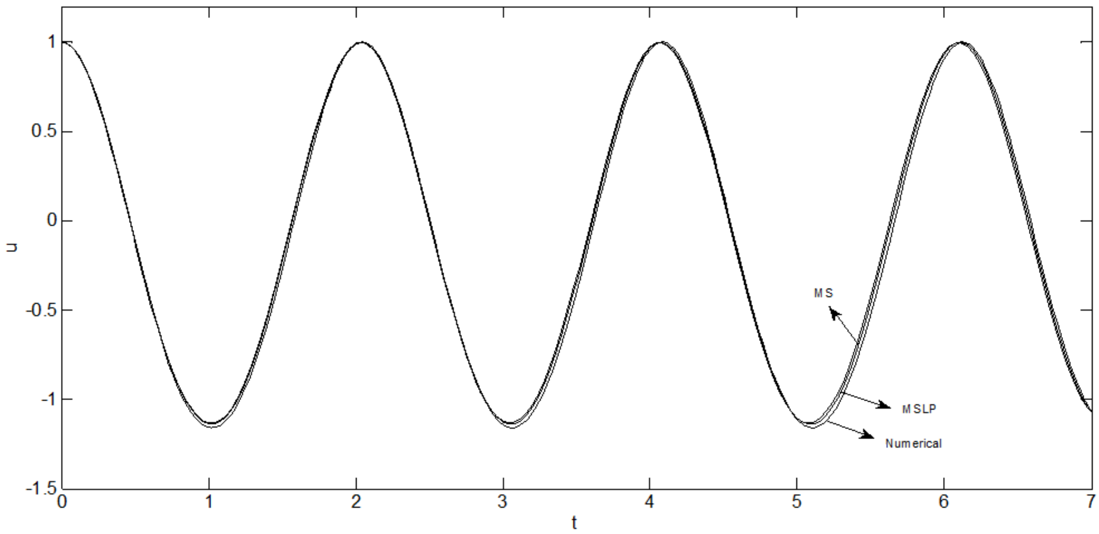

Figure 4. Comparison of numerical solutions and approximate analytical solutions (MS and MSLP) for $\varepsilon=2,\ a_0=1,\ \omega_0=\pi$

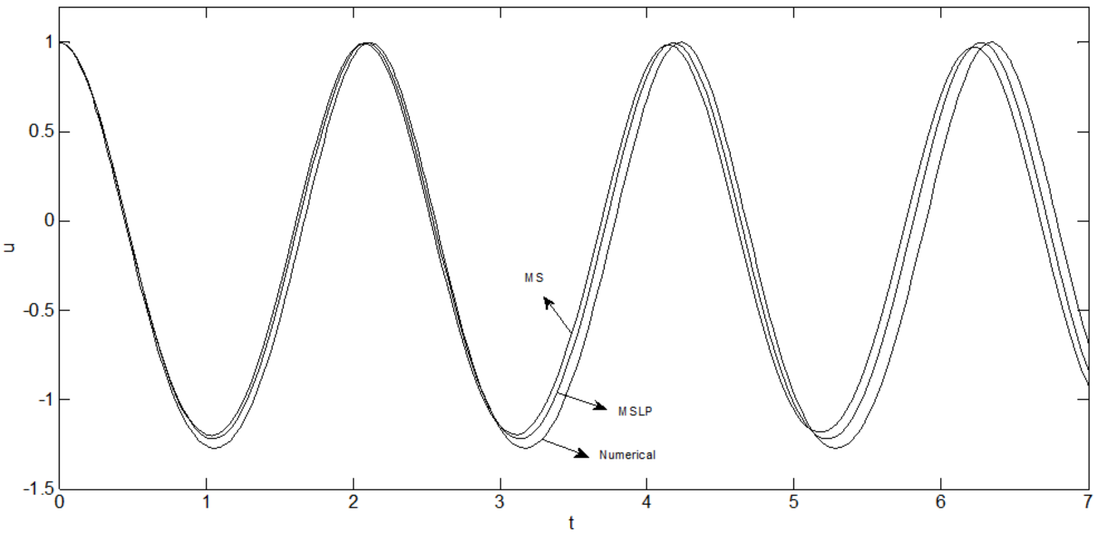

Figure 5. Comparison of numerical solutions and approximate analytical solutions (MS and MSLP) for $\varepsilon = 3,\ a_0 = 1,\ \omega_0 = \pi$

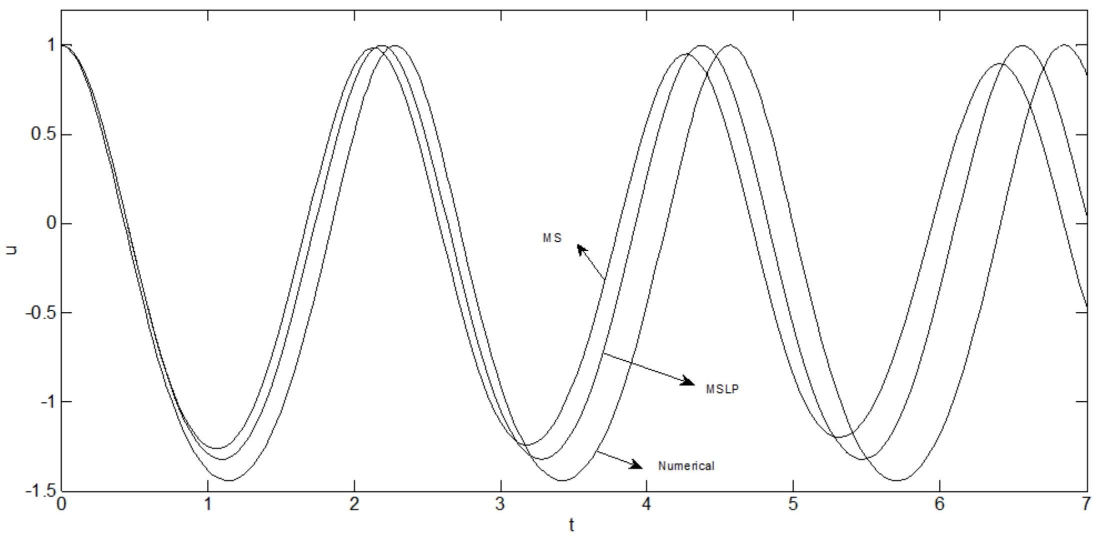

Figure 6. Comparison of numerical solutions and approximate analytical solutions (MS and MSLP) for $\varepsilon = 4,\ a_0 = 1,\ \omega_0 = \pi$

In this section, the approximate solutions are contrasted with the numerical solutions for the quadratic nonlinear model considered. In Figure 4, results are compared for ε=2. The agreement between MSLP and numerical solutions is better than MS solution and the amplitude values of the MS solution yield higher errors. ε=3 is selected in Figure 5 and ε=4 is selected in Figure 6. In Figures 5 and 6, the same trend is observed with more amplification.

4. QUADRATIC NONLINERITY WITH DAMPING

A damping term is added to the previous model

$$\ddot{u}+\omega_0^2 u+2\varepsilon\mu\dot{u}+\varepsilon\alpha u^2=0 \tag{78}$$

with initial conditions

$$u(0)=a_0 \qquad \dot{u}(0)=0 \tag{79}$$

4.1. Multiple Scales Method (MS)

First, the problem is solved with the classical method. Solutions are assumed to be of the form;

$$u=u_0(T_0,T_1,T_2)+\varepsilon u_1(T_0,T_1,T_2)+\varepsilon^2 u_2(T_0,T_1,T_2) \tag{80}$$

where $T_0=t$ is the usual fast time scale, $T_1=\varepsilon t$ and $T_2=\varepsilon^2 t$ are the slow time scales. Time derivatives are defined as

$$\frac{d}{dt}=D_0+\varepsilon D_1+\varepsilon^2 D_2, \qquad \frac{d^2}{dt^2}=D_0^2+2\varepsilon D_0 D_1+\varepsilon^2(D_1^2+2D_0D_2) \tag{81}$$

where $D_n=\partial/\partial T_n$.

If (80) and (81) are substituted into the original equation, the following equations are obtained at each order of ε;

$$O(1) \qquad D_0^2u_0+\omega_0^2u_0=0 \qquad u_0(0)=a_0 \quad D_0u_0(0)=0 \tag{82}$$

$$O(\varepsilon) \qquad D_0^2u_1+\omega_0^2u_1=-2D_0D_1u_0-2\mu D_0u_0-\alpha u_0^2 \qquad u_1(0)=0 \quad (D_0u_1+D_1u_0)(0)=0 \tag{83}$$

$$O(\varepsilon^2) \qquad D_0^2u_2+\omega_0^2u_2=-2D_0D_1u_1-(D_1^2+2D_0D_2)u_0-2\mu(D_0u_1+D_1u_0)-2\alpha u_0u_1 \tag{84}$$

At order 1, the solution may be expressed as

$$u_0=Ae^{i\omega_0T_0}+cc \tag{85}$$

where cc stands for the complex conjugates of the preceding terms and

$$A=\frac{1}{2}ae^{i\beta} \tag{86}$$

The first order solution is obtained in terms of real amplitude and phase

$$u_0=a(T_1,T_2)\cos(\omega_0T_0+\beta(T_1,T_2)) \tag{87}$$

Applying the initial conditions yields

$$\beta(0)=0 \quad a(0)=a_0 \tag{88}$$

Equation (85) is substituted into (83) and secular terms are eliminated

$$D_1A=-\mu A \Rightarrow a=a(T_2)e^{-\mu T_1}, \quad \beta=\beta(T_2) \tag{89}$$

The solution at order ε is

$$u_1=Be^{i\omega_0T_0}+cc+\frac{\alpha}{3\omega_0^2}(A^2e^{2i\omega_0T_0}+cc)-\frac{\alpha}{\omega_0^2}2A\bar{A} \tag{90}$$

This solution may be represented in terms of real amplitude and phase

$$u_1=b\cos(\omega_0T_0+\gamma)+\frac{\alpha}{6\omega_0^2}a^2\cos(2\omega_0T_0+2\beta)-\frac{\alpha}{2\omega_0^2}a^2 \tag{91}$$

where

$$B=\frac{1}{2}ibe^{i\gamma} \tag{92}$$

Applying the initial conditions yields

$$\gamma(0)=Arc\tan\left(-\frac{3\mu\omega_0}{\alpha a_0}\right), \quad b(0)=\frac{a_0}{\omega_0}\sqrt{\frac{\alpha^2a_0^2}{9\omega_0^2}+\mu^2} \tag{93}$$

Equation (85) and (90) are inserted into (84) and secular terms are eliminated,

$$-2i\omega_0 D_1 B - D_1^2 A - 2i\omega_0 D_2 A - 2\mu i\omega_0 B - 2\mu D_1 A + \frac{\alpha^2}{\omega_0^2}(4-\frac{2}{3})A^2\bar{A} = 0 \quad (94)$$

If (82), (92), (93) are inserted into (94), one finally has

$$a = a_0 e^{-\mu T_1}, \quad b = b_0 e^{-\mu T_1}, \quad \gamma = \gamma_0, \quad \beta = -(\frac{\mu^2}{2\omega_0} + \frac{5\alpha^2}{12\omega_0^3}a^2)T_2 \quad (95)$$

The final solution is

$$u = a_0 e^{-\varepsilon\mu t}\cos((\omega_0 - \varepsilon^2(\frac{\mu^2}{2\omega_0} + \frac{5}{12}\frac{\alpha^2}{\omega_0^3}a_0^2 e^{-2\varepsilon\mu t}))t)$$
$$+\varepsilon(\frac{a_0}{\omega_0}\sqrt{\frac{\alpha^2 a_0^2}{9\omega_0^2} + \mu^2}\, e^{-\varepsilon\mu t}\cos(\omega_0 t + Arc\tan(\frac{-3\mu\omega_0}{\alpha a_0}))$$
$$+\frac{\alpha}{6\omega_0^2}a_0^2 e^{-2\varepsilon\mu t}(\cos(2(\omega_0 - \varepsilon^2(\frac{\mu^2}{2\omega_0} + \frac{5}{12}\frac{\alpha^2}{\omega_0^3}a_0^2 e^{-2\varepsilon\mu t}))t) - 3) + O(\varepsilon^2) \quad (96)$$

4.2. Multiple Scales Lindstedt-Poincare Method (MSLP)

The time transformation $\tau = \omega t$ is applied to the model

$$\omega^2 u'' + \omega_0^2 u + 2\varepsilon\mu\omega u' + \varepsilon\alpha u^2 = 0 \quad (97)$$

where prime denotes derivative with respect to the new variable τ . The time scales in this method are slightly different from classical multiple scales

$$T_0 = \tau, \ T_1 = \varepsilon\tau, \ T_2 = \varepsilon^2\tau \quad (98)$$

Expressing the time derivatives

$$\frac{d}{dt} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2 \qquad \frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2(D_1^2 + 2D_0 D_2) \quad (99)$$

with $D_n = \partial / \partial T_n$ and substituting the expansions

$$u = u_0(T_0,T_1,T_2) + \varepsilon u_1(T_0,T_1,T_2) + \varepsilon^2 u_2(T_0,T_1,T_2) \quad (100)$$

$$\omega_0^2 = \omega^2 - \varepsilon\omega_1 - \varepsilon^2\omega_2 \quad (101)$$

into the original equation, the following equations are obtained at each order of approximation;

$$O(1) \quad \omega^2 D_0^2 u_0 + \omega^2 u_0 = 0, \quad u_0(0) = a_0 \quad D_0 u_0(0) = 0 \quad (102)$$

$$O(\varepsilon) \quad \omega^2 D_0^2 u_1 + \omega^2 u_1 = -2\omega^2 D_0 D_1 u_0 + \omega_1 u_0 - \alpha u_0^2 - 2\mu\omega D_0 u_0 \qquad u_1(0) = 0 \quad (D_0 u_1 + D_1 u_0)(0) = 0 \quad (103)$$

$$O(\varepsilon^2) \quad \omega^2 D_0^2 u_2 + \omega^2 u_2 = -2\omega^2 D_0 D_1 u_1 - \omega^2(D_1^2 + 2D_0 D_2)u_0 + \omega_1 u_1 + \omega_2 u_0 - 2\alpha u_0 u_1 - 2\mu\omega(D_0 u_1 + D_1 u_0) \quad (104)$$

The first order solution is

$$u_0 = Ae^{iT_0} + cc = a\cos(T_0 + \beta) \quad (105)$$

Applying the initial conditions yields

$$\beta(0) = 0, \ a(0) = a_0 \quad (106)$$

Equation (105) is substituted into (103) and secular terms are eliminated

$$-2i\omega^2 D_1 A + \omega_1 A - 2\mu\omega i A = 0 \quad (107)$$

If $D_1 A = -\dfrac{\mu}{\omega}A$ is selected, $a = a(T_2)e^{-\frac{\mu}{\omega}T_1}$, β= β ($T_2$) and ω_1=0. Since ω_1 is not complex, this choice is admissible. The solution at order ε is

$$u_1 = Be^{iT_0} + cc + \frac{\alpha}{3\omega^2}(A^2 e^{2iT_0} + cc) - \frac{\alpha}{\omega^2}2A\bar{A} \qquad B = \frac{1}{2}ibe^{i\gamma} \quad (108)$$

In terms of real amplitude and phase, the solution is

$$u_1 = b\cos(T_0+\gamma) + \frac{\alpha}{6\omega^2}a^2\cos(2T_0+2\beta) - \frac{\alpha}{2\omega^2}a^2 \tag{109}$$

Applying the initial conditions yields

$$\gamma(0) = Arc\tan(-\frac{3\mu\omega}{\alpha a_0}),\ b(0) = \sqrt{\frac{\alpha^2}{9\omega^2}a_0^{\ 2} + \mu^2}\ \frac{a_0}{\omega} \tag{110}$$

Equations (105) and (108) are inserted into (104) and secular terms are eliminated

$$-2i\omega^2 D_1 B - 2i\omega^2 D_2 A + \omega_2 A + (\frac{10}{3})\frac{\alpha^2}{\omega^2}A^2\bar{A} = 0 \tag{111}$$

$D_1 B = -\frac{\mu}{\omega}B$ can be assumed. If $D_2A=0$ is selected, ω_2 comes out to be real and this is again an admissible choice. After algebraic calculations, Equation (111) yields

$$a = a_0 e^{-\varepsilon\mu t},\quad b = \sqrt{\frac{\alpha^2}{9\omega^2}a_0^{\ 2} + \mu^2}\ \frac{a_0}{\omega}e^{-\frac{\mu}{\omega}T_1},\quad \gamma = \gamma_0 = \mathrm{Arctan}(-\frac{3\mu\omega}{\alpha a_0}),\quad \beta = \beta_0 = 0,\quad \omega_2 = -\frac{5}{6}\frac{\alpha^2}{\omega^2}a^2 - \mu^2 \tag{112}$$

The frequency is

$$\omega = \sqrt{\frac{1}{2}(\omega_0^{\ 2} - \varepsilon^2\mu^2) + \frac{1}{2}\sqrt{(\omega_0^{\ 2} - \varepsilon^2\mu^2)^2 - \frac{10}{3}\varepsilon^2\alpha^2 a^2}} \tag{113}$$

The final solution in terms of this frequency is

$$u = a_0 e^{-\varepsilon\mu t}\cos(\omega t) + \varepsilon(\frac{a_0}{\omega}\sqrt{\frac{\alpha^2}{9\omega^2}a_0^{\ 2} + \mu^2}\,e^{-\varepsilon\mu t}\cos(\omega t + \tan^{-1}(-\frac{3\mu\omega}{\alpha a_0})) \tag{114}$$

$$+\frac{\alpha a_0^{\ 2}}{6\omega^2}e^{-2\varepsilon\mu t}(\cos(2\omega t) - 3)) + O(\varepsilon^2)$$

For valid solutions, the perturbation criteria is

$$\frac{\varepsilon\alpha a}{6\omega^2} << 1 \tag{115}$$

4.3. Comparisons with the Numerical Solutions

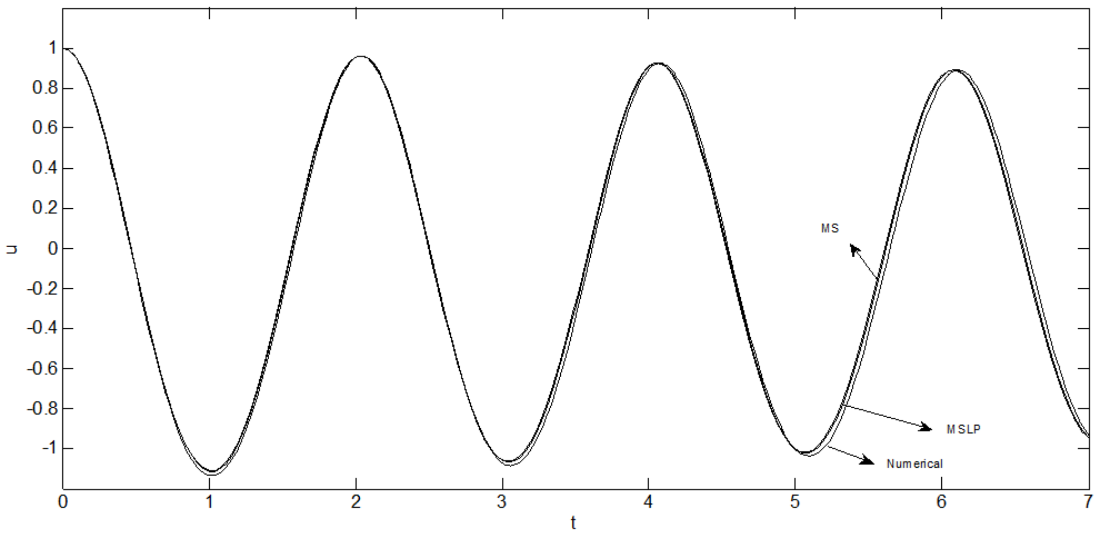

Figure 7. Comparison of numerical solutions and approximate analytical solutions (MS and MSLP) for $\varepsilon = 2,\ a_0 = 1,\ \omega_0 = \pi,\ \mu = 0.01$

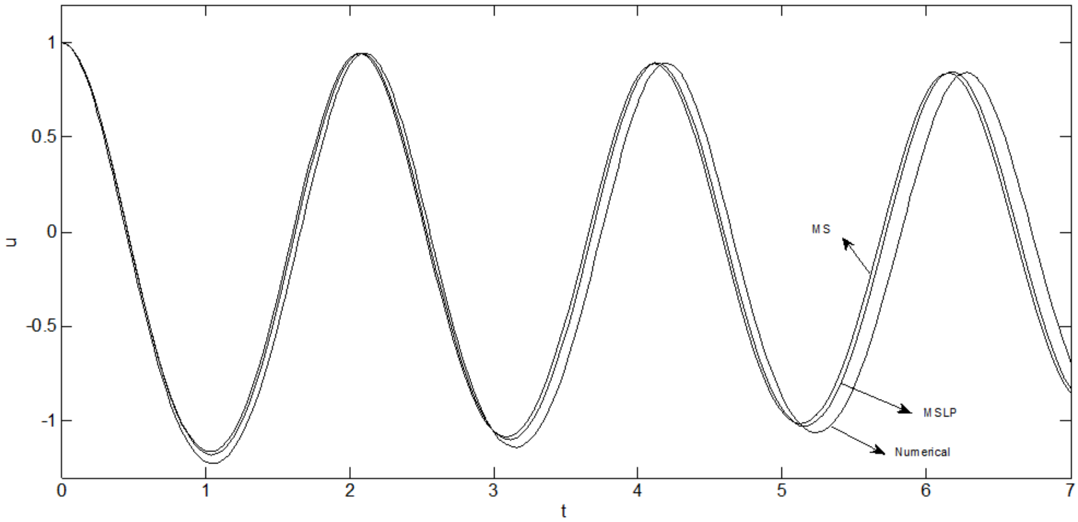

Figure 8. Comparison of numerical solutions and approximate analytical solutions (MS and MSLP) for $\varepsilon = 3,\ a_0 = 1,\ \omega_0 = \pi,\ \mu = 0.01$

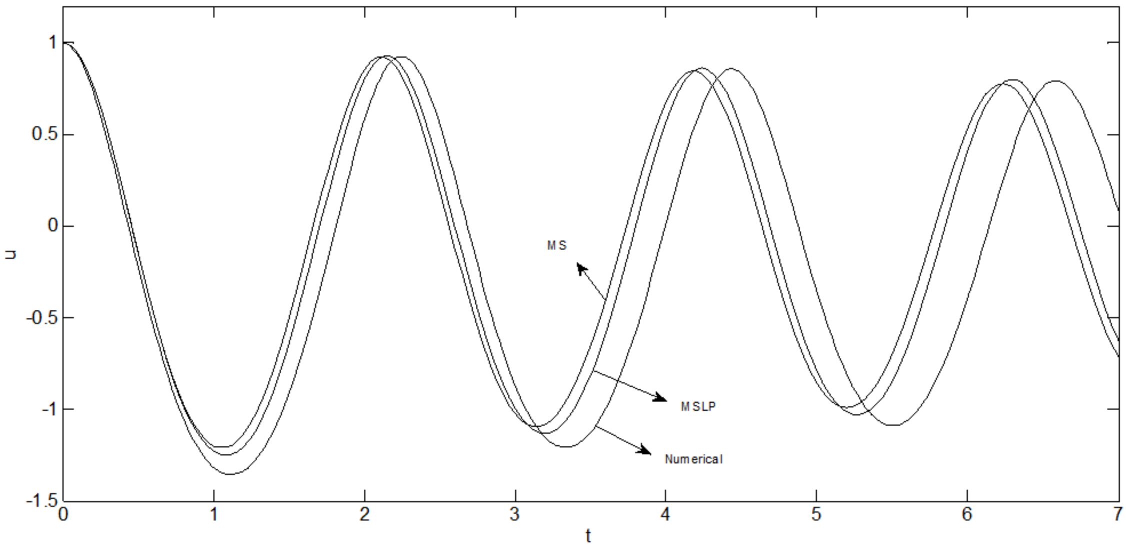

Figure 9. Comparison of numerical solutions and approximate analytical solutions (MS and MSLP) for $\varepsilon = 4,\ a_0 = 1,\ \omega_0 = \pi,\ \mu = 0.01$

The approximate solutions are contrasted with the numerical solutions for the quadratic nonlinear model with damping. In Figure 7, results are compared for ε=2. The agreement between MSLP and numerical solutions is better than MS solution and the amplitude values of the MS solution yield higher errors. ε=3 is selected in Figure 8 and ε=4 is selected in Figure 9. In summary, the MSLP solutions are better compared to the MS solutions.

5. CONCLUDING REMARKS

A recently developed perturbation technique namely the multiple scales Lindstedt Poincare method (MSLP) combining the multiple scales and Lindstedt Poincare methods is applied to three different mathematical models with quadratic nonlinearities.

In conclusion, the MSLP method yields closer solutions to the numerical ones compared to those of MS for strong quadratic nonlinearities.

6. REFERENCES

1. M. Pakdemirli, M.M.F. Karahan, H. Boyacı, A new perturbation algorithm with better convergence properties: Multiple Scales Lindstedt–Poincare Method. *Mathematical and Computational Applications* **14(1)**, 31-44, 2009.
2. M. Pakdemirli and M. M. F. Karahan, A new perturbation solution for systems with strong quadratic and cubic nonlinearities, *Mathematical Methods in the Applied Sciences* **3**, 704–712, 2010.
3. M. Pakdemirli, M. M. F. Karahan and H. Boyacı, Forced Vibrations of Strongly Nonlinear Systems with Multiple Scales Lindstedt Poicare Method, *Mathematical and Computational Appl*ications **16(4)**, 879-889, 2011.