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FERMAT COLLOCATION METHOD FOR NONLINEAR SYSTEM OF FIRST ORDER BOUNDARY VALUE PROBLEMS

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Abstract- In this study, a numerical approach is proposed to obtain approximate solutions of nonlinear system of first order boundary value problem. This technique is essentially based on the truncated Fermat series and its matrix representations with collocation points. Using the matrix method, we reduce the problem to a system of nonlinear algebraic equations. Numerical examples are also given to demonstrate the validity and applicability of the presented technique. The method is easy to implement and produces accurate results.

Keywords- Nonlinear system, First order boundary value problem, Fermat polynomials and series, Collocation points.

1. INTRODUCTION

Ordinary differential systems are encountered in scientific fields such as biology, ecology, chemistry, medicine, physics and engineering. Many scientific applications lead to higher order systems of ordinary differential equations, but it is often difficult or impossible to obtain explicit solutions to higher order or coupled systems of nonlinear ordinary differential equations. The method we shall discuss here is aimed primarily at discovering something of the character of the solutions. For simplicity, we shall concentrate on first order systems; the method may readily be generalized to higher order systems.

In this paper, we will consider the following nonlinear system of first order differential equations:

$$\left.\begin{aligned} a_1(x)y_1' + a_2(x)y_1 + a_3(x)y_2' + a_4(x)y_2 + N_1(y_1, y_2) = g_1(x) \\ b_1(x)y_1' + b_2(x)y_1 + b_3(x)y_2' + b_4(x)y_2 + N_2(y_1, y_2) = g_2(x) \end{aligned}\right\} \quad 0 \le x \le 1 \qquad (1)$$

with boundary conditions

$$y_1(0) = y_2(1) = 0 \text{ or } y_2(0) = y_1(1) = 0 \qquad (2)$$

where $0 \le x \le 1$, N_1 and N_2 are nonlinear functions of y_1 and y_2. Also $a_i(x)$, $b_i(x)$ for $i = 1,\dots,4$ are given continuous functions and g_1 and g_2 are known functions.

Systems of differential equations are often encountered in applications. Most realistic systems of ordinary differential equations do not have analytical solutions so that the numerical technique must be used [1]. They can be readily solved by many methods, such as the simple Taylor series method and fourth order Runge-Kutta method [2,3], the Tau Method [4,5,6] and the Adomian's decomposition method [8,9]. Over the past few years, many new alternatives to the use of traditional methods for the numerical solution of systems of differential equations have been proposed. In ref. [7], the author present the operational approach to the Tau Method for the numerical solution of mixed-order systems of linear ordinary differential equations with polynomial or rational polynomial coefficients, together with initial or boundary conditions. Kaya deals with the implementation of the Adomian's decomposition method in chemical applications [10]. In ref. [11], the Adomian's decomposition method is applied to initial problems for systems of ordinary differential equations in both linear and nonlinear cases. In this paper, we will consider general problems mainly focus on systems of nonlinear differential equations. Since every ordinary differential equation of order n can be written as a system consisting of n first-order ordinary differential equations, we restrict our study to a system of first-order differential equations. In the present work, by modifying and developing matrix and collocation methods studied in [12-17], we want to find the approximate solutions of the system (1) with boundary conditions (2) in the truncated Fermat series form

$$y_{i,N}(x)=\sum_{n=0}^{N} y_{i,n}F_n(x) \quad , \quad i=1,2 \quad , \quad 0\le x\le b. \tag{3}$$

where y_{in}, $(n=0,1,\ldots,N,\ \ i=1,2)$ are unknown coefficients to be determined.

The organization of this paper is as follows: In the next section we describe the matrix representations of each term in the system (1) and (2). In Section 3, we find the fundamental matrix relation of this system. In Section 4, the Fermat collocation method is performed. In Section 5, the accuracy of solution is given and in Section 6, some computational results are given to clarify the method. Section 7 ends this paper with a brief conclusion.

2. FUNDAMENTAL RELATIONS

Let us consider the nonlinear system in the form (1) and find the matrix representations of each term in the system. First we convert the solution defined by (3) and its derivatives, for $n=0,1,\ldots,N$ to the following matrix forms:

$$y_i(x)=\mathbf{F}(x)\mathbf{Y}_i, \quad i=1,2 \quad , \quad [18,19] \tag{4}$$

$$y_i^{(n)}(x)=\mathbf{F}(x)\mathbf{D}^n\mathbf{Y}_i \quad , \quad i=1,2 \tag{5}$$

where,

$$\begin{aligned}\mathbf{F}(x)&=\begin{bmatrix}F_0(x) & F_1(x) & F_2(x) & F_3(x) & \cdots & F_N(x)\end{bmatrix}_{1\times(N+1)}\\ &=\begin{bmatrix}1 & 3x & 9x^2-2 & 27x^3-12x & \cdots & 3xF_{n-1}(x)-2F_{n-2}(x)\end{bmatrix},\end{aligned}$$

$$\mathbf{D}=\begin{bmatrix} 0 & 3 & 0 & 6 & 0 & 12 & 0 & 24 & 0 & \cdots \\ 0 & 0 & 6 & 0 & 12 & 0 & 24 & 0 & 48 & \cdots \\ 0 & 0 & 0 & 9 & 0 & 18 & 0 & 36 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 12 & 0 & 24 & 0 & 48 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 15 & 0 & 30 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 18 & 0 & 36 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 21 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 24 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \end{bmatrix}_{(N+1)\times(N+1)}, \qquad \mathbf{Y}_i=\begin{bmatrix} y_{i,0} \\ y_{i,1} \\ y_{i,2} \\ \vdots \\ y_{i,N} \end{bmatrix}_{(N+1)\times 1}.$$

Nonlinear part of the system (1), $N_i(y_1,y_2)$, $i=1,2$ can be found as $y_i^2(x)$ or $y_i(x)y_j(x)$, $i\neq j$, $i=j=1,2$. Also, we can write the matrix form of these nonlinear expressions

$$y_i^2(x)=\mathbf{F}(x)\mathbf{F}^*(x)\overline{\mathbf{Y}}_{i,i} \tag{6}$$

$$y_i(x)y_j(x)=\mathbf{F}(x)\mathbf{F}^*(x)\overline{\mathbf{Y}}_{i,j} \tag{7}$$

$$y_j(x)y_i(x)=\mathbf{F}(x)\mathbf{F}^*(x)\overline{\mathbf{Y}}_{j,i} \tag{8}$$

where

$$\mathbf{F}^*(x)=\begin{bmatrix} \mathbf{F}(x) & 0 & \cdots & 0 \\ 0 & \mathbf{F}(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{F}(x) \end{bmatrix}_{(N+1)\times(N+1)^2}$$

and

$$\overline{\mathbf{Y}}_{i,i}=\begin{bmatrix} y_{i,0}\mathbf{Y}_i & y_{i,1}\mathbf{Y}_i & \cdots & y_{i,N}\mathbf{Y}_i \end{bmatrix}^T_{(N+1)^2\times 1},$$

$$\overline{\mathbf{Y}}_{i,j}=\begin{bmatrix} y_{j,0}\mathbf{Y}_i & y_{j,1}\mathbf{Y}_i & \cdots & y_{j,N}\mathbf{Y}_i \end{bmatrix}^T_{(N+1)^2\times 1}.$$

3. FUNDAMENTAL MATRIXS EQUATIONS FOR SYSTEM

We are now ready to construct the fundamental matrix equations for the nonlinear system of first order boundary value problem (1). For this purpose, substituting the matrix relations (4)-(8) into system (1) and simplifying, we obtain the system of matrix equations:

$$\left.\begin{aligned} a_1(x)\mathbf{F}(x)\mathbf{D}\mathbf{Y}_1+a_2(x)\mathbf{F}(x)\mathbf{Y}_1+a_3(x)\mathbf{F}(x)\mathbf{D}\mathbf{Y}_2+a_4(x)\mathbf{F}(x)\mathbf{Y}_2+N_1(x)\overline{\mathbf{Y}}_{i,j}=g_1(x) \\ b_1(x)\mathbf{F}(x)\mathbf{D}\mathbf{Y}_1+b_2(x)\mathbf{F}(x)\mathbf{Y}_1+b_3(x)\mathbf{F}(x)\mathbf{D}\mathbf{Y}_2+b_4(x)\mathbf{F}(x)\mathbf{Y}_2+N_2(x)\overline{\mathbf{Y}}_{j,i}=g_2(x) \end{aligned}\right\} \tag{9}$$

Therefore, we can write the matrix representation of the system (9) in the form

$$\left.\begin{aligned}\underbrace{\left[a_1(x)\mathbf{F}(x)\mathbf{D}+a_2(x)\mathbf{F}(x)\right]}_{\mathbf{A}_1(x)}\mathbf{Y}_1+\underbrace{\left[a_3(x)\mathbf{F}(x)\mathbf{D}+a_4(x)\mathbf{F}(x)\right]}_{\mathbf{A}_2(x)}\mathbf{Y}_2+N_1(x)\bar{\mathbf{Y}}_{i,j}&=g_1(x)\\ \underbrace{\left[b_1(x)\mathbf{F}(x)\mathbf{D}+b_2(x)\mathbf{F}(x)\right]}_{\mathbf{B}_1(x)}\mathbf{Y}_1+\underbrace{\left[b_3(x)\mathbf{F}(x)\mathbf{D}+b_4(x)\mathbf{F}(x)\right]}_{\mathbf{B}_2(x)}\mathbf{Y}_2+N_2(x)\bar{\mathbf{Y}}_{j,i}&=g_2(x)\end{aligned}\right\}$$

$$\left.\begin{aligned}\mathbf{A}_1(x)\mathbf{Y}_1+\mathbf{A}_2(x)\mathbf{Y}_2+N_1(x)\bar{\mathbf{Y}}_{i,j}&=g_1(x)\\ \mathbf{B}_1(x)\mathbf{Y}_1+\mathbf{B}_2(x)\mathbf{Y}_2+N_2(x)\bar{\mathbf{Y}}_{j,i}&=g_2(x)\end{aligned}\right\} \tag{10}$$

where

$$\mathbf{A}_1(x)=a_1(x)\mathbf{F}(x)\mathbf{D}+a_2(x)\mathbf{F}(x) \quad , \quad \mathbf{A}_2(x)=a_3(x)\mathbf{F}(x)\mathbf{D}+a_4(x)\mathbf{F}(x)$$

and

$$\mathbf{B}_1(x)=b_1(x)\mathbf{F}(x)\mathbf{D}+b_2(x)\mathbf{F}(x) \quad , \quad \mathbf{B}_2(x)=b_3(x)\mathbf{F}(x)\mathbf{D}+b_4(x)\mathbf{F}(x)\,.$$

Consequently, the fundamental matrix equations of the system (10) can be written in the following compact form

$$\mathbf{P}(x)\mathbf{Y}+\mathbf{N}(x)\bar{\mathbf{Y}}=\mathbf{G}(x) \tag{11}$$

where

$$\mathbf{P}(x)=\begin{bmatrix}\mathbf{A}_1(x) & \mathbf{A}_2(x)\\ \mathbf{B}_1(x) & \mathbf{B}_2(x)\end{bmatrix}_{2\times 2(N+1)} \quad , \quad \mathbf{N}(x)=\begin{bmatrix}N_1(x) & 0\\ 0 & N_2(x)\end{bmatrix}_{2\times 2(N+1)^2}$$

$$\mathbf{Y}=\begin{bmatrix}\mathbf{Y}_1\\ \mathbf{Y}_2\end{bmatrix}_{2(N+1)\times 1} \quad , \quad \bar{\mathbf{Y}}=\begin{bmatrix}\bar{\mathbf{Y}}_{i,j}\\ \bar{\mathbf{Y}}_{j,i}\end{bmatrix}_{2(N+1)^2\times 1} \quad , \quad \mathbf{G}(x)=\begin{bmatrix}g_1(x)\\ g_2(x)\end{bmatrix}_{2\times 1}.$$

4. FERMAT COLLACATION METHOD

In this section, by substituting the collocation points defined by

$$x_s=\frac{b}{N}s \quad , \quad s=0,1,\ldots,N$$

into the fundamental matrix equation (11), we obtain the new system

$$\mathbf{P}(x_s)\mathbf{Y}+\mathbf{N}(x_s)\bar{\mathbf{Y}}=\mathbf{G}(x_s) \tag{12}$$

and therefore, the new fundamental matrix equation

$$\mathbf{W}\mathbf{Y}^* + \mathbf{V}\bar{\mathbf{Y}}^* = \mathbf{G}^* \tag{13}$$

where

$$\mathbf{W} = \begin{bmatrix} \mathbf{P}(x_0) & 0 & \cdots & 0 \\ 0 & \mathbf{P}(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{P}(x_N) \end{bmatrix}_{2(N+1)\times 2(N+1)^2},$$

$$\mathbf{V} = \begin{bmatrix} \mathbf{N}(x_0) & 0 & \cdots & 0 \\ 0 & \mathbf{N}(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{N}(x_N) \end{bmatrix}_{2(N+1)\times 2(N+1)^3}$$

$$\mathbf{Y}^* = \begin{bmatrix} \mathbf{Y} \\ \mathbf{Y} \\ \vdots \\ \mathbf{Y} \end{bmatrix}_{2(N+1)^2\times 1}, \quad \bar{\mathbf{Y}}^* = \begin{bmatrix} \bar{\mathbf{Y}} \\ \bar{\mathbf{Y}} \\ \vdots \\ \bar{\mathbf{Y}} \end{bmatrix}_{2(N+1)^3\times 1}, \quad \mathbf{G}^* = \begin{bmatrix} \mathbf{G}(x_0) \\ \mathbf{G}(x_1) \\ \vdots \\ \mathbf{G}(x_N) \end{bmatrix}.$$

To find matrix representation of boundary conditions given with (2), by using Eq. 4 we can write row matrices as

$$\mathbf{F}(0)\mathbf{Y}_1 = 0 \ , \ \mathbf{F}(1)\mathbf{Y}_2 = 0$$

or

$$\mathbf{F}(0)\mathbf{Y}_2 = 0 \ , \ \mathbf{F}(1)\mathbf{Y}_1 = 0 .$$

Thus, we obtain the matrix forms of the conditions, respectively,

$$\mathbf{U}_0\mathbf{Y} = \mathbf{0} \ \text{or} \ \mathbf{U}_1\mathbf{Y} = \mathbf{0} \tag{14}$$

where

$$\mathbf{U}_0 = \begin{bmatrix} \mathbf{F}(0) & 0 \\ 0 & \mathbf{F}(1) \end{bmatrix}_{2\times 2(N+1)}, \ \mathbf{U}_1 = \begin{bmatrix} \mathbf{F}(1) & 0 \\ 0 & \mathbf{F}(0) \end{bmatrix}_{2\times 2(N+1)}, \ \mathbf{Y} = \begin{bmatrix} \mathbf{Y}_1 \\ \mathbf{Y}_2 \end{bmatrix}_{2(N+1)\times 1}.$$

To obtain the approximate solution of Eq. (1) with boundary conditions (2) in the terms of Fermat polynomials, by replacing the row matrix (14) by the last row of the matrix (13),we obtain the required augmented matrix:

$$\left[\tilde{\mathbf{W}},\tilde{\mathbf{V}};\tilde{\mathbf{G}}^*\right] = \begin{bmatrix} \mathbf{P}(x_0) & 0 & 0 & \cdots & 0 & 0 & , & \mathbf{N}(x_0) & 0 & 0 & \cdots & 0 & 0 & ; & \mathbf{G}(x_0) \\ 0 & \mathbf{P}(x_1) & 0 & \cdots & 0 & 0 & , & 0 & \mathbf{N}(x_1) & 0 & \cdots & 0 & 0 & ; & \mathbf{G}(x_1) \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & , & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & ; & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \mathbf{U}_0 & , & 0 & 0 & 0 & \cdots & 0 & 0 & ; & 0 \end{bmatrix}$$

or

$$\left[\tilde{\mathbf{W}},\tilde{\mathbf{V}};\tilde{\mathbf{G}}^*\right]=\begin{bmatrix} \mathbf{P}(x_0) & 0 & 0 & \cdots & 0 & 0 & , & \mathbf{N}(x_0) & 0 & 0 & \cdots & 0 & 0 & ; & \mathbf{G}(x_0) \\ 0 & \mathbf{P}(x_1) & 0 & \cdots & 0 & 0 & , & 0 & \mathbf{N}(x_1) & 0 & \cdots & 0 & 0 & ; & \mathbf{G}(x_1) \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & , & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & ; & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \mathbf{U}_1 & , & 0 & 0 & 0 & \cdots & 0 & 0 & ; & 0 \end{bmatrix}$$

and the corresponding matrix equation

$$\tilde{\mathbf{W}}\mathbf{Y}^* + \tilde{\mathbf{V}}\bar{\mathbf{Y}}^* = \tilde{\mathbf{G}}^*$$

where

$$\tilde{\mathbf{W}} = \begin{bmatrix} \mathbf{P}(x_0) & 0 & 0 & \cdots & 0 & 0 \\ 0 & \mathbf{P}(x_1) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \mathbf{U}_0 \end{bmatrix} \text{ or } \tilde{\mathbf{W}} = \begin{bmatrix} \mathbf{P}(x_0) & 0 & 0 & \cdots & 0 & 0 \\ 0 & \mathbf{P}(x_1) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \mathbf{U}_1 \end{bmatrix},$$

$$\tilde{\mathbf{V}} = \begin{bmatrix} \mathbf{N}(x_0) & 0 & 0 & \cdots & 0 & 0 \\ 0 & \mathbf{N}(x_1) & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix},\ \tilde{\mathbf{G}}^* = \begin{bmatrix} \mathbf{G}(x_0) \\ \mathbf{G}(x_1) \\ \vdots \\ 0 \end{bmatrix}.$$

The fundamental matrix equation of the system (1) under boundary conditions (2) corresponds to a system of $2(N+1)$ nonlinear algebraic equations with the unknown coefficients $y_{1,n}$ and $y_{2,n}$, $(n=0,1,\ldots,N)$.

Finally, the unknown coefficients are computed by solving this system and they are substituted in Eq. 3. Hence, the Fermat polynomial solutions

$$y_{i,N}(x)=\sum_{n=0}^{N} y_{i,n}F_n(x) \quad , \quad i=1,2 \tag{15}$$

can be obtained.

5. ACCURACY OF SOLUTIONS

We can easily check the accuracy of the above solutions. Since truncated Fermat series (3) is the approximate solution of system (1), when the function $y_{i,N}(x)$, $i=1,2$ and its derivatives are substituted in system (1), the resulting equation must be satisfied approximately; that is, for $x=x_q\in[0,1]$, $q=0,1,2,\ldots$,

$$E_{1,N}(x_q)=\left|a_1(x_q)y_{1,N}{}'+a_2(x_q)y_{1,N}+a_3(x_q)y_{2,N}{}'+a_4(x_q)y_{2,N}+N_1(y_{1,N},y_{2,N})-g_1(x_q)\right|\cong 0,$$

$$E_{2,N}(x_q)=\left|b_1(x_q)y_{1,N}{}'+b_2(x_q)y_{1,N}+b_3(x_q)y_{2,N}{}'+b_4(x_q)y_{2,N}+N_2(y_{1,N},y_{2,N})-g_2(x_q)\right|\cong 0,$$

and $E_{i,N}(x_q) \le 10^{-k_q}$, $i=1,2$ (k_q positive integer).If max $10^{-k_q} = 10^{-k}$ (k_q positive integer) is prescribed, then the truncation limit N is increased until the difference $E_{i,N}(x_q)$ at each of the points becomes smaller than the prescribed 10^{-k} [20].

6. NUMERICAL EXAMPLES

In this section, numerical examples are given to illustrate the accuracy and efficiency of the presented method.

Example 1: Let us first consider the nonlinear system of first boundary value problems

$$\left.\begin{array}{l} y_1' - xy_2' + y_1 y_2 = -x^4 + 2x^3 + x^2 + 3x - 1 \\ y_1' + 2xy_2' - y_1 y_2 = x^4 - 2x^3 - 3x^2 + 4x - 1 \end{array}\right\} \quad , 0 \le x \le 1$$

with boundary condition $y_1(0) = y_2(1) = 0$. The exact solutions of this problem are $y_1(x) = x^2 - x$, $y_2(x) = x - x^2$. Now, let us apply the procedure in Section 4 to obtain this approximate solution. Firstly, we note that

$a_1(x) = b_1(x) = 1, a_3(x) = -x, b_3(x) = 2x$, $a_2(x) = a_4(x) = b_2(x) = b_4(x) = 0$,

$N_1(y_1, y_2) = -N_2(y_1, y_2) = 1, g_1(x) = -x^4 + 2x^3 + x^2 + 3x - 1, g_2(x) = x^4 - 2x^3 - 3x^2 + 4x - 1$.

The set of collocation points for $N = 2$ is computed as $\left\{x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1\right\}$ and the fundamental matrix equation of the problem from Eq. 10 is

$$\left.\begin{array}{l} [\mathbf{F}(x)\mathbf{D}]\mathbf{Y}_1 - x[\mathbf{F}(x)\mathbf{D}]\mathbf{Y}_2 + \bar{\mathbf{Y}}_{i,j} = -x^4 + 2x^3 + x^2 + 3x - 1 \\ [\mathbf{F}(x)\mathbf{D}]\mathbf{Y}_1 + 2x[\mathbf{F}(x)\mathbf{D}]\mathbf{Y}_2 - \bar{\mathbf{Y}}_{j,i} = x^4 - 2x^3 - 3x^2 + 4x - 1 \end{array}\right\}.$$

We can find the compact form of this system from Eq. 11 as

$$\mathbf{P}(x)\mathbf{Y} + \mathbf{N}(x)\bar{\mathbf{Y}} = \mathbf{G}(x)$$

where

$$\mathbf{P}(x) = \begin{bmatrix} 0 & 3 & 18x & 0 & -3x & -18x^2 \\ 0 & 3 & 18x & 0 & 6x & 36x^2 \end{bmatrix}, \ \mathbf{G}(x) = \begin{bmatrix} -x^4 + 2x^3 + x^2 + 3x - 1 \\ x^4 - 2x^3 - 3x^2 + 4x - 1 \end{bmatrix}.$$

The fundamental matrix equation

$$\mathbf{W}\mathbf{Y}^* + \mathbf{V}\bar{\mathbf{Y}}^* = \mathbf{G}^*$$

where

$$\mathbf{W} = \begin{bmatrix} \mathbf{P}(0) & 0 & 0 \\ 0 & \mathbf{P}\left(\frac{1}{2}\right) & 0 \\ 0 & 0 & \mathbf{P}(1) \end{bmatrix}, \ \mathbf{V} = \begin{bmatrix} \mathbf{N}(0) & 0 & 0 \\ 0 & \mathbf{N}\left(\frac{1}{2}\right) & 0 \\ 0 & 0 & \mathbf{N}(1) \end{bmatrix}, \ \mathbf{G}^* = \begin{bmatrix} \mathbf{G}(0) \\ \mathbf{G}\left(\frac{1}{2}\right) \\ \mathbf{G}(1) \end{bmatrix}.$$

The matrix forms of the conditions is

$$\mathbf{U}_0 = \begin{bmatrix} \mathbf{F}(0) & 0 \\ 0 & \mathbf{F}(1) \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 7 \end{bmatrix}.$$

The augmented matrix for this fundamental matrix equation is calculated as

$$\left[\tilde{\mathbf{W}}, \tilde{\mathbf{V}}; \tilde{\mathbf{G}}^*\right] = \begin{bmatrix} \mathbf{P}(0) & 0 & 0 & , & \mathbf{N}(0) & 0 & 0 & ; & \mathbf{G}(0) \\ 0 & \mathbf{P}\left(\frac{1}{2}\right) & 0 & , & 0 & \mathbf{N}\left(\frac{1}{2}\right) & 0 & ; & \mathbf{G}\left(\frac{1}{2}\right) \\ 0 & 0 & \mathbf{U}_0 & , & 0 & 0 & 0 & ; & 0 \end{bmatrix}$$

where

$$\mathbf{P}(0) = \begin{bmatrix} 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \end{bmatrix}, \ \mathbf{P}\left(\frac{1}{2}\right) = \begin{bmatrix} 0 & 3 & 9 & 0 & -\frac{3}{2} & -\frac{18}{4} \\ 0 & 3 & 9 & 0 & 3 & 9 \end{bmatrix},$$

$$\mathbf{G}(0) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \ \mathbf{G}\left(\frac{1}{2}\right) = \begin{bmatrix} \frac{15}{16} \\ \frac{1}{16} \end{bmatrix}, \ \mathbf{U}_0 = \begin{bmatrix} 1 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 & 7 \end{bmatrix}$$

$$\mathbf{N}(0) = \begin{bmatrix} 1 & 0 & -2 & 0 & 0 & 0 & -2 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & -4 \end{bmatrix},$$

$$\mathbf{N}\left(\frac{1}{2}\right) = \begin{bmatrix} 1 & \frac{3}{2} & \frac{1}{4} & \frac{3}{2} & \frac{9}{4} & \frac{3}{8} & \frac{1}{4} & \frac{3}{8} & \frac{1}{16} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\frac{3}{2} & -\frac{1}{4} & -\frac{3}{2} & -\frac{9}{4} & -\frac{3}{8} & -\frac{1}{4} & -\frac{3}{8} & -\frac{1}{16} \end{bmatrix}.$$

By solving this system, the Fermat coefficients matrix is gained as

$\mathbf{Y}_1 = \begin{bmatrix} 0 & -1 & 1 \end{bmatrix}^T$ and $\mathbf{Y}_2 = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}^T$.

Substituting the elements of these column matrixes into Eq. 4, we obtain the solution set in terms of Taylor polynomials as

$y_1(x) = x^2 - x, \qquad y_2(x) = x - x^2$

which are the exact solutions.

Example 2: Other example is the nonlinear system

$$\left.\begin{aligned} y_1' + y_2' - y_1 y_2 &= e^{2x}(x^4 - 2x^3 + x^2) \\ y_1' + y_2' + y_1 y_2 &= -e^{2x}(x^4 - 2x^3 + x^2) \end{aligned}\right\}, \ 0 \le x \le 1$$

with the boundary conditions $y_2(0) = y_1(1) = 0$. The exact solutions of this problem are $y_1(x) = xe^x - x^2e^x$ and $y_2(x) = x^2e^x - xe^x$. Using the procedure in Section 4, we calculate the approximate solutions $y_{1,N}(x)$ and $y_{2,N}(x)$ for $N = 4,6,9$. In Tables 1-2, the exact solutions and approximate solutions obtained by the present method are compared. On the other hand, in Fig. 1-2, the approximate solutions for the present method are shown for different values of N . Additionally, in Table 3, the accuracy of solutions are stated. These results show that if N increases, than the absolute errors decrease more rapidly.

Table 1. Comparison of the numerical errors for $y_1(x)$

x_i	Exact solution	$N=4$ $y_{1,4}(x_i)$	Absolute error	$N=6$ $y_{1,6}(x_i)$	Absolute error	$N=9$ $y_{1,9}(x_i)$	Absolute error
0	0	0	0	0	0	0	0
0.2	0.1954244413	0.1954665557	4.211438E-5	0.1954238977	5.443017E-7	0.1954243366	1.046987E-7
0.4	0.3580379274	0.3594641853	1.426257E-3	0.3580355169	2.426496E-5	0.3580355169	2.410478E-6
0.6	0.4373085117	0.4487862471	1.147773E-2	0.4371230444	1.854672E-4	0.4372949727	1.353896E-5
0.8	0.3560865435	0.4074213187	5.133477E-2	0.3555413638	5.451796E-4	0.3560428022	4.374122E-5
1	0	0.1665531977	1.665531E-1	-0.0001064399	1.064399E-4	-0.0000949677	9.496777E-5

Table 2. Comparison of the numerical errors for $y_2(x)$

x_i	Exact solution	$N=4$ $y_{2,4}(x_i)$	Absolute error	$N=6$ $y_{2,6}(x_i)$	Absolute error	$N=9$ $y_{2,9}(x_i)$	Absolute error
0	0	0	0	0	0	0	0
0.2	-0.1954244413	-0.1954665557	4.211438E-5	-0.1954238977	5.443017E-7	-0.1954243366	1.046987E-7
0.4	-0.3580379274	-0.3594641853	1.426257E-3	-0.3580355169	2.426496E-5	-0.3580355169	2.410478E-6
0.6	-0.4373085117	-0.4487862471	1.147773E-2	-0.4371230444	1.854672E-4	-0.4372949727	1.353896E-5
0.8	-0.3560865435	-0.4074213187	5.133477E-2	-0.3555413638	5.451796E-4	-0.3560428022	4.374122E-5
1	0	-0.1665531977	1.665531E-1	0.0001064399	1.064399E-4	0.0000949677	9.496777E-5

Table 3. Absolute errors of $y_1(x)$ and $y_2(x)$

x_i	$N=9$, $y_{1,9}(x_i)$, $y_{2,9}(x_i)$	$N=11$, $y_{1,11}(x_i)$, $y_{2,11}(x_i)$
0	0	0
0.2	1.046987567E-7	1.046788749E-7
0.4	2.410478372E-6	2.411260662E-6
0.6	1.353896629E-5	1.365234750E-5
0.8	4.374122287E-5	4.597224864E-5
1	9.496777777E-5	1.163503608E-4

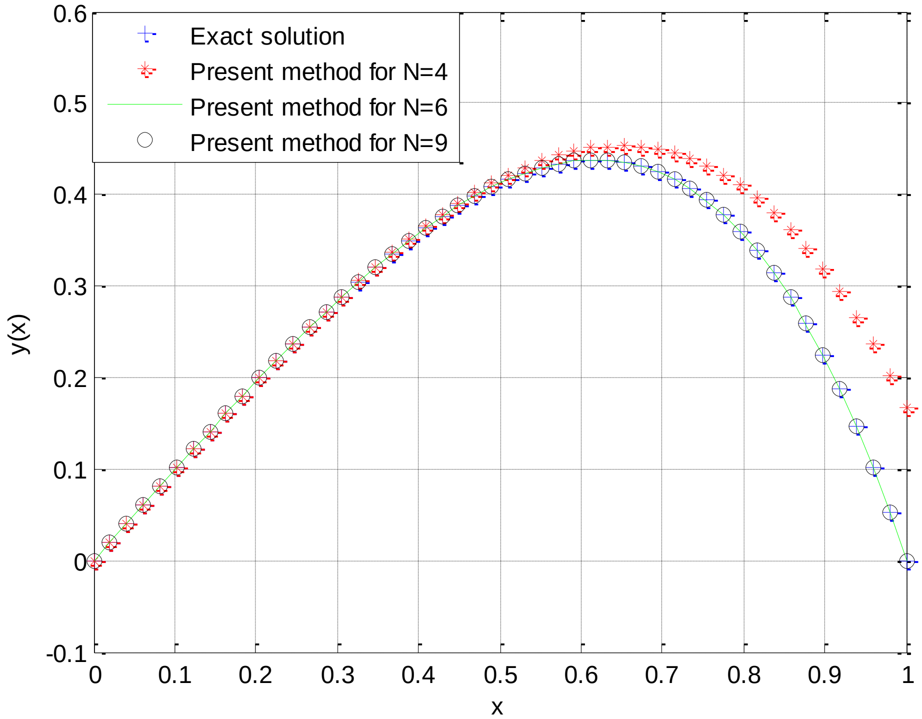

Figure 1. Numerical and exact solution of Example 2 for $y_1(x)$ **N = 4,6,9**

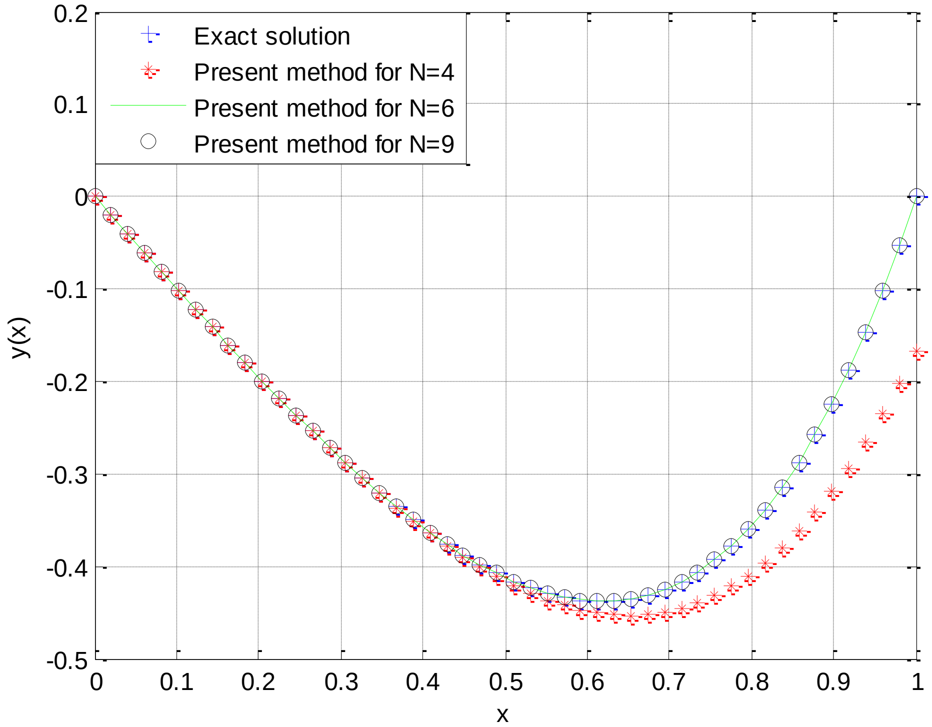

Figure 2. Numerical and exact solution of Example 2 for $y_2(x)$ **N = 4,6,9**

7. CONCLUSION

In this study, a new Fermat matrix collocation method is proposed for nonlinear system of first order boundary value problems. It is observed from Figures and Tables that the method is a simple and powerful tool to obtain the approximate solution. Thus, if *N* is increased, it can be seen that approximate solutions obtained by the mentioned method are close to the exact solutions. One of the considerable advantages of the method is finding the approximate solutions very easily by using the computer program. Shorter computation time and lower operation count results in a reduction of cumulative truncation errors and improvement of overall accuracy. In addition, the method can also be extended to other models in the future.

8. REFERENCES

1. Y. Li, F. Geng, M. Cui, The Analytical Solution of a System of Nonlinear Differential Equations, *International Journal of Mathematical Analysis* **1**(10), 451–462, 2007.

2 W. Cheney, D. Kincaid, *Numerical Mathematics and Computing*, Books/Cole Publishing Company, California, 1985.

3. C.F. Gerald, P.O. Wheatley, *Applied Numerical Analysis*, Addison-Wesley, California,1994.

4. M.R. Crisci, E.Russo, An extension of Ortiz' recursive formulation of the Tau Method to certain linear systems of ordinary differential equations, *Maths, Computation*, **41**, 27-42, 1983.

5. A.E.M. EiMisiery, E.L. Ortiz, Tau-Lines:Anew hybrid approach to the numerical treatment of crack problems based on the Tau Method, *Computer Methods in Applied Mechanics and Engineering* **56**, 265-282, 1986.

6. E.L.Ortiz, The Tau Method, *SIAM Journal of Numerical Analysis* **6**, 480-492, 1969.

7. K.M. Liu C.K. Pan, The Automatic Solution to Systems of Ordinary Differential Equations by the Tau Method, *Computers and Mathematics with Applications* **38**,197-210, 1999.

8. G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method, Kluwer Academic Publishers*, Boston, MA, 1994.

9. G. Adomian, A review of the decomposition method applied mathematics, *Journal of Mathematical Analysis and Applications* **135**, 501-544, 1988.

10. D. Kaya, A reliable method for the numerical solution of the kinetics problems, *Applied Mathematics and Computation* **156**, 261-270, 2004.

11. N. Güzel, M. Bayram, On the numerical solution of stiff systems, *Applied Mathematics and Computation* **170**, 230-236, 2005.

12. M. Sezer, Taylor polynomial solutions of Volterra integral equations, *International Journal of Mathematical Education in Science and Technology* **25(5)**, 625-633, 1994.

13. A. A. Daşcıoğlu, H. C. Yaslan, The solution of high-order nonlinear ordinary differential equations by Chebyshev Series, *Applied Mathematics and Computation* **217,** 5658–5666, 2011.

14. O. R.Işık, M. Sezer, Z. Güney, A rational approximation based on Bernstein polynomials for high order initial and boundary values problems, *Applied Mathematics and Computation* **217,** 9438–9450, 2011.
15. Ş. Yüzbaşı, M. Sezer, B. Kemancı, Numerical solutions of integro-differential equations and application of a population model with an improved Legendre method, *Applied Mathematical Modelling* **37,** 2086–2101, 2013.
16. Ş. Yüzbaşı, On the solutions of a system of linear retarded and advanced differential equations by the Bessel collocation approximation, *Computers and Mathematics with Applications* **63,** 1442–1455, 2012.
17. E. Gökmen, M. Sezer, Taylor collocation method for systems of high-order linear differential–difference equations with variable coefficients, *Ain Shams Engineering Journal* **4**, 117–125, 2013.
18. D. Taştekin, S. Yalçınbaş, Fermat Collocation Method for Solvıng a Class of the First Order Nonlinear Differential Equations, *Journal of Mathematical Sciences and Applications* **2(1)**, 4-9, 2014.
19. S. Yalçınbaş, D. Taştekin, Fermat Collocation Method for Solvıng a Class of the Second Order Nonlinear Differential Equations, *Applied Mathematics and Physics,* **2(2)**, 33-39, 2014.
20. E. Gökmen, M. Sezer, Taylor Collocation Method for Nonlinear System of Second Order Boundary Value Problems, *Düzce University Journal of Science & Technology* **1**, 11–23, 2013.