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SHIFTED JACOBI COLLOCATION METHOD BASED ON OPERATIONAL MATRIX FOR SOLVING THE SYSTEMS OF FREDHOLM AND VOLTERRA INTEGRAL EQUATIONS

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Abstract- This paper aims to construct a general formulation for the shifted Jacobi operational matrices of integration and product. The main aim is to generalize the Jacobi integral and product operational matrices to the solving system of Fredholm and Volterra equations. These matrices together with the collocation method are applied to reduce the solution of these problems to the solution of a system of algebraic equations. The method is applied to solve system of linear and nonlinear Fredholm and Volterra equations. Illustrative examples are included to demonstrate the validity and efficiency of the presented method. Also, several theorems, which are related to the convergence of the proposed method, will be presented.

Key Words- Collocation Method, Shifted Jacobi Polynomials, System of Fredholm and Volterra Integral Equations; Convergence; Operational Integral and Product Matrices

1. INTRODUCTION

Finding the analytical solutions of functional equations has been devoted of attention of mathematicians’ interest in recent years. Several methods are proposed to achieve this purpose, such as [1-18]. Mathematical modeling for many problems in different fields, such as engineering, chemistry, physics and biology, leads to integral equations or system of integral equations. Several methods have been proposed to solve these problems. For example, Variational iteration method [19], differential transform method [20], Nystrom method [21], Haar functions method [22], Homotopy perturbation method [23], Chebyshev wavelet method [24] and many others. Between of present methods, spectral methods have been used to solve different functional equations, because of their high accuracy and easy applying. Specific types of spectral methods that more applicable and widely used, are the Galerkin, collocation, and tau methods [25-29]. Saadatmandi and Dehghan introduced shifted Legendre operational matrix for fractional differential equations [30], Doha derived a new explicit formula for shifted Chebyshev polynomials for fractional differential equations [31], Bhrawy used a quadrature shifted Legendre tau method for fractional differential equations [32]. Recently, Doha introduced shifted Chebyshev operational matrix and applied it with spectral methods for solving problems to initial and boundary conditions [33].

The importance of Sturm-Liouville problems for spectral methods lies in the fact that the spectral approximation of the solution of a functional equation is usually regarded as a finite expansion of eigenfunctions of a suitable Sturm-Liouville problem.

The Jacobi polynomials $P_i^{(\alpha,\beta)}(x)\,(i \geq 0,\ \ \alpha,\beta > -1)$ play important roles in mathematical analysis and its applications [34]. It is proven that Jacobi polynomials are precisely the only polynomials arising as eigenfunctions of a singular Sturm-Liouville problem [35 - 36]. This class of polynomials comprises all the polynomial solution to singular Sturm-Liouville problems on $[-1,1]$. Chebyshev, Legendre, and ultraspherical polynomials are particular cases of the Jacobi polynomials.

In this paper, the shifted Jacobi operational matrices of integration and product is introduced, which is based on Jacobi collocation method for solving numerically the systems of the linear and nonlinear Fredholm and Volterra integral equations on the interval $[0,1]$, to find the approximate solution $u_N(x)$. The each of equation of the systems resulted are collocated at $(N+1)$ nodes of the shifted Jacobi- Gauss interpolation on $(0,1)$. These equations generate $n(N+1)$ linear or nonlinear algebraic equations. The nonlinear systems resulted can be solved using Newton iterative method.

The remainder of this paper is organized as follows: The Jacobi polynomials and their integral and product operational matrices to integral equations are obtained in Section 2. Section 3 is devoted to applying the Jacobi operational matrices for solving system of integral equations. In Section 4, the proposed method is applied to several examples. A conclusion is presented in Section 5.

2. JACOBI POLYNOMIALS AND THEIR OPERATIONAL MATRICES

2.1. Properties of shifted Jacobi polynomials

The Jacobi polynomials, associated with the real parameters $(\alpha,\beta > -1)$ are a sequence of polynomials $P_i^{(\alpha,\beta)}(t), (i = 0,1,2,\ldots)$, each of degree i, are orthogonal with Jacobi weighted function, $w(x) = (1-x)^{\alpha}(x-1)^{\beta}$ over $I = [-1,1]$, and

$$\int_{-1}^{1} P_n^{(\alpha,\beta)}(t)\,P_m^{(\alpha,\beta)}(t)\,w(t)\,dt = h_n\,\delta_{nm},$$

where δ_{nm} is Kroneker function and $h_n = \dfrac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\,n!\,\Gamma(n+\alpha+\beta+1)}$.

These polynomials can be generated with the following recurrence formula;

$$P_i^{(\alpha,\beta)}(t) = \frac{(\alpha+\beta+2i-1)\ \ \alpha^2-\beta^2+t(\alpha+\beta+2i)(\alpha+\beta+2i-2)}{2i(\alpha+\beta+i)(\alpha+\beta+2i-2)}\,P_{i-1}^{(\alpha,\beta)}(t)$$
$$-\frac{(\alpha+i-1)(\beta+i-1)(\alpha+\beta+2i)}{i(\alpha+\beta+i)(\alpha+\beta+2i-2)}\,P_{i-2}^{(\alpha,\beta)}(t), \qquad i = 2,3,\ldots,$$

where,

$$P_0^{(\alpha,\beta)}(t) = 1, \ \text{ and } \ P_1^{(\alpha,\beta)}(t) = \frac{\alpha+\beta+2}{2}\,t + \frac{\alpha-\beta}{2}.$$

In order to use these polynomials on the interval $[0,1]$, shifted Jacobi polynomials are defined by introducing the change of variable $t = 2x - 1$. Let the shifted Jacobi polynomials $P_i^{(\alpha,\beta)}(2x-1)$ be denoted by $R_i^{(\alpha,\beta)}(x)$, then $R_i^{(\alpha,\beta)}(x)$ can be generated from following formula;

$$\begin{aligned} R_i^{(\alpha,\beta)}(x) &= \frac{(\alpha+\beta+2i-1)\ \alpha^2-\beta^2+(2x-1)(\alpha+\beta+2i)(\alpha+\beta+2i-2)}{2i(\alpha+\beta+i)(\alpha+\beta+2i-2)} R_{i-1}^{(\alpha,\beta)}(t) \\ &\quad - \frac{(\alpha+i-1)(\beta+i-1)(\alpha+\beta+2i)}{i(\alpha+\beta+i)(\alpha+\beta+2i-2)} R_{i-2}^{(\alpha,\beta)}(t), \qquad i = 2,3,\ldots, \end{aligned} \tag{1}$$

where,

$$R_0^{(\alpha,\beta)}(x) = 1, \text{ and } R_1^{(\alpha,\beta)}(x) = \frac{\alpha+\beta+2}{2}(2x-1) + \frac{\alpha-\beta}{2}.$$

Remark. Of this polynomials, the most commonly used are the shifted Gegenbauer polynomials, $C_{S,i}{}^{\alpha}(x)$, the shifted Chebyshev polynomials of the first kind, $T_{S,i}(x)$, the shifted Legendre polynomials, $P_{S,i}(x)$, the shifted Chebyshev polynomials of the second kind, $U_{S,i}(x)$. These orthogonal polynomials are related to the shifted Jacobi polynomials by the following relations.

$$C_{S,i}{}^{\alpha}(x) = \frac{i!\,\Gamma(\alpha+\frac{1}{2})}{\Gamma(i+\alpha+\frac{1}{2})} R_i^{(\alpha-\frac{1}{2},\beta-\frac{1}{2})}(x), \qquad T_{S,i}(x) = \frac{i!\,\Gamma(\frac{1}{2})}{\Gamma(i+\frac{1}{2})} R_i^{(-\frac{1}{2},-\frac{1}{2})}(x),$$

$$P_{S,i}(x) = R_i^{(0,0)}(x), \qquad U_{S,i}(x) = \frac{(i+1)!\,\Gamma(\frac{1}{2})}{\Gamma(i+\frac{3}{2})} R_i^{(\frac{1}{2},\frac{1}{2})}(x).$$

The analytic form of the shifted Jacobi polynomials, $R_i^{(\alpha,\beta)}(x)$, is given by

$$R_i^{(\alpha,\beta)}(x) = \sum_{k=0}^{i} \frac{(-1)^{i-k}\Gamma(i+\beta+1)\Gamma(i+k+\alpha+\beta+1)x^k}{\Gamma(k+\beta+1)\Gamma(i+\alpha+\beta+1)(i-k)!\,k!}, \tag{2}$$

Some properties of the shifted Jacobi polynomials are as follows,

$$(1)\ R_i^{(\alpha,\beta)}(0) = (-1)^i \begin{pmatrix} i+\alpha \\ i \end{pmatrix},$$

$$(2)\ R_i^{(\alpha,\beta)}(1) = (-1)^i \begin{pmatrix} i+\beta \\ i \end{pmatrix},$$

$$(3)\ \frac{d^i}{dx^i} R_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n+\alpha+\beta+i+1)}{\Gamma(n+\alpha+\beta+1)} R_{n-i}^{(\alpha+i,\beta+i)}(x).$$

The orthogonality condition of shifted Jacobi polynomials is

$$\int_0^1 R_j^{(\alpha,\beta)}(x)\, R_k^{(\alpha,\beta)}(x)\, W^{(\alpha,\beta)}(x)\, dx = \theta_k\, \delta_{jk}, \tag{3}$$

where $W^{(\alpha,\beta)}(x)$, shifted weighted function, is as follows,

$$W^{(\alpha,\beta)}(x) = x^{\beta}(1-x)^{\alpha}, \quad \text{and} \quad \theta_k = \frac{h_k}{2^{\alpha+\beta+1}}.$$

Lemma 1. The shifted Jacobi polynomial $R_n^{(\alpha,\beta)}(x)$ can be obtained in the form of:

$$R_n^{(\alpha,\beta)}(x) = \sum_{i=0}^{n} p_i^{(n)} x^i,$$

where $p_i^{(n)}$ are

$$p_i^{(n)} = (-1)^{n-i}\binom{n+\alpha+\beta+i}{i}\binom{n+\alpha}{n-i}.$$

Proof. $p_i^{(n)}$ can be obtained as,

$$p_i^{(n)} = \frac{1}{i!}\frac{d^i}{dx^i} R_n^{(\alpha,\beta)}(x)\,|_{x=0}\,.$$

Now, using properties (1) and (3) in above, the lemma can be proved. □

Lemma 2. For $m > 0$, one has

$$\int_0^1 x^m R_j^{(\alpha,\beta)}(x) W^{(\alpha,\beta)}(x)\,dx = \sum_{l=0}^{j} p_l^{(j)} B(m+l+\beta+1, \alpha+1),$$

where $B(s,t)$ is the Beta function and is defined as

$$B(s,t) = \int_0^1 v^{s-1}(1-v)^{t-1}\,dv = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}.$$

Proof. Using Lemma 1 and $W^{(\alpha,\beta)}(x) = x^{\beta}(1-x)^{\alpha}$ one has

$$\begin{aligned}\int_0^1 x^m R_j^{(\alpha,\beta)}(x) W^{(\alpha,\beta)}(x)\,dx &= \sum_{l=0}^{j} p_l^{(j)} \int_0^1 x^m x^l (1-x)^{\alpha} x^{\beta}\,dx \\ &= \sum_{l=0}^{j} p_l^{(j)} \int_0^1 x^{m+l+\beta}\,(1-x)^{\alpha}\,dx \\ &= \sum_{l=0}^{j} p_l^{(j)} B(m+l+\beta+1, \alpha+1). \;\square\end{aligned}$$

2.2. The approximation of functions

Let $\Omega = (0,1)$, and for $r \in \mathbb{N}$ ($\mathbb{N}$ is the set of all non-negative integers), the weighted Sobolov space $H^r_{W^{(\alpha,\beta)}}(\Omega)$ is defined in the usual way and is denoted inner product, semi-norm and norm by $(u,v)_{r,W^{(\alpha,\beta)}}$, $|v|_{r,W^{(\alpha,\beta)}}$ and $\|v\|_{r,W^{(\alpha,\beta)}}$, respectively. In particular, $L^2_{W^{(\alpha,\beta)}}(\Omega) = H^0_{W^{(\alpha,\beta)}}(\Omega)$, $(u,v)_{W^{(\alpha,\beta)}} = (u,v)_{0,W^{(\alpha,\beta)}}$ and $\|v\|_{W^{(\alpha,\beta)}} = \|v\|_{0,W^{(\alpha,\beta)}}$,

$$H^r_{W^{(\alpha,\beta)}}(\Omega) = \{ f \mid f \text{ is integrable and } \|v\|_{r,W^{(\alpha,\beta)}} < \infty \},$$

$$\|v\|^2_{r,W^{(\alpha,\beta)}} = \sum_{k=0}^{r} \|\partial^k_x v\|^2_{W^{(\alpha+k,\beta+k)}},$$

$$|v|_{r,W^{(\alpha,\beta)}} = \|\partial^r_x\|_{W^{(\alpha,\beta)}}.$$

A function $u(x) \in H^r_{W^{(\alpha,\beta)}}(\Omega)$ can be expanded in $\mathbf{P}^{(N,\alpha,\beta)} = \operatorname{span}\{R_0^{(\alpha,\beta)}(x), R_1^{(\alpha,\beta)}(x), \ldots, R_N^{(\alpha,\beta)}(x)\}$ as the below formula,

$$u(x) = \sum_{j=0}^{\infty} c_j R_j^{(\alpha,\beta)}(x), \tag{4}$$

where the coefficients c_j are given by

$$c_j = \frac{1}{\theta_j} \int_0^1 R_j^{(\alpha,\beta)}(x)\, u(x)\, W^{(\alpha,\beta)}(x)\, dx, \quad j = 0,1,2,\ldots\,. \tag{5}$$

By noting in practice, only the first $(N+1)$ terms shifted polynomials are considered, then one has

$$u(x) \simeq u_N(x) = \sum_{j=0}^{N} c_j R_j^{(\alpha,\beta)}(x) = \Phi^T(x) C, \tag{6}$$

where

$$C = [c_0, c_1, \ldots, c_N]^T, \qquad \Phi(x) = [R_0^{(\alpha,\beta)}(x), R_1^{(\alpha,\beta)}(x), \ldots, R_N^{(\alpha,\beta)}(x)]^T. \tag{7}$$

Since $\mathbf{P}^{(N,\alpha,\beta)}$ is a finite dimensional vector space, $u(x)$ has a unique best approximation from $\mathbf{P}^{(N,\alpha,\beta)}$, say $u_N(x) \in \mathbf{P}^{(N,\alpha,\beta)}$, that is

$$\forall y \in \mathbf{P}^{(N,\alpha,\beta)}, \qquad \|u(x) - u_N(x)\|_{W^{(\alpha,\beta)}} \le \|u(x) - y\|_{W^{(\alpha,\beta)}}.$$

In [37] is shown that for any $u(x) \in H^r_{W^{(\alpha,\beta)}}(\Omega)$, $r \in \mathbb{N}$ and $0 \le \mu \le r$, a positive constant C independent of any function, N, α and β exist that

$$\|u(x) - u_N(x)\|_{\mu,W^{(\alpha,\beta)}} \le c\,(N(N+\alpha+\beta))^{\frac{\mu-r}{2}} |u|_{r,W^{(\alpha,\beta)}}. \tag{8}$$

Let $u(x)$ is $N+1$ times continuously differentiable. The following Theorem can present an upper bound for estimating the error.

Theorem 1. Let $u(x) : [x_0, 1] \to \mathbb{R}$ is $N+1$ times continuously differentiable for $x_0 > 0$, and $\mathbf{P}^{(N,\alpha,\beta)} = \operatorname{span}\{R_0^{(\alpha,\beta)}(x), R_1^{(\alpha,\beta)}(x), \ldots, R_N^{(\alpha,\beta)}(x)\}$. If $u_N(x) = \Phi^T(x) A$ is the best approximation to $u(x)$ from $\mathbf{P}^{(N,\alpha,\beta)}$ then the error bound is presented as follows:

$$\left\|u(x)-u_N(x)\right\|_{W^{(\alpha,\beta)}} \le \frac{M\,S^{N+1}}{(N+1)!}\sqrt{B(\alpha+1,\beta+1)},$$

where $M=\max_{x\in[x_0,1]}\left|u^{(N+1)}(x)\right|$ and $S=\max\{x_0,1-x_0\}$.

Proof. Let consider the Taylor expansion

$y(x)=u(x_0)+u'(x_0)(x-x_0)+...+u^{(N)}(x_0)\dfrac{(x-x_0)^N}{N!}$. Therefore

$$\left|u(x)-u_N(x)\right| \le \left|u^{(N+1)}(\xi)\right|\frac{\left|x-x_0\right|^{N+1}}{(N+1)!}, \qquad \xi\in(x_0,1),$$

Since $\Phi^T(x)A$ is the best approximation to $u(x)$ from $\mathbf{P}^{(N,\alpha,\beta)}$, and $y(x)\in\mathbf{P}^{(N,\alpha,\beta)}$ one has

$$\left\|u(x)-u_N(x)\right\|^2_{W^{(\alpha,\beta)}} \le \left\|u(x)-y(x)\right\|^2_{W^{(\alpha,\beta)}} \le \frac{M^2}{((N+1)!)^2}\int_0^1 (x-x_0)^{2(N+1)}W^{(\alpha,\beta)}(x)\,dx.$$

Since $W^{(\alpha,\beta)}(x)$ is always positive in $(0,1)$, by choosing $S=\max\{x_0,1-x_0\}$ one has

$$\left\|u(x)-u_N(x)\right\|^2_{W^{(\alpha,\beta)}} \le \frac{M^2S^{2(N+1)}}{((N+1)!)^2}\int_0^1 W^{(\alpha,\beta)}(x)\,dx = \frac{M^2S^{2(N+1)}}{((N+1)!)^2}B(\alpha+1,\beta+1). \quad \square$$

This error bound shows approximation of polynomials converges to $u(x)$ as $N\to\infty$.

2.3. The Jacobi integral operational matrix

In this subsection, Jacobi operational matrix of the integration is derived. Let

$$\int_0^x \Phi(t)\,dt \simeq P\,\Phi(x), \tag{9}$$

where matrix P is called the Jacobi operational matrix of the integration.

Theorem 3. Let P is $(N+1)\times(N+1)$ operational matrix of integral. Then the elements of this matrix are obtained as

$$P_{ij}=\frac{1}{\theta_j}\sum_{m=0}^{i}\sum_{n=0}^{j}\frac{1}{m+1}p_m^{(i)}p_n^{(j)}B(m+n+\beta+2,\alpha+1), \quad i,j=0,1,\dots N.$$

Proof. Using Eq. (9) and orthogonality property of Jacobi polynomials one has,

$$P=\left(\int_0^x \Phi(t)\,dt,\Phi^T(x)\right)_{W^{(\alpha,\beta)}}\Delta^{-1},$$

where $\left(\int_0^x \Phi(t)\,dt,\Phi^T(x)\right)_{W^{(\alpha,\beta)}}$ and Δ^{-1} are two $(N+1)\times(N+1)$ matrices defined as follows,

$$(\int_0^x \Phi(t)\,dt, \Phi^T(x))_{W^{(\alpha,\beta)}} = \left\{(\int_0^x R_i^{(\alpha,\beta)}(t)\,dt, R_j^{(\alpha,\beta)}(x))_{W^{(\alpha,\beta)}}\right\}_{i,j=0}^{N},$$

$$\Delta^{-1} = \mathrm{diag}\left(\frac{1}{\theta_j}\right)_{j=0}^{N}.$$

Set

$$\rho_{ij} = (\int_0^x R_i^{(\alpha,\beta)}(t)\,dt, R_j^{(\alpha,\beta)}(x))_{W^{(\alpha,\beta)}} = \int_0^1 \left\{\int_0^x R_i^{(\alpha,\beta)}(t)\,dt\right\} R_j^{(\alpha,\beta)}(x)\, W^{(\alpha,\beta)}(x)dx.$$

$\int_0^x R_i^{(\alpha,\beta)}(t)\,dt$ and $R_j^{(\alpha,\beta)}(x)$ by using Lemma 1 can be obtained as

$$\int_0^x R_i^{(\alpha,\beta)}(t)\,dt = \sum_{m=0}^{i} p_m^{(i)} \frac{x^{m+1}}{m+1},$$

$$R_j^{(\alpha,\beta)}(x) = \sum_{n=0}^{j} p_n^{(j)}\, x^n, \qquad i,j = 0,1,\dots,N.$$

Therefore, ρ_{ij} by using Lemma 2 can be obtained as follows,

$$\begin{aligned}\rho_{ij} &= \sum_{m=0}^{i}\sum_{n=0}^{j} \frac{1}{m+1}\, p_m^{(i)}\, p_n^{(j)} \int_0^1 x^{m+1}\, x^n\, (1-x)^\alpha\, x^\beta\, dx \\ &= \sum_{m=0}^{i}\sum_{n=0}^{j} \frac{1}{m+1}\, p_m^{(i)}\, p_n^{(j)}\, B(m+n+\beta+2, \alpha+1).\end{aligned}$$

So, the elements of matrix P is obtained as

$$P_{ij} = \frac{1}{\theta_j} \sum_{m=0}^{i}\sum_{n=0}^{j} \frac{1}{m+1}\, p_m^{(i)} p_n^{(j)} B(m+n+\beta+2,\alpha+1), \quad i,j = 0,1,\dots N. \quad \square$$

Now the following theorem can present an upper bound for estimating the error of integral operator. The error vector E is defined as,

$$E = \int_0^x \Phi(t)\,dt - P\Phi(x) = [E_0, E_1, \dots, E_N],$$

where

$$E_k = \int_0^x R_k^{(\alpha,\beta)}(t)\,dt - \sum_{j=0}^{N} P_{kj} R_j^{(\alpha,\beta)}(x), \quad k = 0,1,\dots,N.$$

Theorem 4. If $E_k = \int_0^x R_k^{(\alpha,\beta)}(t)\,dt - \sum_{j=0}^{N} P_{kj} R_j^{(\alpha,\beta)}(x) \in H^r_{W^{(\alpha,\beta)}}(\Omega)$, then an error bound of integral operator of vector Φ can be expressed by

$$\left\|E_k\right\|_{\mu,W^{(\alpha,\beta)}} \le c^2 (N(N+\alpha+\beta))^{\mu-r} \sum_{i=0}^{k}\sum_{j=0}^{k} \rho_i^{(k)}\, \rho_j^{(k)}\, B(i+j+\beta-r+3, \alpha+r+1).$$

Proof. By using inequality (8), Lemma 1, and setting $u(x)=\int_0^x R_k^{(\alpha,\beta)}(t)\,dt$ one has

$$
\begin{aligned}
\left|u\right|_{r,W^{(\alpha,\beta)}} &= \left\| D^r \int_0^x R_k^{(\alpha,\beta)}(t)\,dt \right\|^2_{W^{(\alpha+r,\beta+r)}} \\
&= \left\| D^r \sum_{i=0}^{k} \frac{1}{i+1}\, p_i^{(k)}\, x^{i+1} \right\|^2_{W^{(\alpha+r,\beta+r)}} \\
&= \left\| \sum_{i=0}^{k} \frac{i!}{\Gamma(i-r+2)}\, p_i^{(k)}\, x^{i-r+1} \right\|^2_{W^{(\alpha+r,\beta+r)}} \\
&= \int_0^1 (\sum_{i=0}^{k} \rho_i^{(k)}\, x^{i-r+1})(\sum_{i=0}^{k} \rho_j^{(k)}\, x^{j-r+1}) W^{(\alpha+r,\beta+r)}(x)\,dx \\
&= \sum_{i=0}^{k}\sum_{j=0}^{k} \rho_i^{(k)}\rho_j^{(k)} \int_0^1 x^{i+j+\beta-r+2}\,(1-x)^{\alpha+r}\,dx \\
&= \sum_{i=0}^{k}\sum_{j=0}^{k} \rho_i^{(k)}\rho_j^{(k)}\, B(i+j+\beta-r+3, \alpha+r+1),
\end{aligned}
$$

where $\rho_i^{(k)} = \dfrac{i!}{\Gamma(i-r+2)}\, p_i^{(k)}$ and the theorem can be proved. □

2.4 The product operational matrix

The following property of the product of two Jacobi function vector will also be applied to solve the Volterra integral equations.

$$\Phi(x)\,\Phi^T(x)\,Y \simeq \tilde{Y}\,\Phi(x) \tag{10}$$

where $\tilde{Y}$ is a $(N+1)\times(N+1)$ product operational matrix and it`s elements are determined in terms of the vector $Y$ `s elements. Using Eq.(11) and by the orthogonality property of Jacobi polynomials the elements $\tilde{Y}_{ij}$ can be calculated as follows,

$$
\begin{aligned}
\tilde{Y}_{ij} &= \frac{1}{\theta_j} \sum_{k=0}^{N} Y_k \int_0^1 (\Phi(x))_i\,(\Phi(x))_j\,(\Phi(x))_k\, W^{(\alpha,\beta)}(x)\,dx \\
&= \frac{1}{\theta_j} \sum_{k=0}^{N} Y_k \int_0^1 R_i^{(\alpha,\beta)}(x)\, R_j^{(\alpha,\beta)}(x)\, R_k^{(\alpha,\beta)}(x)\, W^{(\alpha,\beta)}(x)\,dx \\
&= \frac{1}{\theta_j} Y_k\, h_{ijk},
\end{aligned}
$$

where $h_{ijk} = \int_0^1 R_i^{(\alpha,\beta)}(x)\, R_j^{(\alpha,\beta)}(x)\, R_k^{(\alpha,\beta)}(x)\, W^{(\alpha,\beta)}(x)\,dx.$ □

3. APPLICATIONS OF THE OPERATIONAL MATRICES OF INTEGRATION AND PRODUCT

In this section, the presented operational matrices are applied to solve the system of linear and nonlinear Fredholm and Volterra integral equations.

3.1. The system of Fredholm integral equations

A system of Fredholm integral equations can be presented as follows;

$$u_i(x) = f_i(x) + \sum_{j=1}^{m} \int_0^1 k_{ij}(x,t)\, G_{ij}(u_1(t), u_2(t), \ldots, u_n(t))\, dt, \quad 0 \le x \le 1, \quad i = 1,2,\ldots,n, \tag{11}$$

where $k_{ij}(x,t) \in L^2\ [0,1]\times[0,1]$, f_i are known functions, and G_{ij} are linear or nonlinear functions in terms of unknown functions $u_1(x), u_2(x), \ldots,$ and $u_n(x)$. To solve system (11), the functions $u_i(x)$, $G_{ij}(t)$ and $k_{ij}(x,t)$ can be approximated as follows,

$$\begin{gathered} u_i(x) \simeq \sum_{j=0}^{N} c_j^i R_j^{(\alpha,\beta)}(x) = \Phi^T(x) C_i, \;\; G_{ij}(t) \simeq \Phi^T(t) Y_{ij}, \;\; k_{ij}(x,t) \simeq \Phi^T(x) K_{ij} \Phi(t), \\ i = 1,2,\ldots,n, \quad j = 1,2,\ldots,m, \end{gathered} \tag{12}$$

where K_{ij} and Y_{ij} are $(N+1)\times(N+1)$ known matrices and $(N+1)\times 1$ unknown vectors respectively and

$$C_i = [c_0^i, c_1^i, \ldots, c_N^i]^T, \qquad \Phi(x) = [R_0^{(\alpha,\beta)}(x), R_1^{(\alpha,\beta)}(x), \ldots, R_N^{(\alpha,\beta)}(x)]^T.$$

With substituting approximations (13) in system (12) one has

$$\begin{aligned} \Phi^T(x) C_i &= f_i(x) + \sum_{j=1}^{m} \int_0^1 \Phi^T(x) K_{ij}\, \Phi(t)\, \Phi^T(t) Y_{ij}\, dt \\ &= f_i(x) + \sum_{j=1}^{m} \Phi^T(x) K_{ij} \left\{ \int_0^1 \Phi(t)\, \Phi^T(t) dt \right\} Y_{ij} \\ &= f_i(x) + \sum_{j=1}^{m} \Phi^T(x) K_{ij} D\, Y_{ij}, \quad i = 1,2,\ldots,n, \end{aligned} \tag{13}$$

where D is the following $(N+1)\times(N+1)$ known matrix,

$$D = \int_0^1 \Phi(t)\, \Phi^T(t) dt.$$

The system (13) have $n(N+1)$ unknown coefficients c_j^i. For collocating, $(N+1)$ roots of Jacobi polynomials $R_{(N+1)}^{(\alpha,\beta)}(x)$ are applied and the equations are collocated at them. Unknown coefficients are determined with solving the resulted system of linear or nonlinear algebraic equations.

3.2. System of Volterra integral equations

A system of Volterra integral equations of the first kind can be presented as follows,

$$f_i(x) = \sum_{j=1}^{m} \int_0^x k_{ij}(x,t)\, G_{ij}(u_1(t), u_2(t), \dots, u_n(t))\, dt, \quad 0 \le x \le 1, \quad i = 1,2,\dots,n. \tag{14}$$

By using the approximate relations (12) one has

$$\begin{aligned} f_i(x) &= \sum_{j=1}^{m} \int_0^x \Phi^T(x)\, K_{ij}\, \Phi(t)\, \Phi^T(t) Y_{ij}\, dt \\ &= \sum_{j=1}^{m} \Phi^T(x)\, K_{ij} \left\{ \int_0^x \Phi(t)\, \Phi^T(t)\, Y_{ij} dt \right\} \\ &= \sum_{j=1}^{m} \Phi^T(x)\, K_{ij} \tilde{Y}_{ij} \int_0^x \Phi(t)\, dt \\ &= \Phi^T(x) \left\{ \sum_{j=1}^{m} K_{ij} \tilde{Y}_{ij} \right\} P \Phi(x), \qquad i = 1,2,\dots,n, \end{aligned} \tag{15}$$

where $\tilde{Y}_{ij}$ and P are product and integral operational matrices, respectively.
Also, a system of Volterra integral equations of the second kind can be presented as

$$u_i(x) = f_i(x) + \sum_{j=1}^{m} \int_0^x k_{ij}(x,t)\, G_{ij}(u_1(t), u_2(t), \dots, u_n(t))\, dt, \quad 0 \le x \le 1, \quad i = 1,2,\dots,n. \tag{16}$$

In the same way, the system of following equations is resulted.

$$\Phi^T(x)\, C_i = f_i(x) + \Phi^T(x) \left\{ \sum_{j=1}^{m} K_{ij} \tilde{Y}_{ij} \right\} P \Phi(x), \qquad i = 1,2,\dots,n, \tag{17}$$

By using the first $(N+1)$ roots of Jacobi polynomials $R_{(N+1)}^{(\alpha,\beta)}(x)$ and collocated system (17), unknown coefficients c_j^i are determined.

4. ILLUSTRATIVE EXAMPLES

In this section, the presented method is applied to solve some examples. Comparison between the results of present method with the corresponding analytic solutions is given. For this purpose, the maximum of absolute error is computed.

Example 1. The following system of linear Volterra integral equations of the second kind is considered,

$$\begin{cases} (2x^2+3)u_1(x) = f_1(x) + \int_0^x (x^2 - 2t) u_1(t)\, dt + \int_0^x (t^2 - x) u_2(t)\, dt + \int_0^x 2t\, u_3(t)\, dt, \\ (1-3x^2)u_2(x) = f_2(x) + \int_0^x t(x+1) u_1(t)\, dt + \int_0^x tx(x^2+1) u_2(t)\, dt + \int_0^x (2t^2 + x^3)\, u_3(t)\, dt, \\ (3x^2+6)u_3(x) = f_3(x) + \int_0^x (t - x) u_1(t)\, dt + \int_0^x tx(t^2 - x^3) u_2(t)\, dt + \int_0^x (2tx + t^2)\, u_3(t)\, dt, \end{cases} \tag{18}$$

The exact solutions are $u_1(x) = 5x + 8$, $u_2(x) = 2x^2 - 5$ and $u_3(x) = x^2 + x + 1$. By the applying the technique described in pervious section with $N = 4$, solutions and kernels are approximated as:

$$u_1(x) \simeq \Phi^T(x)C_1, \quad u_2(x) \simeq \Phi^T(x)C_2, \quad u_3(x) \simeq \Phi^T(x)C_3, \quad x^2 - 2t \simeq \Phi^T(x)K_1\Phi(t),$$
$$t^2 - x \simeq \Phi^T(x)K_2\Phi(t), \quad 2t \simeq \Phi^T(x)K_3\Phi(t), \quad t(x+1) \simeq \Phi^T(x)K_4\Phi(t),$$
$$tx(x^2+1) \simeq \Phi^T(x)K_5\Phi(t), \quad 2t^2 + x^3 \simeq \Phi^T(x)K_6\Phi(t), \quad t - x \simeq \Phi^T(x)K_7\Phi(t),$$
$$tx(t^2 - x^3) \simeq \Phi^T(x)K_8\Phi(t), \quad 2tx + t^2 \simeq \Phi^T(x)K_9\Phi(t).$$

The system (18) by using above equations is rewritten as,

$$\begin{cases} (2x^2+3)\Phi^T(x)C_1 - \Phi^T(x)\ \ K_1\tilde{C}_1 + K_2\tilde{C}_2 + K_3\tilde{C}_3\ \ P\Phi(x) \approx 0, \\ (1-3x^2)\Phi^T(x)C_2 - \Phi^T(x)\ \ K_4\tilde{C}_1 + K_5\tilde{C}_2 + K_6\tilde{C}_3\ \ P\Phi(x) \approx 0, \\ (3x^2+6)\Phi^T(x)C_3 - \Phi^T(x)\ \ K_7\tilde{C}_1 + K_8\tilde{C}_2 + K_9\tilde{C}_3\ \ P\Phi(x) \approx 0, \end{cases} \tag{19}$$

where $\tilde{C}_1$, $\tilde{C}_2$ and $\tilde{C}_3$ are the operational matrices of product corresponding to unknown vectors C_1, C_2 and C_3. Now using the roots of $R_5^{(\alpha,\beta)}(x)$ and collocating the system (19), reduces the problem to solve a system of algebraic equations. Unknown coefficients are obtained for some values of parameters α and β. Maximum absolute error for $N = 4$ and different values of α and β has been listed in Table 1.

Table 1. Maximum absolute error for $N = 4$ and different values of α and β for Example 1.

α	β	Error (u_1)	Error (u_2)	Error (u_3)
0	0	4.3318×10^{-19}	5.2481×10^{-19}	3.9353×10^{-19}
$1/2$	$1/2$	7.7181×10^{-19}	1.3351×10^{-18}	6.8203×10^{-19}
$1/2$	$-1/2$	1.1087×10^{-18}	2.0769×10^{-18}	2.2193×10^{-19}
$-1/2$	$1/2$	6.0000×10^{-19}	3.7000×10^{-18}	5.0000×10^{-19}
$1/10$	$1/10$	4.6320×10^{-19}	1.9372×10^{-18}	2.2075×10^{-19}
$-1/4$	$-1/4$	4.4035×10^{-19}	1.2295×10^{-18}	2.3332×10^{-19}
1	1	1.0000×10^{-19}	3.2610×10^{-19}	1.7659×10^{-19}
$1/2$	2	2.2000×10^{-18}	2.9000×10^{-18}	2.9096×10^{-19}
$-1/2$	$3/2$	3.1000×10^{-18}	4.0000×10^{-18}	6.0000×10^{-19}

Table 1 shows that a good approximation can be achieved for the exact solutions by using a few terms of shifted Jacobi polynomials for various values of parameters α and β.

Example 2. Consider the following system of linear Fredholm integral equation of second kind,

$$\begin{cases} 2u_1(x) + 3u_2(x) = f_1(x) + \int_0^1 (x+t)u_1(t)dt + \int_0^1 xt\, u_2(t)dt, \\ 3u_1(x) - 4u_2(x) = f_2(x) + \int_0^1 (2xt-1)u_1(t)dt + \int_0^1 (x-t)\, u_2(t)dt, \end{cases} \tag{20}$$

The exact solutions are $u_1(x) = e^x$ and $u_2(x) = \dfrac{1}{2-x}$. With $N = 15$, solutions and kernels are approximated as:

$$u_1(x) \simeq \Phi^T(x)C_1, \quad u_2(x) \simeq \Phi^T(x)C_2, \quad x+t \simeq \Phi^T(x)K_1\Phi(t), \quad xt \simeq \Phi^T(x)K_2\Phi(t),$$

$$2xt - 1 \simeq \Phi^T(x)K_3\Phi(t), \quad x - t \simeq \Phi^T(x)K_4\Phi(t), \quad D = \int_0^1 \Phi(t)\Phi^T(t)dt.$$

The system (20) by using above equations is rewritten as,

$$\begin{cases} 2\Phi^T(x)C_1 + 3\Phi^T(x)C_2 - \Phi^T(x)K_1DC_1 - \Phi^T(x)K_2DC_2 - f_1(x) \approx 0, \\ 3\Phi^T(x)C_1 - 4\Phi^T(x)C_2 - \Phi^T(x)K_3DC_1 - \Phi^T(x)K_4DC_2 - f_2(x) \approx 0. \end{cases} \tag{21}$$

Now using the roots of $R_{16}^{(\alpha,\beta)}(x)$ and collocating the system (21), reduces the problem to solve a system of linear algebraic equations. Solving the system (21), the unknown coefficients will be obtained. Table 2 displays the maximum absolute errors for various α and β with $N = 15$.

Table 2. Maximum absolute error for $N = 15$ and different values of α and β for Example 2.

α	β	$\text{Error}(u_1)$	$\text{Error}(u_2)$
0	0	5.0000×10^{-19}	3.9674×10^{-12}
$1/2$	$1/2$	6.9086×10^{-15}	9.3060×10^{-12}
$1/2$	$-1/2$	8.9852×10^{-15}	1.5421×10^{-11}
$-1/2$	$1/2$	3.9721×10^{-15}	1.0900×10^{-11}
$1/4$	$1/4$	2.4026×10^{-15}	6.2759×10^{-12}
$-1/10$	$-1/10$	5.1630×10^{-16}	3.2245×10^{-12}
1	1	2.5056×10^{-14}	1.7895×10^{-11}
2	2	1.1916×10^{-13}	4.7403×10^{-11}
$3/2$	4	1.1211×10^{-12}	4.6047×10^{-10}

Example 3. Consider the following system of linear Fredholm integral equation of second kind,

$$\begin{cases} u_1(x) = f_1(x) - \int_0^1 t\cos(x)\,u_1(t)\,dt - \int_0^1 x\,\mathrm{c}\sin(t)\,u_2(t)\,dt, \\ u_2(x) = f_2(x) - \int_0^1 e^{xt^2}\,u_1(t)\,dt - \int_0^1 (x+t)\,u_2(t)\,dt. \end{cases} \tag{22}$$

The exact solutions are $u_1(x) = x$ and $u_2(x) = \cos(x)$. With $N = 15$, solutions and kernels are approximated as:

$$u_1(x) \simeq \Phi^T(x)C_1, \quad u_2(x) \simeq \Phi^T(x)C_2, \quad t\cos(x) \simeq \Phi^T(x)K_1\Phi(t), \quad x\sin(t) \simeq \Phi^T(x)K_2\Phi(t),$$

$$\exp(xt^2) \simeq \Phi^T(x)K_3\Phi(t), \quad (x+t) \simeq \Phi^T(x)K_4\Phi(t), \quad D = \int_0^1 \Phi(t)\,\Phi^T(t)dt.$$

The system (22) by using above equations is rewritten as,

$$\begin{cases} \Phi^T(x)C_1 + \Phi^T(x)K_1DC_1 + \Phi^T(x)K_2DC_2 - f_1(x) \approx 0, \\ \Phi^T(x)C_2 + \Phi^T(x)K_3DC_1 + \Phi^T(x)K_4DC_2 - f_2(x) \approx 0. \end{cases} \tag{23}$$

Now using the roots of $R_{16}^{(\alpha,\beta)}(x)$ and collocating the system (23), reduces the problem to solve a system of linear algebraic equations. Solving the system (23), the unknown coefficients will be obtained. Table 3 displays the maximum absolute errors for various α and β with $N = 15$.

Table 3. Maximum absolute error for $N = 15$ and different values of α and β for Example 3.

α	β	$\mathrm{Error}(u_1)$	$\mathrm{Error}(u_2)$
0	0	1.6312×10^{-19}	2.9000×10^{-19}
$1/2$	$1/2$	1.9606×10^{-19}	7.5000×10^{-19}
$1/2$	$-1/2$	7.2797×10^{-20}	1.1800×10^{-18}
$-1/2$	$1/2$	1.5170×10^{-19}	4.1000×10^{-19}
$1/4$	$1/4$	1.0582×10^{-19}	4.0000×10^{-19}
$-1/10$	$-1/10$	7.9882×10^{-20}	2.2000×10^{-19}
1	1	4.7951×10^{-19}	1.1800×10^{-18}
2	2	9.2145×10^{-19}	5.8800×10^{-18}
$3/2$	4	2.9567×10^{-17}	7.6829×10^{-17}

Example 4. In this example, the following nonlinear Volterra integral equation of first kind is considered,

$$\begin{cases} \int\limits_0^x \left(u_1(t) + (x-t)u_1(t)u_2(t)\right) dt = f_1(x), \\ \int\limits_0^x \left(u_2(t) + (x-t)u_1(t)u_2(t)\right) dt = f_2(x). \end{cases} \tag{24}$$

The exact solutions are $u_1(x) = x + e^x$ and $u_1(x) = x - e^x$. With $N = 15$, solutions and kernels are approximated as:

$$u_1(x) \simeq \Phi^T(x)C_1, \quad u_2(x) \simeq \Phi^T(x)C_2, \quad 1 \simeq \Phi^T(x)K_1\Phi(t), \quad x - t \simeq \Phi^T(x)K_2\Phi(t),$$

$$u_1(x)u_2(x) \simeq C_1^T\tilde{C}_2\Phi(x),$$

The system (24) by using above equations is rewritten as,

$$\begin{cases} \Phi^T(x)\;\; K_1\tilde{C}_1 + K_2\tilde{E}\;\; P\Phi(x) - f_1(x) \approx 0, \\ \Phi^T(x)\;\; K_1\tilde{C}_2 + K_2\tilde{E}\;\; P\Phi(x) - f_2(x) \approx 0, \end{cases} \tag{25}$$

where $\tilde{C}_1$, $\tilde{C}_2$ and $\tilde{E}$ are the operational matrices of product corresponding with the vectors C_1, C_2 and $E = \tilde{C}_2^T C_1$, respectively. Now using the roots of $R_{11}^{(\alpha,\beta)}(x)$ and collocating the each equation of system (25), reduces the problem to solve a system of nonlinear algebraic equations. Solving the system (25) by Newton iterative method, the unknown coefficients will be obtained. Table 4 displays the maximum absolute errors for various α and β with $N = 10$.

Table 4. Maximum absolute error for $N = 10$ and different values of α and β for Example 4.

α	β	$\mathrm{Error}(u_1)$	$\mathrm{Error}(u_2)$
0	0	3.8374×10^{-8}	3.8374×10^{-8}
$1/2$	$1/2$	5.2040×10^{-8}	5.2040×10^{-8}
$1/2$	$-1/2$	7.7868×10^{-8}	7.7868×10^{-8}
$-1/2$	$1/2$	1.6827×10^{-8}	1.6827×10^{-8}
$1/4$	$1/4$	4.5072×10^{-8}	1.6827×10^{-8}
$-1/10$	$-1/10$	3.5742×10^{-8}	3.5742×10^{-8}
1	1	6.6291×10^{-8}	6.6291×10^{-8}
2	2	9.5316×10^{-8}	9.5316×10^{-8}
$3/2$	4	2.4651×10^{-8}	2.4651×10^{-8}

Example 5. Fifth example covers the system of nonlinear Volterra integral equation of second kind,

$$\begin{cases} f_1(x) + \int\limits_0^x u_1(t)u_2(t)\,dt = u_1(x), \\ f_2(x) + \int\limits_0^x u_1^2(t)\,dt + \int\limits_0^x u_2^2(t)\,dt = u_2(x). \end{cases} \tag{26}$$

The exact solutions are $u_1(x) = \cos(x)$ and $u_1(x) = \sin(x)$. Solutions and kernels are approximated as:

$$u_1(x) \simeq \Phi^T(x)C_1, \quad u_2(x) \simeq \Phi^T(x)C_2, \quad 1 \simeq \Phi^T(x)K_1\Phi(t),$$

$$u_1(x)u_2(x) \simeq C_1^T\tilde{C}_2\Phi(x), \quad u_1^2(x) \simeq C_1^T\tilde{C}_1\Phi(x), \quad u_2^2(x) \simeq C_2^T\tilde{C}_2\Phi(x),$$

The system (26) by using above equations is rewritten as,

$$\begin{cases} \Phi^T(x)C_1 - \Phi^T(x)K_1\tilde{E}_1P\Phi(x) - f_1(x) \approx 0, \\ \Phi^T(x)C_2 - \Phi^T(x)K_1 \;\; \tilde{E}_2 + \tilde{E}_3 \;\; P\Phi(x) - f_2(x) \approx 0, \end{cases} \tag{27}$$

where $\tilde{E}_1$, $\tilde{E}_2$ and $\tilde{E}_3$ are the operational matrices of product corresponding with the vectors $\tilde{C}_2^TC_1$, $\tilde{C}_1^TC_1$ and $E = \tilde{C}_2^TC_2$, respectively. The maximum absolute errors for $\alpha = \beta = 0$ and $N = 7,10,15$ are listed in Table 5. Also, using the roots of $R_{16}^{(\alpha,\beta)}(x)$ and collocating the each equation of system (27), reduces the problem to solve a system of nonlinear algebraic equations. Solving the system (27) by Newton iterative method, the unknown coefficients will be obtained. Table 6 displays the maximum absolute errors for various α and β with $N = 15$.

Table 5. Maximum absolute error for $N = 7,10,15$ and $\alpha = \beta = 0$ for Example 5.

N	Error(u_1)	Error(u_2)
7	1.2965×10^{-8}	1.6179×10^{-9}
10	8.1745×10^{-13}	6.2637×10^{-14}
15	2.1000×10^{-19}	2.0000×10^{-19}

Table 6. Maximum absolute error for $N = 15$ and different values of α and β for Example 5.

α	β	Error(u_1)	Error(u_2)
$1/2$	$1/2$	6.7470×10^{-11}	1.5516×10^{-10}
$1/2$	$-1/2$	4.3616×10^{-12}	1.1754×10^{-11}
$-1/2$	$1/2$	2.8278×10^{-10}	1.6280×10^{-10}
1	1	4.4401×10^{-12}	2.4305×10^{-11}
2	2	3.5172×10^{-11}	8.2047×10^{-11}
4	4	1.5692×10^{-8}	2.8878×10^{-8}
$1/4$	$-1/10$	3.4748×10^{-11}	5.6251×10^{-11}

5. CONCLUTION

In this paper, the shifted Jacobi collocation method was employed to solve a class of systems of Fredholm and Volterra integral equations of first and second kinds. First, a general formulation for the Jacobi operational matrix of integral has been derived. This matrix is used to approximate numerical solution of system of linear and nonlinear Volterra integral equations. Proposed approach was based on the shifted Jacobi collocation method. The solutions obtained using the proposed method shows that this method is a powerful mathematical tool for solving the integral equations. Proving the convergence of the method, consistency and stability are ensured automatically. Moreover, only a small number of shifted Jacobi polynomials are needed to obtain a satisfactory result.

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