Mathematical and Computational Applications

MDPI

Article

On the Mixed Dirichlet–Steklov-Type and Steklov-Type Biharmonic Problems in Weighted Spaces

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Received: 31 January 2019; Accepted: 16 February 2019; Published: 18 February 2019

Abstract: We studied the properties of generalized solutions in unbounded domains and the asymptotic behavior of solutions of elliptic boundary value problems at infinity. Moreover, we studied the unique solvability of the mixed Dirichlet–Steklov-type and Steklov-type biharmonic problems in the exterior of a compact set under the assumption that generalized solutions of these problems has a bounded Dirichlet integral with weight $|x|^a$. Depending on the value of the parameter a, we obtained uniqueness (non-uniqueness) theorems of these problems or present exact formulas for the dimension of the space of solutions.

Keywords: biharmonic operator; mixed Dirichlet–Steklov-type problem; Steklov-type problem; Dirichlet integral; weighted spaces

MSC: 35J35; 35J40; 31B30

1. Introduction

Let Ω be an unbounded domain in $\mathbb{R}^n$, $n \geq 2$, $\Omega = \mathbb{R}^n \setminus \overline{G}$ with the boundary $\partial\Omega \in C^2$, where G is a bounded simply connected domain (or a union of finitely many such domains) in $\mathbb{R}^n$, $0 \in G$, $\overline{\Omega} = \Omega \cup \partial\Omega$ is the closure of Ω, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$.

In Ω, we consider the following problems for the biharmonic equation

$$\Delta^2 u = 0 \tag{1}$$

with the mixed Dirichlet–Steklov-type boundary conditions

$$u\big|_{\Gamma_1} = \frac{\partial u}{\partial \nu}\Big|_{\Gamma_1} = 0, \quad \frac{\partial u}{\partial \nu}\Big|_{\Gamma_2} = \left(\frac{\partial \Delta u}{\partial \nu} + \tau\, u\right)\Big|_{\Gamma_2} = 0, \tag{2}$$

and the Steklov-type boundary conditions

$$\frac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = \left(\frac{\partial \Delta u}{\partial \nu} + \tau\, u\right)\Big|_{\partial\Omega} = 0, \tag{3}$$

where $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \partial\Omega$, $\Gamma_1 \cap \Gamma_2 = \varnothing$, $\mathrm{mes}_{n-1}\,\Gamma_1 \neq 0$, $\nu = (\nu_1, \ldots, \nu_n)$ is the outer unit normal vector to $\partial\Omega$, $\tau \in C(\partial\Omega)$, $\tau \geq 0$, $\tau \not\equiv 0$, and $\tau > 0$ on a set of positive $(n-1)$-dimensional measure on $\partial\Omega$.

As is well known that, if Ω is an unbounded domain, one should additionally characterize the behavior of the solution at infinity. As a rule, to this end, one usually poses either the condition that the Dirichlet (energy) integral is finite or a condition on the character of vanishing of the modulus of the solution as $|x| \to \infty$. Such conditions at infinity are natural and were studied by several authors (e.g., [1–3]).

The behavior of solutions of the Dirichlet problem for the biharmonic equation as $|x| \to \infty$ is considered in [4,5], where estimates for $|u(x)|$ and $|\nabla u(x)|$ as $|x| \to \infty$ are obtained under certain geometric conditions on the domain boundary.

Elliptic problems with parameters in the boundary conditions have been called Steklov or Steklov-type problems since their first appearance in [6]. For the biharmonic operator, these conditions were first considered the authors of [7–9], who studied the isoperimetric properties of the first eigenvalue.

Note that standard elliptic regularity results are available in [10]. The monograph covers higher order linear and nonlinear elliptic boundary value problems, mainly with the biharmonic or polyharmonic operator as leading principal part. The underlying models and, in particular, the role of different boundary conditions are explained in detail. As for linear problems, after a brief summary of the existence theory and L^p and Schauder estimates, the focus is on positivity. The required kernel estimates are also presented in detail.

In [10,11], the spectral and positivity preserving properties for the inverse of the biharmonic operator under Steklov and Navier boundary conditions are studied. These are connected with the first Steklov eigenvalue. It is shown that the positivity preserving property is quite sensitive to the parameter involved in the boundary condition. Moreover, positivity of the Steklov boundary value problem is linked with positivity under boundary conditions of Dirichlet and Navier type.

In [12], the boundary value problems for the biharmonic equation and the Stokes system are studied in a half space, and, using the Schwartz reflection principle in weighted L^q-space, the uniqueness of solutions of the Stokes system or the biharmonic equation is proved.

We also point out [13–15], in which using the methods of complex analysis the Dirichlet and Neumann problems for the polyharmonic equation are explicitly solved in the unit disc of the complex plane. The solution is obtained by modifying the related Cauchy–Pompeiu representation with the help of the polyharmonic Green function.

In the present note, this condition is the boundedness of the weighted Dirichlet integral:

$$D_a(u,\Omega) \equiv \int_\Omega |x|^a \sum_{|\alpha|=2} |\partial^\alpha u(x)|^2\, dx < \infty, \quad a \in \mathbb{R}.$$

In various classes of unbounded domains with finite weighted Dirichlet (energy) integral, one of the authors [16–29] studied uniqueness (non-uniqueness) problem and found the dimensions of the spaces of solutions of boundary value problems for the elasticity system and the biharmonic (polyharmonic) equation.

By developing an approach based on the use of Hardy type inequalities [1–3,30], in the present note, we obtain a uniqueness (non-uniqueness) criterion for a solution of the mixed Dirichlet–Steklov-type and Steklov-type problems for the biharmonic equation.

Notation: $C_0^\infty(\Omega)$ is the space of infinitely differentiable functions in Ω with compact support in Ω.

We denote by $H^m(\Omega,\Gamma)$, $\Gamma \subset \overline{\Omega}$, the Sobolev space of functions in Ω obtained by the completion of $C^\infty(\overline{\Omega})$ vanishing in a neighborhood of Γ with respect to the norm

$$||u; H^m(\Omega,\Gamma)|| = \left(\int_\Omega \sum_{|\alpha|\le m} |\partial^\alpha u(x)|^2 dx \right)^{1/2}, \quad m = 1, 2,$$

where $\partial^\alpha \equiv \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, $\alpha_i \geq 0$ are integers, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$; if $\Gamma = \emptyset$, we denote $H^m(\Omega,\Gamma)$ by $H^m(\Omega)$.

$\overset{\circ}{H}{}^m(\Omega)$ is the space obtained by the completion of $C_0^\infty(\Omega)$ with respect to the norm $||u(x); H^m(\Omega)||$.

$\overset{\circ}{H}{}^{m}_{loc}(\Omega)$ is the space obtained by the completion of $C_0^\infty(\Omega)$ with respect to the family of semi-norms

$$\|u; H^m(\Omega \cap B_0(R))\| = \left(\int\limits_{\Omega \cap B_0(R)} \sum_{|\alpha| \le m} |\partial^\alpha u(x)|^2 \, dx \right)^{1/2}$$

for all open balls $B_0(R) := \{x : |x| < R\}$ in $\mathbb{R}^n$ for which $\Omega \cap B_0(R) \ne \emptyset$.

Let $\binom{n}{k}$ be the (n,k) binomial coefficient, $\binom{n}{k} = 0$ for $k > n$.

2. Definitions and Auxiliary Statements

Definition 1. *A solution of the homogenous biharmonic Equation (1) in Ω is a function $u \in H^2_{loc}(\Omega)$ such that, for every function $\varphi \in C_0^\infty(\Omega)$, the following integral identity holds:*

$$\int_\Omega \Delta u \, \Delta\varphi \, dx = 0.$$

Lemma 1. *Let u be a solution of Equation (1) in Ω such that $D_a(u, \Omega) < \infty$. Then,*

$$u(x) = P(x) + \sum_{\beta_0 < |\alpha| \le \beta} \partial^\alpha \Gamma(x) C_\alpha + u^\beta(x), \quad x \in \Omega, \tag{4}$$

where $P(x)$ is a polynomial, $\operatorname{ord} P(x) < m_0 = \max\{2, 2 - n/2 - a/2\}$, $\beta_0 = 2 - n/2 + a/2$, $\Gamma(x)$ *is the fundamental solution of Equation (1), $C_\alpha = \text{const}$, $\beta \ge 0$ is an integer, and the function u^β satisfies the estimate:*

$$|\partial^\gamma u^\beta(x)| \le C_{\gamma\beta} |x|^{3-n-\beta-|\gamma|}, \ C_{\gamma\beta} = \text{const},$$

for every multi-index γ.

Remark 1. *As is known [31], the fundamental solution $\Gamma(x)$ of the biharmonic equation has the form*

$$\Gamma(x) = \begin{cases} C|x|^{4-n}, \ if\ 4 - n < 0 \ or\ n\ is\ odd, \\ C|x|^{4-n} \ln |x|, \ if\ 4 - n \ge 0 \ and\ n\ is\ even. \end{cases}$$

Proof. Consider the function $v(x) = \theta_N(x) u(x)$, where $\theta_N(x) = \theta(|x|/N), \theta \in C^\infty(\mathbb{R}^n)$, $0 \le \theta \le 1$, $\theta(s) = 0$ for $s \le 1$, $\theta(s) = 1$ for $s \ge 2$, while $N \gg 1$ and $G \subset \{x : |x| < N\}$. We extend v to $\mathbb{R}^n$ by setting $v = 0$ on $G = \mathbb{R}^n \setminus \overline{\Omega}$.

Then, the function v belongs to $C^\infty(\mathbb{R}^n)$ and satisfies the equation

$$\Delta^2 v = f,$$

where $f \in C_0^\infty(\mathbb{R}^n)$ and $\operatorname{supp} f \subset \{x : |x| < 2N\}$. It is easy to see that $D_a(v, \mathbb{R}^n) < \infty$.

We can now use Theorem 1 of [32] since it is based on Lemma 2 of [32], which imposes no constraint on the *sign* of σ. Hence, the expansion

$$v(x) = P(x) + \sum_{\beta_0 < |\alpha| \le \beta} \partial^\alpha \Gamma(x) C_\alpha + v^\beta(x),$$

holds for each a, where $P(x)$ is a polynomial of order $\operatorname{ord} P(x) < m_0 = \max\{2, 2 - n/2 - a/2\}$, $\beta_0 = 2 - n/2 + a/2$, $C_\alpha = \text{const}$ and

$$|\partial^\gamma v^\beta(x)| \le C_{\gamma\beta} |x|^{3-n-\beta-|\gamma|}, \quad C_{\gamma\beta} = \text{const}.$$

Therefore, by the definition of v, we obtain Equation (4). The proof of Lemma 1 is complete. □

3. Main Results

3.1. The Mixed Dirichlet–Steklov-Type Biharmonic Problem

Definition 2. *By a solution of the mixed Dirichlet–Steklov-type problem in Equations (1) and (2) we mean a function* $u \in \overset{\circ}{H}{}^2_{loc}(\Omega, \Gamma_1) \cap \overset{\circ}{H}{}^1_{loc}(\Omega)$, $\partial u/\partial \nu = 0$ *on* Γ_2, *such that, for every function* $\varphi \in \overset{\circ}{H}{}^2_{loc}(\Omega, \Gamma_1) \cap C_0^\infty(\mathbb{R}^n)$, $\partial \varphi/\partial \nu = 0$ *on* Γ_2, *the following integral identity holds:*

$$\int_\Omega \Delta u \, \Delta \varphi \, dx - \int_{\Gamma_2} \tau \, u \, \varphi \, ds = 0. \tag{5}$$

Theorem 1. *The mixed Dirichlet–Steklov-type problem in Equations (1) and (2) with the condition* $D(u, \Omega) < \infty$ *has* $n + 1$ *linearly independent solutions.*

Proof. For any nonzero vector A in $\mathbb{R}^n$, we construct a generalized solution u_A of the biharmonic Equation (1) with the boundary conditions

$$u_A(x)\big|_{\Gamma_1} = (Ax)\big|_{\Gamma_1}, \quad \frac{\partial u_A(x)}{\partial \nu}\bigg|_{\Gamma_1} = \frac{\partial (Ax)}{\partial \nu}\bigg|_{\Gamma_1}, \quad \frac{\partial u_A}{\partial \nu}\bigg|_{\Gamma_2} = \left(\frac{\partial \Delta u_A}{\partial \nu} + \tau \, u_A\right)\bigg|_{\Gamma_2} = 0, \tag{6}$$

and the condition

$$\chi(u_A, \Omega) \equiv \begin{cases} \displaystyle\int_\Omega \left(\frac{|u_A|^2}{|x|^4} + \frac{|\nabla u_A|^2}{|x|^2} + |\nabla\nabla u_A|^2\right) dx < \infty \\ \qquad\qquad\qquad\qquad\qquad \text{for } n > 4, \\ \displaystyle\int_\Omega \left(\frac{|u_A|^2}{||x|^2 \ln|x||^2} + \frac{|\nabla u_A|^2}{||x| \ln|x||^2} + |\nabla\nabla u_A|^2\right) dx < \infty \\ \qquad\qquad\qquad\qquad\qquad \text{for } 2 \le n \le 4, \end{cases} \tag{7}$$

for $A, x \in \mathbb{R}^n$, where Ax denotes the standard scalar product of A and x.

Such a solution of the problem in Equations (1) and (6) can be constructed by the variational method [31], minimizing the functional

$$\Phi(v) = \frac{1}{2} \int_\Omega |\Delta v|^2 \, dx$$

in the class of admissible functions $\{v \colon v \in H^2(\Omega),\, v(x)\big|_{\Gamma_1} = (Ax)\big|_{\Gamma_1},\, \frac{\partial v(x)}{\partial \nu}\big|_{\Gamma_1} = \frac{\partial (Ax)}{\partial \nu}\big|_{\Gamma_1},\, \frac{\partial v}{\partial \nu}\big|_{\Gamma_2} = \left(\frac{\partial \Delta v}{\partial \nu} + \tau \, v\right)\big|_{\Gamma_2} = 0,\ v \text{ is compactly supported in } \overline{\Omega}\}$.

The validity of the condition in Equation (7) as a consequence of the Hardy inequality follows from the results in [1–3].

Now, for any arbitrary number $e \neq 0$, we construct a generalized solution u_e of Equation (1) with the boundary conditions

$$u_e\big|_{\Gamma_1} = e, \quad \frac{\partial u_e}{\partial \nu}\bigg|_{\Gamma_1} = 0, \quad \frac{\partial u_e}{\partial \nu}\bigg|_{\Gamma_2} = \left(\frac{\partial \Delta u_e}{\partial \nu} + \tau \, u_e\right)\bigg|_{\Gamma_2} = 0, \tag{8}$$

and the condition

$$\chi(u_e,\Omega) \equiv \begin{cases} \displaystyle\int_\Omega \left(\frac{|u_e|^2}{|x|^4} + \frac{|\nabla u_e|^2}{|x|^2} + |\nabla\nabla u_e|^2 \right) dx < \infty \\ \hfill \text{for } n > 4, \\ \displaystyle\int_\Omega \left(\frac{|u_e|^2}{||x|^2 \ln |x||^2} + \frac{|\nabla u_e|^2}{||x| \ln |x||^2} + |\nabla\nabla u_e|^2 \right) dx < \infty \\ \hfill \text{for } 2 \le n \le 4. \end{cases} \tag{9}$$

The solution of the problem in Equations (1) and (8) is also constructed by the variational method with the minimization of the corresponding functional in the class of admissible functions $\{v : v \in H^2(\Omega),\ v\big|_{\Gamma_1} = e,\ \frac{\partial v}{\partial \nu}\big|_{\Gamma_1} = 0,\ \frac{\partial v}{\partial \nu}\big|_{\Gamma_2} = \left(\frac{\partial \Delta v}{\partial \nu} + \tau\, v\right)\big|_{\Gamma_2} = 0,$ where v is compactly supported in $\overline{\Omega}\}$.

The condition in Equation (9) as a consequence of the Hardy inequality follows from the results in [1–3].

Consider the function $v = (u_A - Ax) - (u_e - e)$.

Obviously, v is a solution of the problem in Equations (1) and (2):

$$\Delta^2 v = 0, \quad x \in \Omega, \quad v\big|_{\Gamma_1} = \frac{\partial v}{\partial \nu}\bigg|_{\Gamma_1} = 0, \quad \frac{\partial v}{\partial \nu}\bigg|_{\Gamma_2} = \left(\frac{\partial \Delta v}{\partial \nu} + \tau\, v\right)\bigg|_{\Gamma_2} = 0.$$

One can easily see that $v \not\equiv 0$ and $D(v,\Omega) < \infty$.

To each nonzero vector $\mathbf{A} = (A_0, A_1, \ldots, A_n)$ in $\mathbb{R}^{n+1}$, there corresponds a nonzero solution $v_{\mathbf{A}} = (v_{A_0}, v_{A_1}, \ldots, v_{A_n})$ of the problem in Equations (1) and (2) with the condition $D(v_{\mathbf{A}},\Omega) < \infty$, and, moreover,

$$v_{\mathbf{A}} = u_A - u_e - Ax + e.$$

Let $A_0, A_1, \ldots, A_n$ be a basis in $\mathbb{R}^{n+1}$. Let us prove that the corresponding solutions $v_{A_0}, v_{A_1}, \ldots, v_{A_n}$ are linearly independent. Let

$$\sum_{i=0}^{n} C_i v_{A_i} \equiv 0, \qquad C_i = \text{const}\,.$$

Set $W \equiv \sum_{i=1}^{n} C_i A_i x - C_0 e$. We have

$$W = \sum_{i=1}^{n} C_i u_{A_i} - C_0 u_e,$$

$$\int_\Omega |x|^{-2} |\nabla W|^2\, dx < \infty, \quad n > 4,$$

$$\int_\Omega ||x| \ln |x||^{-2} |\nabla W|^2\, dx < \infty, \quad 2 \le n \le 4.$$

Let us show that

$$W \equiv \sum_{i=1}^{n} C_i A_i x - C_0 e \equiv 0.$$

Let $T = \sum_{i=0}^{n} C_i A_i = (t_0, \ldots, t_n)$, where $A_0 = -e$. Then,

$$\int_\Omega |x|^{-2} |\nabla W|^2\, dx = \int_\Omega |x|^{-2} (t_1^2 + \cdots + t_n^2)\, dx = \infty, \quad n > 4,$$

$$\int_\Omega ||x| \ln |x||^{-2} |\nabla W|^2\, dx = \int_\Omega ||x| \ln |x||^{-2} (t_1^2 + \cdots + t_n^2)\, dx = \infty, \quad 2 \le n \le 4,$$

if $T \neq 0$.

Consequently, $T = \sum_{i=0}^{n} C_i A_i = 0$, and since the vectors $A_0, A_1, \dots, A_n$ are linearly independent, we obtain $C_i = 0,\ i = 0, 1, \dots, n$.

Thus, the Dirichlet–Steklov-type problem in Equations (1) and (2) with the condition $D(u, \Omega) < \infty$ has at least $n + 1$ linearly independent solutions.

Let us prove that each solution u of the problem in Equations (1) and (2) with the condition $D(u, \Omega) < \infty$ can be represented as a linear combination of the functions $v_{A_0}, v_{A_1}, \dots, v_{A_n}$, i.e.,

$$u = \sum_{i=0}^{n} C_i v_{A_i}, \qquad C_i = \text{const}\,.$$

Since $A_0, A_1, \dots, A_n$ is a basis in $\mathbb{R}^{n+1}$, it follows that there exists constants $C_0, C_1, \dots, C_n$ such that

$$A = \sum_{i=0}^{n} C_i A_i.$$

We set

$$u_0 \equiv u - \sum_{i=0}^{n} C_i v_{A_i}.$$

Obviously, the function u_0 is a solution of the problem in Equations (1) and (2), and $D(u_0, \Omega) < \infty$, $\chi(u_0, \Omega) < \infty$.

Let us show that $u_0 \equiv 0,\ x \in \Omega$. To this end, we substitute the function $\varphi(x) = u_0(x)\theta_N(x)$ into the integral identity in Equation (5) for the function u_0, where $\theta_N(x) = \theta(|x|/N)$, $\theta \in C^\infty(\mathbb{R})$, $0 \le \theta \le 1$, $\theta(s) = 0$ for $s \ge 2$ and $\theta(s) = 1$ for $s \le 1$; then, we obtain

$$\int_\Omega (\Delta u_0)^2 \theta_N(x)\, dx + \int_{\Gamma_2} \tau\, |u_0|^2 \theta_N(x)\, ds = -J_1(u_0) - J_2(u_0), \tag{10}$$

where

$$J_1(u_0) = 2 \int_\Omega \Delta u_0\, \nabla u_0\, \nabla\theta_N(x)\, dx, \qquad J_2(u_0) = \int_\Omega u_0\, \Delta u_0\, \Delta\theta_N(x)\, dx.$$

By applying the Cauchy–Schwarz inequality and by taking into account the conditions $D(u_0, \Omega) < \infty$ and $\chi(u_0, \Omega) < \infty$, one can easily show that $J_1(u_0) \to 0$ and $J_2(u_0) \to 0$ as $N \to \infty$. Consequently, by passing to the limit as $N \to \infty$ in Equation (10), we obtain

$$\int_\Omega (\Delta u_0)^2\, \theta_N(x)\, dx + \int_{\Gamma_2} \tau\, |u_0|^2 \theta_N(x)\, ds \to 0.$$

Using the integral identity

$$\int_\Omega (\Delta u_0)^2\, dx + \int_{\Gamma_2} \tau\, |u_0|^2\, ds = 0,$$

we find that if $u_0(x)$ is a solution of the homogeneous problem in Equations (1) and (2), then $\Delta u_0 = 0$.

Therefore, we have

$$\Delta u_0 = 0, \quad x \in \Omega,$$

$$u_0\big|_{\Gamma_1} = \frac{\partial u_0}{\partial \nu}\bigg|_{\Gamma_1} = 0, \quad \frac{\partial u_0}{\partial \nu}\bigg|_{\Gamma_2} = \left(\frac{\partial \Delta u_0}{\partial \nu} + \tau\, u_0\right)\bigg|_{\Gamma_2} = 0.$$

Hence, it follows ([33] Ch.2) that $u_0 = 0$ in Ω. The relation

$$\int_{\partial\Omega} \tau\, |u_0|^2\, ds = 0$$

implies that $u_0 \equiv 0$ on a set of a positive measure on $\partial\Omega$. The proof of the theorem is complete. □

Theorem 2. *The mixed Dirichlet–Steklov-type problem in Equations (1) and (2) with the condition $D_a(u,\Omega) < \infty$ has:*

(i) the trivial solution for $n-2 \le a < \infty$, $n > 4$;

(ii) n linearly independent solutions for $n-4 \le a < n-2$, $n > 4$;

(iii) $n+1$ linearly independent solutions for $-n \le a < n-4$, $n > 4$; and

(iv) $k(r,n)$ linearly independent solutions for $-2r+2-n \le a < -2r+4-n$, $r > 1$, $n > 4$, where

$$k(r,n) = \binom{r+n}{n} - \binom{r+n-4}{n}.$$

The proof of Theorem 2 is based on Lemma 1 about the asymptotic expansion of the solution of the biharmonic equation and the Hardy type inequalities for unbounded domains [1–3]. In Case (iv), we need to determine the number of linearly independent solutions of the biharmonic Equation (1), the degree of which not exceed the fixed number.

It is well know that the dimension of the space of all polynomials in $\mathbb{R}^n$ of degree $\le r$ is equal to $\binom{r+n}{n}$ [34]. Then, the dimension of the space of all biharmonic polynomials in $\mathbb{R}^n$ of degree $\le r$ is equal to

$$\binom{r+n}{n} - \binom{r+n-4}{n},$$

since the biharmonic equation is the vanishing of some polynomial of degree $r-4$ in $\mathbb{R}^n$. If we denote by $k(r,n)$ the number of linearly independent polynomial solutions of Equation (1) whose degree do not exceed r and by $l(r,n)$ the number of linearly independent homogeneous polynomials of degree r, that are solutions of Equation (1), then

$$k(r,n) = \sum_{s=0}^{r} l(s,n),$$

where

$$l(s,n) = \binom{s+n-1}{n-1} - \binom{s+n-5}{n-1}, \qquad s > 0.$$

Further, we prove that the mixed Dirichlet–Steklov-type problem in Equations (1) and (2) with the condition $D_a(u,\Omega) < \infty$ for $-2r+2-n \le a < -2r+4-n$ has equally $k(r,n)$ linearly independent solutions.

3.2. The Steklov-Type Biharmonic Problem

Definition 3. *A function u is a solution of the Steklov-type biharmonic problem in Equations (1) and (3), if $u \in H^2_{loc}(\Omega)$, $\partial u/\partial \nu = 0$ on $\partial\Omega$, such that for every function $\varphi \in C_0^\infty(\mathbb{R}^n)$, $\partial\varphi/\partial\nu = 0$ on $\partial\Omega$, the following integral identity holds*

$$\int_\Omega \Delta u\, \Delta\varphi\, dx - \int_{\partial\Omega} \tau\, u\, \varphi\, ds = 0.$$

Theorem 3. *The Steklov-type biharmonic problem in Equations (1) and (3) with the condition $D(u,\Omega) < \infty$ has $n+1$ linearly independent solutions.*

Theorem 4. *The Steklov-type biharmonic problem in Equations (1) and (3) with the condition $D_a(u,\Omega) < \infty$ has:*

(i) n linearly independent solutions for $n-4 \le a < \infty$;

(ii) $n+1$ linearly independent solutions for $-n \le a < n-4$; and

(iii) $k(r,n)$ linearly independent solutions for $-2r+2-n \le a < -2r+4-n$, $r > 1$, where

$$k(r,n) = \binom{r+n}{n} - \binom{r+n-4}{n}.$$

The results of the article were presented at the 7th International Conference on Mathematical Modeling in Physical Sciences (7th ICMSQUARE, August 27–31, 2018, Moscow, Russia).

Acknowledgments: I am grateful to my anonymous reviewers for carefully reading the manuscript and for their suggestions, which I will consider in the future.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Kondratiev, V.A.; Oleinik, O.A. Boundary value problems for the system of elasticity theory in unbounded domains. Korn's inequalities. *Rus. Math. Surv.* **1988**, *43*, 65–119. [CrossRef]
2. Kondratiev, V.A.; Oleinik, O.A. Hardy's and Korn's Inequality and their Application. *Rend. Mat. Appl.* **1990**, *10*, 641–666.
3. Konkov, A.A. On the dimension of the solution space of elliptic systems in unbounded domains. *Rus. Acad. Sci. Sb. Math.* **1995**, *80*, 411–434. [CrossRef]
4. Kondratiev, V.A.; Kopacek, I.; Oleinik, O.A. On asymptotic properties of solutions of the biharmonic equation. *Differ. Uravn.* **1981**, *17*, 1886–1899.
5. Kondratiev, V.A.; Oleinik, O.A. Estimates for solutions of the Dirichlet problem for biharmonic equation in a neighbourhood of an irregular boundary point and in a neighbourhood of infinity Saint-Venant's principle. *Proc. R. Soc. Edinb.* **1983**, *93*, 327–343. [CrossRef]
6. Stekloff, W. Sur les problemes fondamentaux de la physique mathematique. *Ann. Sci. de l'E.N.S.* **1902**, *19*, 191–259. [CrossRef]
7. Brock, F. An isoperimetric inequality for eigenvalues of the Stekloff problem. *Z. Angew. Math. Mech.* **2001**, *81*, 69–71. [CrossRef]
8. Kuttler, J.R.; Sigillito, V.G. Inequalities for membrane and Stekloff eigenvalues. *J. Math. Anal. Appl.* **1968**, *23*, 148–160. [CrossRef]
9. Payne, L.E. Some isoperimetric inequalities for harmonic functions. *SIAM J. Math. Anal.* **1970**, *1*, 354–359. [CrossRef]
10. Gazzola, F.; Grunau, H.-C.; Sweers, G. *Polyharmonic Boundary Value Problems: Positivity Preserving and Nonlinear Higher Order Elliptic Equations in Bounded Domains*; Springer-Verlag: Berlin/Heidelberg, Germany, 2010.
11. Gazzola, F.; Sweers, G. On positivity for the biharmonic operator under Steklov boundary conditions. *Arch. Rational Mech. Anal.* **2008**, *188*, 399–427. [CrossRef]
12. Farwig, R. A note on the reflection principle for the biharmonic equation and the Stokes system. *Acta Appl. Math.* **1994**, *34*, 41–51. [CrossRef]
13. Begehr, H. Dirichlet problems for the biharmonic equation. *Gen. Math.* **2005**, *13*, 65–72.
14. Begehr, H.; Vu, T.N.H.; Zhang, Z.-X. Polyharmonic Dirchlet problems. *Proc. Steklov Math. Inst.* **2006**, *255*, 13–34. [CrossRef]
15. Begehr, H.; Vagenas, C.J. Iterated Neumann problem for the higher order Poisson equation. *Math. Nachr.* **2006**, *279*, 38–57. [CrossRef]
16. Matevosyan, O.A. On solutions of boundary value problems for a system in the theory of elasticity and for the biharmonic equation in a half–space. *Differ. Uravn.* **1998**, *34*, 803–808.
17. Matevosyan, O.A. The exterior Dirichlet problem for the biharmonic equation: Solutions with bounded Dirichlet integral. *Math. Notes* **2001**, *70*, 363–377. [CrossRef]
18. Matevossian, O.A. Solutions of exterior boundary value problems for the elasticity system in weighted spaces. *Sb. Math.* **2001**, *192*, 1763–1798. [CrossRef]
19. Matevossian, H.A. On solutions of mixed boundary-value problems for the elasticity system in unbounded domains. *Izvestiya Math.* **2003**, *67*, 895–929. [CrossRef]
20. Matevosyan, O.A. On solutions of a boundary value problem for a polyharmonic equation in unbounded domains. *Rus. J. Math. Phys.* **2014**, *21*, 130–132. [CrossRef]
21. Matevossian, H.A. On solutions of the Dirichlet problem for the polyharmonic equation in unbounded domains. *P-Adic Numbers Ultrametr. Anal. Appl.* **2015**, *7*, 74–78. [CrossRef]
22. Matevosyan, O.A. Solution of a mixed boundary value problem for the biharmonic equation with finite weighted Dirichlet integral. *Differ. Equ.* **2015**, *51*, 487–501. [CrossRef]

23. Matevossian, O.A. On solutions of the Neumann problem for the biharmonic equation in unbounded domains. *Math. Notes* **2015**, *98*, 990–994. [CrossRef]
24. Matevosyan, O.A. On solutions of the mixed Dirichlet–Navier problem for the polyharmonic equation in exterior domains. *Rus. J. Math. Phys.* **2016**, *23*, 135–138. [CrossRef]
25. Matevosyan, O.A. On solutions of one boundary value problem for the biharmonic equation. *Differ. Equ.* **2016**, *52*, 1379–1383. [CrossRef]
26. Matevossian, H.A. On the biharmonic Steklov problem in weighted spaces. *Rus. J. Math. Phys.* **2017**, *24*, 134–138. [CrossRef]
27. Matevossian, H.A. On solutions of the mixed Dirichlet–Steklov problem for the biharmonic equation in exterior domains. *P-Adic Numbers Ultrametr. Anal. Appl.* **2017**, *9*, 151–157. [CrossRef]
28. Matevossian, H.A. On the Steklov-type biharmonic problem in unbounded domains. *Rus. J. Math. Phys.* **2018**, *25*, 271–276. [CrossRef]
29. Matevossian, H.A. On the polyharmonic Neumann problem in weighted spaces. *Complex Var. Elliptic Equ.* **2019**, *64*, 1–7. [CrossRef]
30. Egorov, Y.V.; Kondratiev, V.A. *On Spectral Theory of Elliptic Operators*; Birkhauser: Basel, Switzerland, 1996.
31. Sobolev, S.L. *Some Applications of Functional Analysis in Mathematical Physics*; American Mathematical Society: Providence, RI, USA, 1991.
32. Kondratiev, V.A.; Oleinik, O.A. On the behavior at infinity of solutions of elliptic systems with a finite energy integral. *Arch. Ration. Mech. Anal.* **1987**, *99*, 75–99. [CrossRef]
33. Gilbarg, D.; Trudinger, N. *Elliptic Partial Differential Equations of Second Order*; Springer-Verlag: Berlin, Germany, 1977.
34. Mikhlin, S.G. *Linear Partial Differential Equations*; Vyssaya Shkola: Moscow, Russia, 1977. (In Russian)