Mathematical and Computational Applications

MDPI

Article

Augmented Lagrangian Approach to the Newsvendor Model with Component Commonality

Abdelouahed Hamdi [1,*] and Lotfi Tadj [2]

[1] Department of Mathematics, Statistics and Physics, Qatar University, Doha 2713, Qatar
[2] Department of Industrial Engineering, Alfaisal University, Riyadh 11533, Saudi Arabia; ltadj@Alfaisal.edu
* Correspondence: abhamdi@qu.edu.qa

Received: 17 April 2019; Accepted: 23 May 2019; Published: 24 May 2019

check for updates

Abstract: Component commonality is a well-known approach in manufacturing, where the same components are used for multiple products. It has been implemented by many established companies such as Airbus, Kodak, Toyota, etc. We consider a standard two-product inventory model with a common component. The demands for the products are independent random variables. Instead of the usual approach to minimize the total shortage quantity, we propose to minimize the total shortage cost. The resulting problem is a non-convex nonlinear mathematical program. We illustrate the use of a primal-dual proximal method to solve this problem by obtaining numerically the optimal allocations of components. In particular, we show that a higher unit shortage cost induces a higher allocation.

Keywords: stochastic inventory systems; component commonality; augmented Lagrangian

---

1. Introduction

Nowadays, many companies have increased the number of variations in products to develop new markets and niches and to serve successfully highly distinct customer demands. However, most companies cannot afford to keep in stock all components needed to assemble all the types of these products. A simple way to reduce inventory levels is to incorporate common components that replace unique components in several final products. This is known as component commonality.

Component commonality has been successfully implemented by many companies such as Airbus (aircraft families), Caterpillar (Ò785D Mining Truck), Volkswagen's MQB platform (including VW Golf, Audi A3, and Seat Octavia), GM's Ecotec development program, The Joint Strike Fighter program, Black and Decker's electric hand tools, Kodak cameras, IBM, HP, Toyota, etc.

The component commonality can reduce inventory levels as well as the number of slow/non-moving components in the warehouse so it reduces the risk of perishability. This, in turn, reduces the direct and indirect costs of inventory and manufacturing. In fact, the most important benefits of component commonality have been identified as follows: (a) a reduction in unit production costs due to economies of scale; (b) savings in inventory holding or shortage costs due to risk pooling; (c) reduced product development lead-time and development cost since components that have been developed for one product do not have to be tested when included in other products; and (d) reduced administrative cost because of fewer components to manage.

These benefits can be realized best by a manufacturing strategy where parts and subassemblies are made or acquired to forecasts, while the final assembly of products is delayed until customer orders have been received. The production system using this sort of strategy is commonly referred to as assemble-to-order (ATO) system. Such a strategy is likely to emerge when component lead-time is relatively long while the assembly process is short, and when considerable commonality of components exists among the products.

On the other hand, the commonality of components is a double-edged sword that can lead to a wide fallout. For example, Toyota's accelerator pedal is common across many Toyota models globally. If anything goes wrong, the number of affected vehicles is enormous. The commonality of components can also block the production process for many finished products. Among the important drawbacks of component commonality are: (a) a lack of product distinctiveness, (i.e., mass confusion); (b) a lack of innovation and creativity; (c) an increase in the weight/cost of components and products; and (d) a compromised product performance.

The model we are considering in this paper is a standard newsvendor problem with a common component. The newsvendor (also called newsboy or single-period) problem aims at determining optimal inventory levels for a product characterized by fixed prices and uncertain demand. Various aspects of the newsvendor problem have been considered in previous research. Among recent relevant papers are, for example the articles [1–6]. The newsvendor model under study turned out to be non-convex. We show how to use the proximal multipliers method to derive the optimal components allocation.

The rest of the paper is organized as follows. The next section covers the review of the existing literature. Sections 3 and 4 deal with Models N and C, respectively. In Section 5, we describe our algorithmic approach to solve Model C. Section 6 presents computational results and Section 7 describes some managerial implications. Section 8 concludes the paper.

2. Literature Review

The two main topics that are relevant to our model and that we want to review briefly here are the component commonality and the primal-dual proximal methods.

2.1. Component Commonality

The benefits of component commonality for cost savings were first identified in the papers by Rutenberg [7] and Rutenberg and Shaftel [8]. They recognized the economies of scale in production from using a common product module for multiple products. Since then, the benefits and trade-offs involved in the decision to incorporate commonality have been well studied. In recent literature, we find researchers integrating component commonality with other production issues (see, e.g., [9–20]). In addition, the recent literature contains several success stories about commonality applications (e.g., [9,12–14,16,21,22]). For more extensive reviews on component commonality, we refer to the recent surveys [23–27].

Our work follows that pioneered by Baker et al. [28], who considered the following two models, usually called Models N and C. Model N, which has no common component, consists of two end products, and each end product comprises two different components that are normalized so that one component of each type is needed to make one end product (see Figure 1). In Model C, which has a common component, Component 7 replaces Components 4 and 5 (see Figure 2).

Baker et al. [28] assumed that the demands for two end products followed independent uniform distributions. Gerchak et al. [29] extended the model of Baker et al. [28] to an arbitrary number of products and any joint demand distribution. Each product had one unique component and a common one. Eynan and Rosenblatt [30] considered the model of Baker et al. [28] in the case where the common component was more expensive. They also used independent uniformly distributed demands. In Eynan [31], the demands for end products were correlated, uniformly distributed. Jönsson and Silver [32] also dealt with the model of Baker et al. [28]. They considered a different cost function and assumed the demands for the end products were independent and normally distributed. The same model was also discussed by Jönsson and Silver [33] and Jönsson et al. [34]. Fu and Fong [35] showed that the objective function of Jönsson and Silver [32] was convex for any unbounded continuous demand distributions. Fong et al. [36] considered the same product structure as Baker et al. [28] but assumed that not all components had the same cost. They also assumed Erlang distributed product demands. Fu et al. [37] considered the same structure and supposed mixed Erlang distributed product demands. We show the contribution and motivation of this paper by summarizing in Table 1 the previous relevant research.

Table 1. Summary of the relevant research.

Author(s)	Number of Products	Demands Distributions	Common Component	Cost Function
Baker et al. [28]	two	independent uniform	all costs equal	max service measure
Gerchak et al. [29]	arbitrary	any joint distribution	all costs equal	max expected profit
Eynan & Rosenblatt [30]	two	independent uniform	more expensive	min inventory costs
Eynan [31]	two	correlated uniform	all costs equal	min inventory costs
Jönsson & Silver [32]	two	independent normal	all costs equal	min expected units shortage
Jönsson & Silver [33]	arbitrary	independent discrete	all costs equal	min expected units shortage
Fu & Fong [35]	two	independent continuous	all costs equal	min expected units shortage
Fong et al. [36]	two	independent Erlang	more expensive	min expected units shortage
Fu et al. [37]	two	mixed independent Erlang	more expensive	min expected units shortage

The latest paper that came to our attention on this topic is that of Deza et al. [38] who showed that lowering component commonality may yield a higher type-II service level.

Since Jönsson and Silver [32], and all the papers that have followed, the objective function to minimize was the shortage quantity. However, it is customary in operations research and management science to minimize costs, not quantities. For this reason, we choose to study Models N and C with the objective to minimize the expected shortage cost; see Labro [23] for a review of the costs and benefits of commonality identified in the related literature. One might think that it should not make any difference whether shortage quantity or shortage cost is minimized, when the two products have the same unit shortage cost. In addition, it would seem intuitive that different unit shortage costs would lead to different component allocations to the products. While investigating these two hypotheses, the objective function considered turned out to be not necessarily convex. To cope with the non-convexity of the model, we propose to use what is known in mathematical programming literature as the proximal multipliers method (see Rockafellar [39] and references therein), a mixture of the augmented Lagrangian method with the proximal point algorithm.

2.2. Primal-Dual Proximal Methods

The proximal point algorithm (PPA) was first introduced by Moreau [40] and was studied by Martinet [41]. It has brought some stability to many classical methods of mathematical programming. For the convex minimization problem:

$$\min \{ f(x) \; : \; x \in \Re^n \} , \tag{1}$$

the PPA finds the solution of the following unconstrained regularized problem:

$$\min \left\{ f(x) + \frac{1}{2c} \|x - x^n\|^2 \; : \; x \in \Re^n \right\}, \tag{2}$$

where $c > 0$ is the proximal real parameter. The added quadratic term produces a regularization effect on the new objective function that should be minimized. Therefore ,some good theoretical and numerical solutions can be obtained.

Rockafellar [39,42] proved that a straightforward application of the PPA to the dual problem of a generic convex programming problem is equivalent to the *multipliers method* of Hestenes–Powell (Hestenes [43] and Powell [44]) also called the *augmented Lagrangian method*. The priority to considering an augmented Lagrangian rather than the ordinary one, may be found in the fact that the problem is not convex. The augmented Lagrangian formulation strongly convexifies the problem and therefore ensures the existence of a saddle-point of the constrained problem for non-convex problems.

As a consequence, the classical Usawa-type algorithms will converge by using a constant step gradient scheme.

Powell [44] and Hestenes [43] presented independently a new method based on a modified Lagrangian. The nonlinear programming problem

$$\min \{ f(x) \ : \ f_i(x) \leq 0 \ \ i = 1, \cdots, p, \ x \in \Re^n \}, \tag{3}$$

was reworked in a combination of primal-dual methods and penalty approaches by Hestenes in such a way to obtain the penalized Lagrangian:

$$\mathcal{L}_{\mathcal{A}}(x, u, \lambda) = f(x) + \frac{1}{2\lambda} \sum_{i=1}^{p} \left\{ \max \left(u_i + \lambda f_i(x), \ 0 \right)^2 - u_i^2 \right\}, \tag{4}$$

where $u = (u_1, ..., u_p)$ is the Lagrange multipliers vector and $\lambda > 0$ is an arbitrary penalty factor. Hestenes proposed to solve a sequence of unconstrained minimization problems using the penalized Lagrangian. The kth unconstrained minimization is

$$\min_x \mathcal{L}_{\mathcal{A}}(x, u^k, \lambda), \tag{5}$$

where u^k is the current estimate of the Lagrange multipliers vector. After each minimization, u^k is updated by the following formula:

$$u_i^{k+1} = \max \left(u_i^k + \lambda f_i(x^k), \ 0 \right), \quad i = 1, \cdots, p, \tag{6}$$

where x^k solves Equation (5).

To avoid the ill-conditioning usually associated with penalty methods, the penalty parameter λ should not increase to infinity. The dual iteration tends to converge rapidly, making the algorithm very efficient. Buys [45] proved the local convergence in the non-convex case under second-order sufficient conditions. He also showed that the unconstrained minimizations need not be exact. In his book, Luenberger [46] briefly explored the dual aspect of the multipliers method and gave an interpretation that the dual iteration is a gradient iteration to maximize the dual function. Rockafellar [47,48] provided an extension to the unique theoretical result of Arrow et al. [49], concerning saddle points in non-convex programming. This extension states that, under the second-order sufficient optimality conditions and under the strict complementarity condition, one can show the existence of a saddle point for a certain class of generalized Lagrangians. Rockafellar introduced the notions of quadratic increase and stability to show that, by using an augmented duality (replace the ordinary Lagrangian by the augmented Lagrangian), the saddle points exist and the dual problem need not be constrained in the case of problems with inequality constraints.

A summary of the augmented Lagrangian method with all its variants up to 1976 can be found in Bertsekas [50].

Therefore, in this paper, we consider the problem of Baker et al. [28]. Our contribution is two-fold. First, we alter the model by modifying the objective function: instead of minimizing the shortage quantity, we minimize the shortage cost. Second, since the new objective function turns out to be non-convex, we solve the model using the proximal multiplier method.

3. Model N

Consider the ATO system shown in Figure 1 where two different products consist each of a "unique" component (numbered 3 and 6, respectively) and a "similar" component (numbered 4 and 5, respectively).

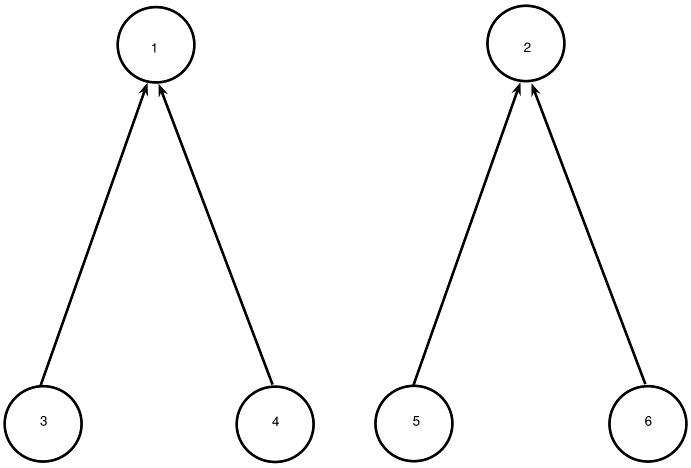

Figure 1. Model N.

The demands for end products are independent, random variables denoted by X and Y with respective density functions $f_X(x)$ and $f_Y(y)$ and respective means $E[X]$ and $E[Y]$. A budget constraint among the components is taken into account and the budget limit (total number of units available among all components) is fixed and denoted by T.

Notation 1. *To simplify equations, it is customary to use the following notation; see, for example, Jönsson and Silver [32], Fu and Fong [35], Fong et al. [36], and Fu et al. [37].*

$$\begin{aligned}
\phi_1(a) &= \int_a^\infty f_X(x)dx, \\
\phi_2(a) &= \int_a^\infty x f_X(x)dx, \\
\phi_3(a) &= \int_a^\infty f_Y(y)dy, \\
\phi_4(a) &= \int_a^\infty y f_Y(y)dy, \\
F_1(a_1, a_2, a_3) &= \int_{a_3-a_2}^{a_1} f_X(x)\phi_3(a_3 - x)dx, \\
F_2(a_1, a_2, a_3) &= \int_{a_3-a_2}^{a_1} f_X(x)[(x - a_3)\phi_3(a_3 - x) + \phi_4(a_3 - x)]dx.
\end{aligned}$$

The decision variable $S_j(j = 3, 4, 5, 6)$, represents the allocation to component j, and, in an optimal allocation, $S_3 = S_4$ and $S_5 = S_6$. Denote by $g_j(j = 1, 2)$, the unit shortage cost for product j. Since the objective is to minimize the expected shortage cost, we need to solve the following mathematical program

$$\begin{aligned}
\min Z &= g_1 I_1 + g_2 I_2 \\
\text{subject to} \quad & S_3 + S_6 = \frac{T}{2} \\
& S_3, S_6 \geq 0.
\end{aligned} \tag{7}$$

Here, T is the sum of components available for allocation and $I_j(j = 1, 2)$ represents the shortage quantities. Since Product 1 shortage happens when the demand X is larger than the number of components allocated S_3, the shortage quantity I_1 is given by:

$$I_1 = \int_{S_3}^\infty (x - S_3) f_X(x)dx = \phi_2(S_3) - S_3\phi_1(S_3). \tag{8}$$

Similarly, and using the constraint in Equation (7), we have the shortage quantity I_2:

$$I_2 = \int_{S_6}^{\infty} (y - S_6) f_Y(y) dy = \phi_4 \left(\frac{T}{2} - S_3 \right) - \left(\frac{T}{2} - S_3 \right) \phi_3 \left(\frac{T}{2} - S_3 \right). \tag{9}$$

Therefore, the objective function is given by

$$Z = g_1 \left[\phi_2(S_3) - S_3 \phi_1(S_3) \right] + g_2 \left[\phi_4 \left(\frac{T}{2} - S_3 \right) - \left(\frac{T}{2} - S_3 \right) \phi_3 \left(\frac{T}{2} - S_3 \right) \right]. \tag{10}$$

Note that

$$\frac{\partial I_1}{\partial S_3} = -\phi_1(S_3) \quad \text{and} \quad \frac{\partial I_2}{\partial S_3} = \phi_3 \left(\frac{T}{2} - S_3 \right).$$

The first-order optimality condition $\frac{\partial Z}{\partial S_3} = 0$ is equivalent to

$$g_1 \phi_1(S_3) = g_2 \phi_3 \left(\frac{T}{2} - S_3 \right). \tag{11}$$

Note that

$$\frac{\partial^2 I_1}{\partial S_3^2} = f_X(S_3) \quad \text{and} \quad \frac{\partial^2 I_2}{\partial S_3^2} = f_Y \left(\frac{T}{2} - S_3 \right).$$

Therefore,

$$\frac{\partial^2 Z}{\partial S_3^2} = g_1 f_X(S_3) + g_2 f_Y \left(\frac{T}{2} - S_3 \right) \geq 0, \tag{12}$$

and the second-order optimality condition guarantees that the solution S_3^* to Equation (11) yields a global minimum.

Illustration 1. *Assume the demand for Product 1 is uniformly distributed on the interval* $(0, u_1)$ *and the demand for Product 2 is uniformly distributed on the interval* $(0, u_2)$. *Then,*

$$f_X(x) = \frac{1}{u_1}, x \in (0, u_1); \qquad \phi_1(a) = \frac{u_1 - a}{u_1}; \qquad \phi_2(a) = \frac{u_1^2 - a^2}{2u_1};$$
$$f_Y(y) = \frac{1}{u_2}, y \in (0, u_2); \qquad \phi_3(a) = \frac{u_2 - a}{u_2}; \qquad \phi_4(a) = \frac{u_2^2 - a^2}{2u_2}.$$

The necessary optimality condition in Equation (11) yields the optimal number of components to allocate to Product 1

$$S_3^* = \frac{g_2 u_1 T/2 + (g_1 - g_2) u_1 u_2}{g_1 u_2 + g_2 u_1},$$

while the constraint in Equation (7) yields the optimal number of components to allocate to Product 2

$$S_6^* = \frac{g_1 u_2 T/2 - (g_1 - g_2) u_1 u_2}{g_1 u_2 + g_2 u_1}.$$

The total shortage cost is then computed as

$$Z^* = \frac{g_1 (u_1 - S_3^*)^2}{2u_1} + \frac{g_2 (u_2 - S_6^*)^2}{2u_2}.$$

4. Model C

Again, there are two end products and each end product results from the assembly of two different components. However, a common component, Component 7, is used to replace the similar Components 4 and 5 of the previous model (see Figure 2).

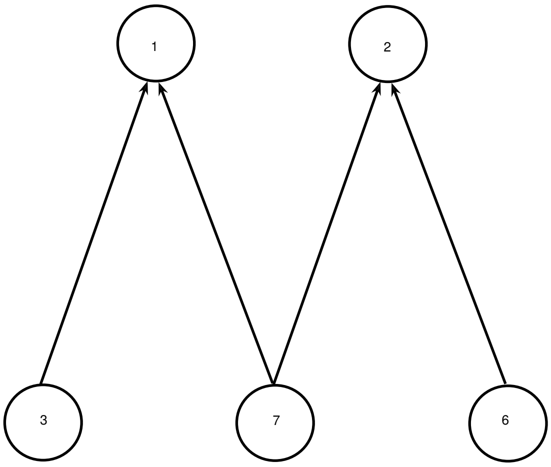

Figure 2. Model C.

The decision variable $S_j (j = 3, 6, 7)$ represents the allocation to component j. Since the objective is to minimize the expected shortage cost, we need to solve the following mathematical program

$$\begin{aligned} \min Z &= \text{expected shortage cost} \\ \text{subject to} \quad & S_3 + S_6 + S_7 = T \\ & S_3 \leq S_7 \\ & S_6 \leq S_7 \\ & S_3 + S_6 \geq S_7 \\ & S_3, S_6, S_7 \geq 0. \end{aligned} \tag{13}$$

To write the objective function, we first calculate the different amounts of shortage. The first type of shortage happens when the demand X for Product 1 exceeds S_3. The amount of shortage is

$$\begin{aligned} I_1 &= \int_{S_3}^{\infty} \int_0^{S_7 - S_3} (x - S_3) f_X(x) f_Y(y) dy dx \\ &= [1 - \phi_3(S_7 - S_3)] [\phi_2(S_3) - S_3 \phi_1(S_3)]. \end{aligned}$$

The second type of shortage happens when the demand Y for Product 2 exceeds S_6. The amount of shortage is

$$\begin{aligned} I_2 &= \int_0^{S_7 - S_6} \int_{S_6}^{\infty} (y - S_6) f_X(x) f_Y(y) dy dx \\ &= [1 - \phi_1(S_7 - S_6)] [\phi_4(S_6) - S_6 \phi_3(S_6)]. \end{aligned}$$

The third type of shortage happens when the combined demand $X + Y$ for Products 1 and 2 exceeds S_7. The amounts of shortage are given by three integrals. The first one is

$$\begin{aligned} I_3 &= \int_{S_3}^{\infty} \int_{S_7 - S_3}^{\infty} (x + y - S_7) f_X(x) f_Y(y) dy dx \\ &= \phi_2(S_3)\phi_3(S_7 - S_3) + [\phi_4(S_7 - S_3) - S_7 \phi_3(S_7 - S_3)] \phi_1(S_3). \end{aligned}$$

The other two integrals can be combined into a single one to get

$$\begin{aligned} I_4 &= \int_{S_7-S_6}^{S_3} \int_{S_6}^{\infty} (x+y-S_7) f_X(x) f_Y(y) dy dx \\ &\quad + \int_{S_7-S_6}^{S_3} \int_{S_7-x}^{S_6} (x+y-S_7) f_X(x) f_Y(y) dy dx \\ &= F_2(S_3, S_6, S_7). \end{aligned}$$

For more details on the calculations of these integrals, we refer, for example, to Jönsson and Silver [32], Fu and Fong [35], Fong et al. [36], and Fu et al. [37]. Denote by g_1, g_2, and g_{12} the unit shortage costs corresponding to the three types of shortages. Then, the objective function is given by

$$\begin{aligned} Z &= g_1 I_1 + g_2 I_2 + g_{12}(I_3 + I_4) \\ &= g_1 \Big\{ [1 - \phi_3(S_7 - S_3)] [\phi_2(S_3) - S_3 \phi_1(S_3)] \Big\} \\ &\quad + g_2 \Big\{ [1 - \phi_1(S_7 - S_6)] [\phi_4(S_6) - S_6 \phi_3(S_6)] \Big\} \\ &\quad + g_{12} \Big\{ \phi_2(S_3) \phi_3(S_7 - S_3) + [\phi_4(S_7 - S_3) - S_7 \phi_3(S_7 - S_3)] \phi_1(S_3) + F_2(S_3, S_6, S_7) \Big\}. \end{aligned}$$

Using the constraint in Equation (13), the problem can be reformulated as follows:

$$\begin{aligned} \min Z &= g_1 \Big\{ [1 - \phi_3(T - 2S_3 - S_6)] [\phi_2(S_3) - S_3 \phi_1(S_3)] \Big\} \\ &\quad + g_2 \Big\{ [1 - \phi_1(T - S_3 - 2S_6)] [\phi_4(S_6) - S_6 \phi_3(S_6)] \Big\} \\ &\quad + g_{12} \Big\{ \phi_2(S_3) \phi_3(T - 2S_3 - S_6) + [\phi_4(T - 2S_3 - S_6) \\ &\qquad - (T - S_3 - S6) \phi_3(T - 2S_3 - S_6)] \phi_1(S_3) + F_2(S_3, S_6, T - S_3 - S6) \Big\} \\ \text{subject to} &\quad 2S_3 + S_6 - T \leq 0 \\ &\quad S_3 + 2S_6 - T \leq 0 \\ &\quad T - 2S_3 - 2S_6 \leq 0 \\ &\quad S_3, S_6 \geq 0. \end{aligned}$$

The traditional approach for this type of optimization problems is to introduce the Lagrangian function:

$$\begin{aligned} \mathcal{L}(S_3, S_6, u_1, u_2, u_3) &= g_1 I_1 + g_2 I_2 + g_{12}(I_3 + I_4) \\ &\quad + u_1(2S_3 + S_6 - T) + u_2(S_3 + 2S_6 - T) + u_3(T - 2S_3 - S_6) \\ &\quad + u_4 S_3 + u_5 S_6. \end{aligned}$$

The Karush–Kuhn–Tucker (KKT) conditions are thus as follows:

$$\begin{aligned}
\frac{\partial L}{\partial S_3} &= 0,\\
\frac{\partial L}{\partial S_6} &= 0,\\
u_1(2S_3+S_6-T) &= 0,\\
u_2(S_3+2S_6-T) &= 0,\\
u_3(T-2S_3-S_6) &= 0,\\
u_4S_3 &= 0,\\
u_5S_6 &= 0,\\
u_1,u_2,u_3,u_4,u_5 &\geq 0,
\end{aligned}$$

where the first two equations are equivalent to

$$g_1\frac{\partial I_1}{\partial S_3}+g_2\frac{\partial I_2}{\partial S_3}+g_{12}\frac{\partial(I_3+\partial I_4)}{\partial S_3}+2u_1+u_2-2u_3+u_4=0,$$
$$g_1\frac{\partial I_1}{\partial S_6}+g_2\frac{\partial I_2}{\partial S_6}+g_{12}\frac{\partial(I_3+\partial I_4)}{\partial S_6}+u_1+2u_2-2u_3+u_5=0,$$

respectively. To write these two equations explicitly, we calculate the first partial derivatives of the first shortage type

$$\begin{aligned}
\frac{\partial I_1}{\partial S_3} &= -2f_Y(T-2S_3-S_6)\left[\phi_2(S_3)-S_3\phi_1(S_3)\right]-\left[1-\phi_3(T-2S_3-S_6)\right]\phi_1(S_3),\\
\frac{\partial I_1}{\partial S_6} &= -f_Y(T-2S_3-S_6)\left[\phi_2(S_3)-S_3\phi_1(S_3)\right],
\end{aligned}$$

the second shortage type

$$\begin{aligned}
\frac{\partial I_2}{\partial S_3} &= -f_X(T-S_3-2S_6)\left[\phi_4(S_6)-S_6\phi_3(S_6)\right],\\
\frac{\partial I_2}{\partial S_6} &= -2f_X(T-S_3-2S_6)\left[\phi_4(S_6)-S_6\phi_3(S_6)\right]-\left[1-\phi_1(T-S_3-2S_6)\right]\phi_3(S_6),
\end{aligned}$$

and the third shortage type

$$\begin{aligned}
\frac{\partial(I_3+I_4)}{\partial S_3} &= 2\left[\phi_2(S_3)-S_3\phi_1(S_3)\right]f_Y(T-2S_3-S_6)\\
&\quad+\left[\phi_4(S_6)-S_6\phi_3(S_6)\right]f_X(T-S_3-2S_6)\\
&\quad+\phi_3(T-2S_3-S_6)\phi_1(S_3)+F_1(S_3,S_6,T-S_3-S_6),\\
\frac{\partial(I_3+I_4)}{\partial S_6} &= \left[\phi_2(S_3)-S_3\phi_1(S_3)\right]f_Y(T-2S_3-S_6)\\
&\quad+2\left[\phi_4(S_6)-S_6\phi_3(S_6)\right]f_X(T-S_3-2S_6)\\
&\quad+\phi_3(T-2S_3-S_6)\phi_1(S_3)+F_1(S_3,S_6,T-S_3-S_6).
\end{aligned}$$

We note that the sums of these first derivatives are, respectively, given by

$$\begin{aligned}
\frac{\partial(I_1+I_2+I_3+I_4)}{\partial S_3} &= 2\phi_1(S_3)\phi_3(T-2S_3-S_6)-\phi_1(S_3)+F_1(S_3,S_6,T-S_3-S_6),\\
\frac{\partial(I_1+I_2+I_3+I_4)}{\partial S_6} &= \phi_1(S_3)\phi_3(T-2S_3-S_6)-\phi_3(S_6)+\phi_3(S_6)\phi_1(T-S_3-2S_6)\\
&\quad +F_1(S_3,S_6,T-S_3-S_6),
\end{aligned}$$

which agree with the results of Fu et al. [37]. Once the KKT system has been solved, the next step is to check whether the critical point obtained yields a global minimum.

5. Numerical Approach to Model C

To check the second-order optimality conditions, we calculate the second partial derivatives as follows:

$$\begin{aligned}
\frac{\partial^2 I_1}{\partial S_3^2} &= 4f_Y'(T-2S_3-S_6)\left[\phi_2(S_3)-S_3\phi_1(S_3)\right]\\
&\quad +\left[1-\phi_3(T-2S_3-S_6)\right]f_X(S_3)+4\phi_1(S_3)f_Y(T-2S_3-S_6)\\
\frac{\partial^2 I_1}{\partial S_6^2} &= f_Y'(T-2S_3-S_6)\left[\phi_2(S_3)-S_3\phi_1(S_3)\right]\\
\frac{\partial^2 I_1}{\partial S_3\partial S_6} &= 2f_Y'(T-2S_3-S_6)\left[\phi_2(S_3)-S_3\phi_1(S_3)\right]+f_Y(T-2S_3-S_6)\phi_1(S_3)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 I_2}{\partial S_3^2} &= f_X'(T-S_3-2S_6)\left[\phi_4(S_6)-S_6\phi_3(S_6)\right]\\
\frac{\partial^2 I_2}{\partial S_6^2} &= 4f_X'(T-S_3-2S_6)\left[\phi_4(S_6)-S_6\phi_3(S_6)\right]\\
&\quad +\left[1-\phi_1(T-S_3-2S_6)\right]f_Y(S_6)+4\phi_3(S_6)f_X(T-S_3-2S_6)\\
\frac{\partial^2 I_2}{\partial S_3\partial S_6} &= 2f_X'(T-S_3-2S_6)\left[\phi_4(S_6)-S_6\phi_3(S_6)\right]+f_X(T-S_3-2S_6)\phi_3(S_6)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2(I_3+I_4)}{\partial S_3^2} &= -4\left[\phi_2(S_3)-S_3\phi_1(S_3)\right]f_Y'(T-2S_3-S_6)\\
&\quad -\left[\phi_4(S_6)-S_6\phi_3(S_6)\right]f_X'(T-S_3-2S_6)+\phi_3(S_6)f_X(T-S_3-2S_6)\\
&\quad +\int_{T-S_3-2S_6}^{S_3} f_X(x)f_Y(T-S_3-S_6-x)dx
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2(I_3+I_4)}{\partial S_6^2} &= -4\left[\phi_4(S_6)-S_6\phi_3(S_6)\right]f_X'(T-S_3-2S_6)\\
&\quad -\left[\phi_2(S_3)-S_3\phi_1(S_3)f_Y'(T-2S_3-S_6)\right]+\phi_1(S_3)f_Y(T-2S_3-S_6)\\
&\quad +\int_{T-S_3-2S_6}^{S_3} f_X(x)f_Y(T-S_3-S_6-x)dx.
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2(I_3+I_4)}{\partial S_3\partial S_6} &= \phi_1(S_3)f_Y(T-2S_3-S_6)+\phi_3(S_6)f_X(T-S_3-2S_6)\\
&\quad -2\Big[\phi_2(S_3)-S_3\phi_1(S_3)\Big]f'_Y(T-2S_3-S_6)\\
&\quad -2\Big[\phi_4(S_6)-S_6\phi_3(S_6)\Big]f'_X(T-S_3-2S_6)\\
&\quad +\int_{T-S_3-2S_6}^{S_3} f_X(x)f_Y(T-S_3-S_6-x)dx.
\end{aligned}$$

We note that the sums of these second derivatives are, respectively, given by

$$\begin{aligned}
\frac{\partial^2(I_1+I_2+I_3+I_4)}{\partial S_3^2} &= 4\phi_1(S_3)f_Y(T-2S_3-S_6)+[1-\phi_3(T-2S_3-S_6)]f_X(S_3)\\
&\quad +\phi_3(S_6)f_X(T-S_3-2S_6)+\int_{T-S_3-2S_6}^{S_3} f_X(x)f_Y(T-S_3-S_6-x)dx,\\
\frac{\partial^2(I_1+I_2+I_3+I_4)}{\partial S_3\partial S_6} &= 2\phi_1(S_3)f_Y(T-2S_3-S_6)+2\phi_3(S_6)f_X(T-S_3-2S_6)\\
&\quad +\int_{T-S_3-2S_6}^{S_6} f_X(x)f_Y(T-S_3-S_6-x)dx,\\
\frac{\partial^2(I_1+I_2+I_3+I_4)}{\partial S_6^2} &= 4\phi_3(S_3)f_X(T-S_3-2S_6)+[1-\phi_1(T-S_3-2S_6)]f_Y(S_6)\\
&\quad +\phi_1(S_6)f_Y(T-2S_3-S_6)+\int_{T-S_3-2S_6}^{S_3} f_X(x)f_Y(T-S_3-S_6-x)dx,
\end{aligned}$$

which agree with the results of Fu and Fong [35]. Now, since g_1, g_2, and g_{12} are positive, it suffices that I_1, I_2, and I_3+I_4 be convex functions of S_3, S_6 for the objective function to be convex. However, calculating the different determinants shows that these functions may not be convex. Consider I_1, for example. The first leading principal minor $\frac{\partial^2 I_1}{\partial S_3^2}$ of the Hessian matrix is obviously positive. The second leading principal minor

$$\begin{aligned}
\frac{\partial^2 I_1}{\partial S_3^2}\cdot\frac{\partial^2 I_1}{\partial S_6^2}-\left[\frac{\partial^2 I_1}{\partial S_3\partial S_6}\right]^2 &= [1-\phi_1(T-2S_3-S_6)]\,f_X(S_3)f'_Y(T-2S_3-S_6)\,[\phi_2(S_3)-S_3\phi_1(S_3)]\\
&\quad -f_Y(T-2S_3-S_6)^2\phi_1(S_3)^2,
\end{aligned}$$

is not necessarily positive. To cope with the non-convexity of our model, we propose to mix the augmented Lagrangian method with the proximal point technique, what is known in mathematical programming literature as the proximal multipliers method (see Rockafellar [39] and the references therein).

Let $u=(u_1,u_2,u_3)$ denote the Lagrange multipliers and consider the augmented Lagrangian

$$\begin{aligned}
\mathcal{L}_{\mathcal{A}}(S_3,S_6,u,\lambda) &= g_1I_1+g_2I_2+g_{12}(I_3+I_4)\\
&\quad +\frac{1}{2\lambda}\Bigg\{\Big[\max\Big(u_1+\lambda(2S_3+S_6-T),\,0\Big)^2-u_1^2\Big]\\
&\quad +\Big[\max\Big(u_2+\lambda(S_3+2S_6-T)^2,\,0\Big)-u_2^2\Big]\\
&\quad +\Big[\max\Big(u_3+\lambda(T-2S_3-2S_6)^2,\,0\Big)-u_3^2\Big]\Bigg\}.
\end{aligned}$$

Algorithm 1 shows how the proximal multipliers method is implemented. The primal and dual variables are initialized in the first step. The primal variables are calculated in Step 2 while Step 3 updates the dual variables.

Algorithm 1: Implementation of the proximal multipliers method.

1. Initialize: $(S_3^0, S_6^0) \in^2, u^0 = (u_1^0, u_2^0, u_3^0) \in^3$: $c > 0,\ \lambda > 0,\ k = 0$.
2. Compute:

$$(S_3^{k+1}, S_6^{k+1}) := \arg \min_{(S_3,S_6)\in^2} \left\{ \mathcal{L}_{\mathcal{A}}(S_3, S_6, u^k, \lambda) + \frac{1}{2c} \|(S_3, S_6) - (S_3^k, S_6^k)\|^2 \right\}.$$

3. Update the multipliers:

$$\begin{aligned} u_1^{k+1} &= \max\left(u_1^k + \lambda(2S_3^k + S_6^k - T),\ 0\right), \\ u_2^{k+1} &= \max\left(u_2^k + \lambda(S_3^k + 2S_6^k - T),\ 0\right), \\ u_3^{k+1} &= \max\left(u_3^k + \lambda(T - 2S_3^k - 2S_6^k),\ 0\right). \end{aligned}$$

Remark 1. *It is well established nowadays in the literature that there is no optimal way to tune the penalty parameter λ and the proximal parameter c. Many heuristics are available to update these parameters. In this case, we used a fixed parameters strategy and, when the feasibility was slow, we increased the parameter λ by some factors.*

6. Illustrative Examples

Assume demands follow independent non-identical Erlang distributions with parameters (α_1, β_1) and (α_2, β_2), respectively:

$$f_X(x) = \frac{\beta_1^{\alpha_1} x^{\alpha_1 - 1} e^{-\beta_1 x}}{(\alpha_1 - 1)!}, \quad x > 0, \quad \text{and} \quad f_Y(y) = \frac{\beta_2^{\alpha_2} y^{\alpha_2 - 1} e^{-\beta_2 y}}{(\alpha_2 - 1)!}, \quad y > 0.$$

In this case, we have

$$\phi_1(a) = e^{-\beta_1 a} \sum_{m=0}^{\alpha_1 - 1} \frac{(a\beta_1)^m}{m!}, \qquad \phi_3(a) = e^{-\beta_2 a} \sum_{m=0}^{\alpha_2 - 1} \frac{(a\beta_2)^m}{m!},$$

$$\phi_2(a) = \frac{\alpha_1}{\beta_1} e^{-\beta_1 a} \sum_{m=0}^{\alpha_1} \frac{(a\beta_1)^m}{m!}, \qquad \phi_4(a) = \frac{\alpha_2}{\beta_2} e^{-\beta_2 a} \sum_{m=0}^{\alpha_2} \frac{(a\beta_2)^m}{m!},$$

$$\begin{aligned} F_1(a_1, a_2, a_3) &= \frac{\beta_1^{\alpha_1} e^{-\beta_2 a_3}}{(\alpha_1 - 1)!} \sum_{m=0}^{\alpha_2 - 1} \beta_2^m \sum_{i=0}^{m} \frac{a_3^i (-1)^{m-i}}{i!(m-i)!} I(a_1, a_2, a_3; m - i + \alpha_1, \beta_1 - \beta_2), \\ F_2(a_1, a_2, a_3) &= \frac{\beta_1^{\alpha_1} e^{-\beta_2 a_3}}{(\alpha_1 - 1)!} \left[\sum_{m=0}^{\alpha_2 - 1} \beta_2^m \sum_{i=0}^{m} \frac{a_3^i (-1)^{m+1-i}}{i!(m+1-i)!} I(a_1, a_2, a_3; m - i + \alpha_1 - 1, \beta_1 - \beta_2) \right. \\ &\quad \left. + \sum_{m=0}^{\alpha_2} \beta_2^m \sum_{i=0}^{m} \frac{a_3^i (-1)^{m-i}}{i!(m-i)!} I(a_1, a_2, a_3; m - i + \alpha_1, \beta_1 - \beta_2) \right], \end{aligned}$$

where

$$I(a_1, a_2, a_3; n, \beta_1 - \beta_2) = \begin{cases} \dfrac{a_1^n - (a_3 - a_2)^n}{n}, & \beta_1 = \beta_2, \\ \dfrac{(n-1)!}{(\beta_1 - \beta_2)^n} \left[e^{-(\beta_1-\beta_2)(a_3-a_2)} \sum_{j=0}^{n-1} \dfrac{[(a_3 - a_2)(\beta_1 - \beta_2)]^j}{j!} \right. & \\ \left. - e^{-(\beta_1-\beta_2)a_1} \sum_{j=0}^{n-1} \dfrac{[a_1(\beta_1 - \beta_2)]^j}{j!} \right], & \beta_1 \neq \beta_2. \end{cases}$$

For Model N, the optimal allocation S_3^* is the numerical solution of Equation (11), which is equivalent in this case to

$$g_1 e^{-\beta_1 S_3} \sum_{m=0}^{\alpha_1 - 1} \frac{(S_3 \beta_1)^m}{m!} = g_2 e^{\beta_2 \left(\frac{T}{2} - S_3\right)} \sum_{m=0}^{\alpha_2 - 1} \frac{\left[\left(\frac{T}{2} - S_3 \right) \beta_2 \right]^m}{m!}.$$

The optimal allocation S_6^* is found using Equation (7) as

$$S_6^* = \frac{T}{2} - S_3^*.$$

The optimal expected shortage cost is given by

$$Z^* = g_1 \left[\phi_2(S_3^*) - S_3^* \phi_1(S_3^*) \right] + g_2 \left[\phi_4(S_6^*) - S_6^* \phi_3(S_6^*) \right].$$

For Model C, we implemented Algorithm 1 above.

We now present some numerical results. We take the following values for the parameters of the Erlang distributions of demands: $\alpha_1 = 5$ and $\beta_1 = 1$ for Product 1 and $\alpha_2 = 5$ and $\beta_2 = 0.5$. Assuming there is a total of $T = 50$ components and the following unit shortage costs $g_1 = 20$, $g_2 = 10$, and $g_{12} = 15$; the results obtained are shown in Table 2.

Table 2. Optimal allocations and optimal costs for each model.

Unit Shortage Costs	Model C			Model N		
	S_{3C}	S_{6C}	Z_C	S_{3N}	S_{6N}	Z_N
$g_1 = 1, g_2 = 1, g_{12} = 1$	8.3333	16.6667	8.5318	5.2310	19.7690	0.3868
$g_1 = 20, g_2 = 10, g_{12} = 15$	9.1207	15.8793	91.3565	6.1513	18.8487	4.8524

When the unit shortage costs are the same, the allocations are $S_{3C} = 8.3333$ and $S_{6C} = 16.6667$ and, when the unit shortage costs are different, then $S_{3C} = 9.1207$ and $S_{6C} = 15.8793$. Recalling that S_{3C} components are used to make Product 1 and S_{6C} components are used to make Product 2, we see that allocation S_{3C} has increased by 9.45% while allocation S_{6C} has decreased by 4.72%. This is because Product 1 has a higher shortage cost than Product 2. Thus, a higher unit shortage cost induces a higher allocation. The relative gain of S_{3C} and the relative reduction of S_{6C} are not equal because the demand parameters are different.

Effect of g_1

To answer the question whether the shortage cost g_1 affects the sizes of relative gain and relative reduction, we gave g_1 different values and obtained the results shown in Table 3.

As the shortage cost of component S_{3C} increases while the shortage cost of component S_{6C} remains constant, both the relative gain and the relative reduction increase. This intuitively makes sense as to avoid a shortage situation, a product that is more and more expensive would be allocated more and more components.

Table 3. Effect of g_1 on relative gain of S_{3C} and relative reduction of S_{6C}.

Shortage Cost g_1	10	20	30	40	50	60	70	80	90	100
% gain of S_{3C}	0.00	3.56	5.64	7.11	8.25	9.18	9.97	10.65	11.25	11.79
% reduction of S_{6C}	0.00	−1.78	−2.82	−3.56	−4.13	−4.59	−4.99	−5.33	−5.63	−5.89

The next three figures further compare our results with those of previous research. Figures 3–5 show that, as g_1 increases, allocation S_{3C} increases and then becomes constant; allocation S_{6C} decreases and then becomes constant; and the cost increases to become constant. The quantities S_{3C}, S_{6C}, and Z_C are always constant if the shortage costs are the same as in previous research.

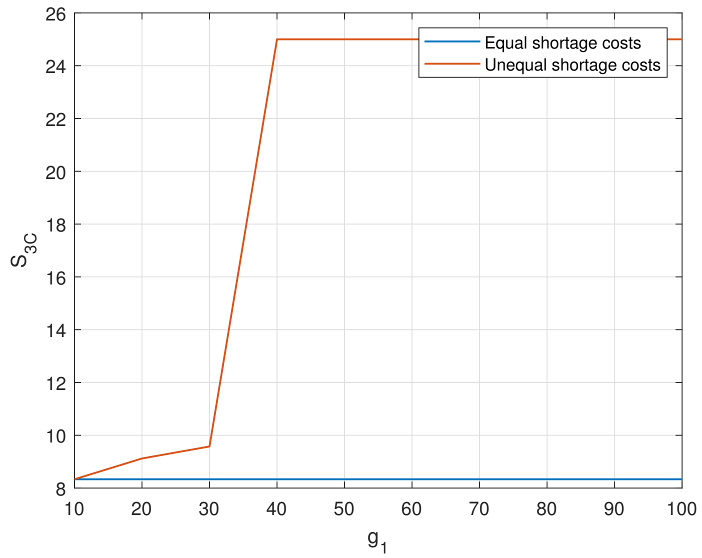

Figure 3. Effect of g_1 on S_{3C}.

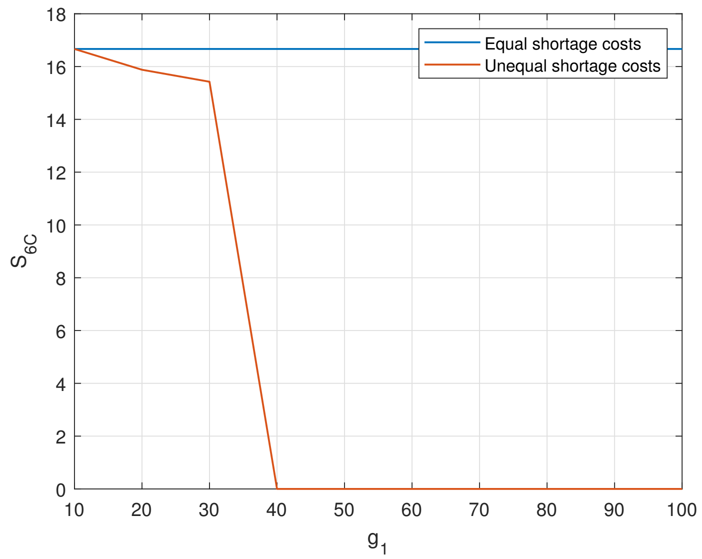

Figure 4. Effect of g_1 on S_{6C}.

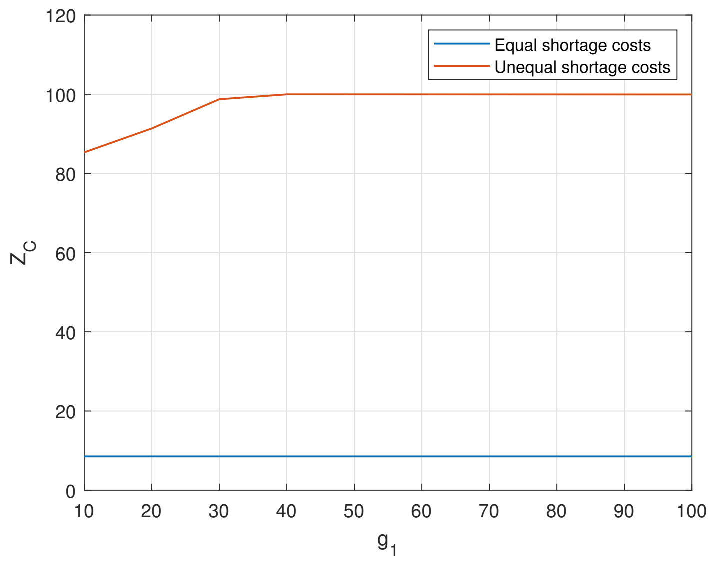

Figure 5. Effect of g_1 on Z_C.

7. Managerial Aspects and Implications

Some of the managerial implications of the commonality of components to manufacturing the different products as presented in this research paper could impact the industry in general and any manufacturing organization in particular, which are explained as follows:

1. The commonality of the components to remanufacturing the products reduces the inventory carrying cost. The quantity and variety of the parts to be kept and maintained in the warehouse reduces to a greater extent as compared to non-commonality of the parts and components. This will significantly improve the commercial viability of a firm as inventory carrying cost will be reduced to a greater extent.
2. The movement of the common components would also be faster as most of the products would be using the same component. Thus, the probability of an item becoming dead stock becomes negligible even if some products of the product line of a firm are not in high demand.
3. If no commonality, then inventory management and control shall require extra efforts and different means. That further results into more difficulty in managing components and extra cost. Thus, commonality can provide a competitive edge to a firm in the era of globalization.
4. The commonality of the components can reduce the requirement of extra inventory. This may boost the manufacturing companies to implement the concept of "Just-in-Time", which may further result into extra profit margins to a firm.
5. The shortage of common components can block the production of many products. The commonality of an item thus may result in higher shortage cost. This paper has tried to find out this shortage impact on any manufacturing firm. This impact may further be extended to the service industry. The shortage of any common component is difficult to afford. Thus, the common component becomes the critical component for the firm.
6. This research paper has tried to find out some measures and solutions to the shortage problem with the help of quantitative techniques. This paper is able to develop a mathematical solution to achieve the desired objectives of maximum commonality of items and minimal inventory of components. Thus, it takes the manufacturing firm to a better position of efficient management and control of its inventory.
7. Technical, precise, and high accuracy components should not be made common for many final products. If something goes wrong with the common component, it could affect the production of many finished products. Since the chances of design error in simple components is lower, it can be afforded to make simple components common.

8. Conclusions

We have considered in this paper a standard two-product newsboy problem with a common component. The aim is to derive the optimal components allocation that minimizes the total expected shortage cost. Our objective differs from the objective of previous studies that minimize the shortage quantity instead of the shortage cost. We have assumed that the unit shortage costs are known with certainty while the demand rates for the two products are random variables.

It is customary to solve minimization problems with a nonlinear objective function and linear constraints by using the first-order optimality conditions to obtain critical points and the second-order optimality conditions to check the convexity of the objective function. This procedure fails when the objective function is non-convex, which is the case for our problem. This paper shows the way to deal with such a problem. The proximal multipliers method has been adapted to the mathematical problem and implemented in an illustrative example. The optimal components allocations are obtained numerically.

The augmented Lagrangian approach is not without limitations. For example, it is impossible to obtain the optimal components allocations in closed-form. In addition, sensitivity analysis on the system parameters can only be conducted numerically, as shown in the illustrative example.

For future research, the procedure applied in this paper could be applied to more complex product structures or to other non-convex mathematical problems. In addition, we have assumed that the total number T of components to allocate is constant and known with certainty. It may be worth investigating the case where T is a random variable.

Acknowledgments: The publication of this article was funded by the Qatar National Library.

Author Contributions: A.H. and L.T. conceived the investigations; L.T. wrote the first draft; A.H. analyzed the results; A.H. and L.T. edited and reviewed the last version of the paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Ahmadi, M.; Shavandi, H. Dynamic pricing in a production system with multiple demand classes. *Appl. Math. Model.* **2015**, *39*, 2332–2344. [CrossRef]
2. Teng, H.-M.; Hsu, P.-H.; Wee, P.-H. An optimization model for products with limited production quantity. *Appl. Math. Model.* **2015**, *39*, 1867–1874. [CrossRef]
3. Sarathi, G.P.; Sarmah, S.P.; Jenamani, M. An integrated revenue sharing and quantity discounts contract for coordinating a supply chain dealing with short life-cycle products. *Appl. Math. Model.* **2014**, *38*, 4120–4136. [CrossRef]
4. Yu, Y.; Zhu, J.; Wang, C. A newsvendor model with fuzzy price-dependent demand. *Appl. Math. Model.* **2013**, *37*, 2644–2661. [CrossRef]
5. Abdel-Malek, L.L.; Otegbeye, M. Separable programming/duality approach to solving the multi-product newsboy/gardener problem with linear constraints. *Appl. Math. Model.* **2013** *37*, 4497–4508. [CrossRef]
6. Pezeshki, Y.; Jokar, M.R.A.; Baboli, A.; Campagne, J.P. Coordination mechanism for capacity reservation by considering production time, production rate and order quantity. *Appl. Math. Model.* **2013**, *37*, 8796–8812. [CrossRef]
7. Rutenberg, D.P. Design commonality to reduce multi-item inventory: Optimal depth of a product line. *Oper. Res.* **1969**, *19*, 491–509. [CrossRef]
8. Rutenberg, D.P.; Shaftel, T. Product design: Subassemblies for multiple markets. *Manag. Sci.* **1971**, *18*, 220–231. [CrossRef]
9. Wickenberg, J.; Stamlin, R.; Persson, M.; Borjesson, S. Challenges for increasing component commonality in platforms. In Proceedings of the EuroMOT, Tampere, Finland, 19–20 September 2011.
10. Chern, C.-C.; Yang, I.-C. A heuristic master planning algorithm for supply chains that consider substitutions and commonalities. *Expert Syst. Appl.* **2011**, *38*, 14918–14934. [CrossRef]
11. Bernstein, F.; Kök, A.G.; Xie, L. The role of component commonality in product assortment decisions. *Manuf. Serv. Oper. Manag.* **2011**, *13*, 261–270. [CrossRef]

12. Subramanian, R.; Ferguson, M.E.; Toktay, L.B. Remanufacturing and the component commonality decision. *Prod. Oper. Manag.* **2013**, *22*, 36–53. [CrossRef]
13. Alizon, F.; Shooter, S.B.; Simpson, T.W. Improving an existing product family based on commonality/diversity, modularity, and cost. *Des. Stud.* **2007**, *28*, 387–409. [CrossRef]
14. Lin, Y.; Ma, S.; Liu, L. A multi-period inventory model of component commonality with lead time. In Proceedings of the IEEE International Conference on Service Operations and Logistics, and Informatics, Shanghai, China, 21–23 June 2006; pp. 352–357.
15. Kranenburg, A.A.; Van Houtum, G.J. Effect of commonality on spare parts provisioning costs for capital goods. *Int. J. Prod. Econ.* **2007**, *108*, 221–227. [CrossRef]
16. Wazed, M.A.; Ahmed, S.; Yusoff, N. Impacts of common components on production system in an uncertain environment. *Sci. Res. Essay* **2009**, *4*, 1505–1517.
17. Eynan, A.; Rosenblatt, M.J. The impact of component commonality on composite assembly policies. *Nav. Res. Logist.* **2007**, *54*, 615–622. [CrossRef]
18. Song, J.-S.; Zhao, Y. The value of component commonality in a dynamic inventory system with lead times. *Manuf. Serv. Oper. Manag.* **2009**, *11*, 493–508. [CrossRef]
19. Nonås, S.L. Finding and identifying optimal inventory levels for systems with common components. *Eur. J. Oper. Res.* **2009**, *193*, 98–119. [CrossRef]
20. Boysen, N.; Scholl, A. A general solution framework for component-commonality problems. *BuR Bus. Res.* **2009**, *2*, 86–106. [CrossRef]
21. Jans, R.; Degraeve, Z.; Schepens, L. Analysis of an industrial component commonality problem. *Eur. J. Oper. Res.* **2008**, *186*, 801–811. [CrossRef]
22. Altfeld, N.; Hinckeldeyn, J.; Kreutzfeldt, J.; Gust, P. Impacts on supply chain management through component commonality and postponement: A case study. In Proceedings of the International MultiConference of Engineers and Computer Scientists, Hong Kong, 16–18 March 2011.
23. Labro, E. The cost effect of component commonality. *Manuf. Serv. Oper. Manag.* **2004**, *6*, 358–367. [CrossRef]
24. Fixson, S.K. Modularity and commonality research: Past developments and future opportunities. *Concurr. Eng.* **2007**, *15*, 85–111. [CrossRef]
25. Wazed, M.A.; Ahmed, S.; Yusoff, N. Commonality models in manufacturing resources planning: State-of-the-art and future directions. *Eur. J. Sci. Res.* **2008**, *23*, 421–435.
26. Wazed, M.A.; Ahmed, S.; Yusoff, N. A review of manufacturing resources planning models under different uncertainties: State-of-the-art and future directions. *S. Afr. J. Ind. Eng.* **2010**, *21*, 17–33. [CrossRef]
27. Deflem, Y.; van Nieuwenhuyse, I. Optimal pooling of inventories with substitution: A literature review. *Rev. Bus. Econ.* **2011**, *56*, 345–374. [CrossRef]
28. Baker, K.R.; Magazine, M.J.; Nuttle, H.L.W. The effect of commonality on safety stocks in a simple inventory model. *Manag. Sci.* **1986**, *32*, 982–988. [CrossRef]
29. Gerchak, Y.; Magazine, M.J.; Gamble, A.B. Component commonality with service level requirements. *Manag. Sci.* **1988**, *34*, 753–760. [CrossRef]
30. Eynan, A.; Rosenblatt, M.J. Component commonality effects on inventory costs. *IIE Trans.* **1996**, *28*, 93–104. [CrossRef]
31. Eynan, A. The impact of demands' correlation on the effectiveness of component commonality. *Int. J. Prod. Res.* **1996**, *34*, 1581–1602. [CrossRef]
32. Jönsson, H.; Silver, E.A. Optimal and heuristic solutions for a simple common component inventory problem. *Eng. Costs Prod. Econ.* **1989a**, *16*, 257–267.
33. Jönsson, H.; Silver, E.A. Common component inventory problems with a budget constraint: Heuristics and upper bounds. *Eng. Costs Prod. Econ.* **1989b**, *18*, 71–81.
34. Jönsson, H.; Jornsten, K.; Silver, E.A. Application of the scenario aggregation approach to a two-stage stochastic, common component inventory problem with a budget constraint. *Eur. J. Oper. Res.* **1993**, *68*, 196–211. [CrossRef]
35. Fu, H.; Fong, D.K.H. A note on the convexity of the objective function for a simple common component inventory problem. *Int. J. Prod. Econ.* **1998**, *55*, 143–148. [CrossRef]
36. Fong, D.K.H.; Fu, H.; Li, Z. Efficiency in shortage reduction when using a more expensive common component. *Comput. Oper. Res.* **2004**, *31*, 123–138. [CrossRef]

37. Fu, H.; Ramasesh, R.; Fong, D.K.H. Component commonality and shortage reduction under a mixed Erlang distributed demand. *Int. J. Oper. Res.* **2006**, *3*, 36–46.
38. Deza, A.; Huang, K.; Liang, H.; Wang, X.J. On component commonality for periodic review assemble-to-order systems. *Ann. Oper. Res.* **2018**, *265* 29–46. [CrossRef]
39. Rockafellar, R.T. Augmented Lagrangians and applications of the proximal point algorithm in convex programming. *Math. Oper. Res.* **1976**, *1*, 97–116. [CrossRef]
40. Moreau, J.J. Proximité et dualité dans un espace de Hilbert. *Bulletin de la Société Mathématique de France* **1965**, *93*, 273–299. [CrossRef]
41. Martinet, B. *Algorithmes pour la Résolution de Problèmes d'Optimisation et de Mini-Max*. Thèse d'Etat, Université de Grenoble, Grenoble, France, 1972.
42. Rockafellar, R.T. The multiplier method of Hestenes and Powell applied to convex programming. *J. Optim. Theory Appl.* **1973**, *12*, 555–562. [CrossRef]
43. Hestenes, M.R. Multiplier and gradient methods. *J. Optim. Theory Appl.* **1969**, *4*, 303–320. [CrossRef]
44. Powell, M.J.D. *A Method for Nonlinear Constraints in Minimization Problems*; Fletcher, R., Ed.; Academic Press: New York, NY, USA, 1969.
45. Buys, J.D. Dual Algorithms for Constrained Optimization Problems. Ph.D. Thesis, University of Leiden, Leiden, The Netherlands, 1972.
46. Luenberger, D.G. *Introduction to Linear and Nonlinear Programming*, 2nd ed.; Addison-Wesley: Boston, MA, USA, 1984.
47. Rockafellar, R.T. A dual approach to solving nonlinear programming problems by unconstrained optimization. *Math. Program.* **1974a**, *5*, 354–373. [CrossRef]
48. Rockafellar, R.T. Augmented Lagrange multiplier function and duality in non convex programming. *SIAM J. Control* **1974b**, *12*, 268–285. [CrossRef]
49. Arrow, K.J.; Gould, F.J.; Howe, S.M. A general saddle point result for constrained optimization. *Math. Program.* **1973**, *5*, 225–234. [CrossRef]
50. Bertsekas, D.P. On the method of multipliers for convex programming. *IEEE Trans. Autom. Control* **1973**, *AC 20*, 385–388. [CrossRef]