Communication

Mixed Boundary Value Problems for the Elasticity System in Exterior Domains

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Abstract: We study the properties of solutions of the mixed Dirichlet–Robin and Neumann–Robin problems for the linear system of elasticity theory in the exterior of a compact set and the asymptotic behavior of solutions of these problems at infinity under the assumption that the energy integral with weight $|x|^a$ is finite for such solutions. We use the variational principle and depending on the value of the parameter a, obtain uniqueness (non-uniqueness) theorems of the mixed problems or present exact formulas for the dimension of the space of solutions.

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ be an unbounded domain, $\Omega = \mathbb{R}^n \setminus \overline{G}$ with the boundary $\partial\Omega \in C^1$, where G is a bounded simply connected domain (or a union of finitely many such domains) in $\mathbb{R}^n$, $n \geq 2$, $\Omega \cup \partial\Omega = \overline{\Omega}$ is the closure of Ω, $x = (x_1, \ldots, x_n)$, *and* $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$.

In the domain Ω, we consider the linear system of elasticity theory

$$Lu \equiv (Lu)_i = \sum_{j,k,h=1}^{n} \frac{\partial}{\partial x_k}\left(a_{kh}^{ij}\frac{\partial u_j}{\partial x_h}\right) = 0, \quad i = 1, \ldots, n. \tag{1}$$

Here and in what follows, we assume summation from 1 to n over repeated indices. We also assume that the coefficients are constant and the following conditions hold:

$$a_{kh}^{ij} = a_{hk}^{ji} = a_{ih}^{kj}, \quad \lambda_1|\xi|^2 \leq a_{kh}^{ij}\xi_k^i\xi_h^j \leq \lambda_2|\xi|^2,$$

where ξ is an arbitrary symmetric matrix, $\{\xi_k^i\}$, $\xi_k^i = \xi_i^k$, $|\xi|^2 = \xi_k^i\xi_k^i$, and λ_1, λ_2 are positive constants.

In this paper, we study the following mixed boundary value problems:

(A) Mixed Dirichlet–Robin problem: Find a vector–function u satisfying Equation (1) in Ω and the homogenous boundary conditions:

$$u\big|_{\Gamma_1} = 0, \quad (\sigma(u) + \tau(x)u)\big|_{\Gamma_2} = 0. \tag{2}$$

(B) Mixed Neumann–Robin problem: Find a vector–function u satisfying Equation (1) in Ω and the homogenous boundary conditions:

$$\sigma(u)\big|_{\Gamma_1} = 0, \quad (\sigma(u) + \tau(x)u)\big|_{\Gamma_2} = 0, \tag{3}$$

where $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \partial\Omega$, $\Gamma_1 \cap \Gamma_2 = \varnothing$, $mes_{n-1}\Gamma_1 \neq 0$, $u = (u_1, \ldots, u_n)$, $\sigma(u) = (\sigma_1(u), \ldots, \sigma_n(u))$, $\sigma_i(u) \equiv a_{kh}^{ij}\frac{\partial u_j}{\partial x_h}\nu_k$, $i = 1, \ldots, n$, $\nu = (\nu_1, \ldots, \nu_n)$ is the unit outward normal vector to $\partial\Omega$, $\tau(x)$ is an infinitely differentiable function on $\partial\Omega$ with uniformly bounded derivatives, and $\tau \geq 0$, $\tau \not\equiv 0$.

Boundary-value problems for the elasticity system in bounded domains have been well studied in Fichera's monograph [1]. In [2,3], Kondratiev and Oleinik established generalizations of the Korn and Hardy inequalities for bounded domains and a wide class of unbounded domains, by means of which the main boundary value problems for the elasticity system have been investigated.

In [4], new asymptotically sharp Korn and Korn-type inequalities are proved in thin curved domains with a non-constant thickness, which are used in studying the buckling of compressed shells, in particular, when calculating the critical buckling load. In [5], the authors considered shells with zero Gaussian curvature, namely shells with one principal curvature zero and the other one having a constant sign. The authors proved that the best constant in the first Korn inequality scales as thickness to the power $3/2$ for a wide range of boundary conditions at the thin edges of the shell. Their method is to prove, for each of the three mutually orthogonal two-dimensional cross-sections of the shell, a "first-and-a-half Korn inequality"—a hybrid between the classical first and second Korn inequalities.

We also note the work in [6], where the authors considered the general class of external cusps determined by combining the corresponding regions of John. For this class, weighted Korn inequalities are proved using simple reasoning.

As we know, when Ω is unbounded, it is also necessary to characterize the behavior of a solution at infinity. As a rule, this is done using a finiteness condition of the Dirichlet integral $D(u, \Omega)$ or the energy integral $E(u, \Omega)$, or a condition on the nature of the decay of the modulus of a solution as $|x| \to \infty$. Such conditions at infinity are natural and were studied by several authors (e.g., [2,3,7,8]).

In this note, we study the properties of generalized solutions of the mixed Dirichlet–Robin and Neumann–Robin problems for the elasticity system in the exterior of a compact set with a finiteness condition of the weighted energy integral:

$$E_a(u, \Omega) \equiv \int_\Omega |x|^a \sum_{i,j=1}^{n} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 dx < \infty, \quad a \in \mathbb{R}.$$

In various classes of unbounded domains with finite weighted energy (Dirichlet) integral, the authors of [9–20] studied uniqueness (non-uniqueness) problem and found the dimensions of the spaces of solutions of boundary value problems for the elasticity system and the biharmonic (polyharmonic) equation.

Developing the approach based on inequalities of Korn and Hardy type [2,3,8], we succeed here in obtaining a uniqueness (non-uniqueness) criterion for solutions of the Neumann–Robin problem for the elasticity system. To construct solutions, we use a variational method, that is, we minimize the corresponding functional in the class of admissible functions.

Notation: $C_0^\infty(\Omega)$ is the space of infinitely differentiable functions in Ω with compact support in Ω.

We denote by $H^1(\Omega, \Gamma), \Gamma \subset \overline{\Omega}$ the Sobolev space of functions in Ω obtained by the completion of $C^\infty(\overline{\Omega})$ vanishing in a neighborhood of Γ with respect to the norm

$$||u; H^1(\Omega, \Gamma)|| = \left(\int_\Omega \sum_{|\alpha| \leq 1} |\partial^\alpha u|^2 dx \right)^{1/2},$$

where $\partial^\alpha u = \partial^{|\alpha|} u / \partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi–index, $\alpha_j \geq 0$ are integers, and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. If $\Gamma = \emptyset$, we denote $H^1(\Omega, \Gamma)$ by $H^1(\Omega)$.

$\overset{\circ}{H}{}^1(\Omega)$ is the space of functions in Ω obtained by the completion of $C_0^\infty(\Omega)$ with respect to the norm $||u; H^1(\Omega)||$.

$\overset{\circ}{H}{}^1_{loc}(\Omega)$ is the space of functions in Ω obtained by the completion of $C_0^\infty(\Omega)$ with respect to the system of semi-norms $||u; H^1(G_0)||$, where $G_0 \subset \overline{\Omega}$ is any compact subset.

Let

$$D(u, \Omega) = \int_\Omega \sum_{i,j=1}^n \left(\frac{\partial u_i}{\partial x_j} \right)^2 dx, \quad E(u, \Omega) = \int_\Omega \sum_{i,j=1}^n \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 dx.$$

By the cone K in $\mathbb{R}^n$ with vertex at x_0, we mean a domain such that if, $x - x_0 \in K$, then $\lambda(x - x_0) \in K$ for all $\lambda > 0$. We assume that the origin $x_0 = 0$ lies outside $\overline{\Omega}$.

2. Definitions and Auxiliary Statements

Definition 1. *A solution of the system in Equation* (1) *in* Ω *is a vector-valued function* $u \in H^1_{loc}(\Omega)$ *such that, for any vector-valued function* $\varphi \in C_0^\infty(\Omega)$, *the following integral identity holds*

$$\int_\Omega a_{kh}^{ij} \frac{\partial u_j}{\partial x_h} \frac{\partial \varphi_i}{\partial x_k} dx = 0.$$

Lemma 1. *Let u be a solution of the system in Equation* (1) *in* Ω *such that* $E_a(u, \Omega) < \infty$. *Then,*

$$u(x) = P(x) + \sum_{\beta_0 < |\alpha| \leq \beta} \partial^\alpha \Gamma(x) C_\alpha + u^\beta(x), \ x \in \Omega, \tag{4}$$

where $P(x)$ *is a polynomial,* ord $P(x) \leq m = \max\{1, 1 - n/2 - a/2\}$, $\Gamma(x)$ *is the fundamental solution of the system in Equation* (1), $C_\alpha = $ const, $\beta_0 = 1 - n/2 + a/2$, $\beta \geq 0$ *is an integer, and the function* u^β *satisfies the estimate*

$$|\partial^\gamma u^\beta(x)| \leq C_{\gamma\beta}(a, u)|x|^{1-n-\beta-|\gamma|}$$

for every multi-index γ, $C_{\gamma\beta} = $ const.

Remark 1. *It is known [21] that there exists a fundamental solution* $\Gamma(x)$, *which for* $n > 2$ *has the following estimate*

$$|\partial^\alpha \Gamma(x)| \leq C(\alpha)|x|^{2-n-|\alpha|}, \quad C(\alpha) = \text{const}.$$

For $n = 2$, *the fundamental solution has a representation* $\Gamma(x) = S(x) \ln |x| + T(x)$, *where* $S(x)$ *and* $T(x)$ *are square matrices of order 2 whose entries are homogeneous functions of order zero [22].*

Proof of Lemma 1. Consider the vector-valued function $v(x) = \theta_N(x)u(x)$, where $\theta_N(x) = \theta(|x|/N)$, $\theta \in C^\infty(\mathbb{R}^n)$, $0 \leq \theta \leq 1$, $\theta(s) = 0$ for $s \leq 1$, $\theta(s) = 1$ for $s \geq 2$, and also $N \gg 1$ and $G \subset \{x : |x| < N\}$. We extend v to $\mathbb{R}^n$ by setting $v = 0$ on $\overline{G} = \mathbb{R}^n \setminus \Omega$. Then, the vector-valued function v belongs to $C^\infty(\mathbb{R}^n)$ and satisfies the system

$$Lv = F_i, \quad i = 1, \ldots, n,$$

where $F_i \in C_0^\infty(\mathbb{R}^n)$, supp $F_i \subset \{x : |x| < 2N\}$. It is easy to see that $E_a(v, \mathbb{R}^n) < \infty$. If $a + n \neq 0$, then Korn's inequality ([2], [§ 3, inequality (1)]) implies that $v(x) = w(x) + Ax$, where A is a constant skew-symmetric matrix and w satisfies $D_a(w, \Omega) < \infty$.

We can now use Theorem 1 of [7] since it is based on Lemma 2 of [7], which imposes no constraints on the sign of σ'. Hence, the expansion

$$w(x) = P_0(x) + \sum_{\beta_0 < |\alpha| \leq \beta} \partial^\alpha \Gamma(x) C_\alpha + w^\beta(x)$$

holds for each a, where $P_0(x)$ is a polynomial, ord $P_0(x) \leq \max\{1, 1 - n/2 - a/2\}$, $C_\alpha = \text{const}$, $\beta_0 = 1 - n/2 + a/2$ and

$$|\partial^\gamma w^\beta(x)| \leq C_{\gamma\beta} |x|^{1-n-\beta-|\gamma|}, \qquad C_{\gamma\beta} = \text{const}.$$

Hence, by the definition of v, we obtain Equation (4) with $P(x) = P_0(x) + Ax$.

Now, let $a + n = 0$. Then, for each $\varepsilon > 0$,

$$E_{-n-\varepsilon}(v, \mathbb{R}^n) \leq E_{-n}(v, \mathbb{R}^n) < \infty.$$

By Korn's inequality ([2], [§ 3, inequality (1)]), there exists a constant skew-symmetric matrix A such that

$$D_{-n-\varepsilon}(v - Ax, \mathbb{R}^n) \leq CE_{-n-\varepsilon}(v, \mathbb{R}^n) < \infty,$$

where the constant C is independent of $v(x)$. Hence, using Theorem 1 of [7], we get

$$v(x) - Ax = P_0(x) + \sum_{\beta_0 < |\alpha| \leq \beta} \partial^\alpha \Gamma(x) C_\alpha + v^\beta(x),$$

where $P_0(x)$ is a polynomial, ord $P_0(x) \leq 1$, $C_\alpha = \text{const}$, $\beta_0 = 1 - n/2 + a/2$ and

$$|\partial^\gamma v^\beta(x)| \leq C_{\gamma\beta} |x|^{1-n-\beta-|\gamma|}, \qquad C_{\gamma\beta} = \text{const}.$$

Thus,

$$v(x) - Ax = P_0(x) + \sum_{\beta_0 < |\alpha| \leq \beta} \partial^\alpha \Gamma(x) C_\alpha + v^\beta(x),$$

which proves the lemma for $a = -n$. $\square$

Lemma 2. *Let u be a solution of the system in Equation (1) in Ω such that $E_a(u, \Omega) < \infty$ for some $a \geq 0$. Then, for all $x \in \Omega$, the equality in Equation (4) holds with u^β, satisfying an estimate similar to that in Lemma 1; in addition, $P(x) = Ax + B$, where A is a constant skew-symmetric matrix and B is a constant vector.*

Proof. Let u be a solution of the system in Equation (1) in Ω. Then, by Lemma 1, we have

$$u(x) = P(x) + R(x),$$

where $P(x)$ is a polynomial, ord $P(x) \leq 1$, and

$$R(x) = \sum_{\beta_0 < |\alpha| \leq \beta} \partial^\alpha \Gamma(x) C_\alpha + u^\beta(x), \qquad R(x) = O(|x|^{2-n}), \qquad n > 2.$$

Let us prove that $P(x) = Ax + B$, where A is a constant skew-symmetric matrix and B is a constant vector. Obviously, if $E_a(u, \Omega) < \infty$ and $a \geq 0$, then $E(u, \Omega) < \infty$.

Assume that $n > 2$. It is easy to verify that $E(R(x), \Omega) < \infty$ for $n > 2$. Hence, $E(P(x), \Omega) < \infty$ by the triangle inequality.

Let $P(x) = Ax + B$, that is, $P_i(x) = \sum_{j=1}^{n} a_{ij}x_j + b_i$. Then,

$$E(P(x), \Omega) = \int_{\Omega} \sum_{i,j=1}^{n} (a_{ij} + a_{ji})^2 dx,$$

where a_{ij} are the entries of A. The last integral converges if and only if $a_{ij} = -a_{ji}$, that is, A is a constant skew-symmetric matrix.

We consider now the case $n = 2$. It is known [22] that $\Gamma(x) = S(x) \ln |x| + T(x)$, where $S(x)$ and $T(x)$ are (2×2)-matrices whose entries are homogeneous functions of order zero. Then, $\nabla \Gamma(x) = O(|x|^{-1} \ln |x|)$, and, therefore, $\nabla R(x) = O(|x|^{-1} \ln |x|)$.

Assume that there exists k and l such that $a_{kl} + a_{lk} \neq 0$. Then,

$$|e_{kl}(u)| = |a_{kl} + a_{lk} + O(|x|^{-1} \ln |x|)| \geq \frac{1}{2}|a_{kl} + a_{lk}| \quad \text{for} \quad |x| \gg 1.$$

Hence,

$$\infty > E(u, \Omega) = \int_{\Omega} \sum_{i,j=1}^{n} |e_{ij}(u)|^2 \, dx \geq \int_{\Omega} |e_{kl}(u)|^2 \, dx \geq \frac{1}{4} \int_{|x|>H} |a_{kl} + a_{lk}|^2 \, dx = \infty.$$

This contradiction demonstrates that $a_{kl} = -a_{lk}$ for all k and l, which completes the proof. $\square$

Definition 2. *A solution of the mixed Dirichlet–Robin problem in Equations (1) and (2) is a vector-valued function $u \in \overset{\circ}{H}^1_{loc}(\Omega, \Gamma_1)$, such that, for each vector-valued function $\varphi \in \overset{\circ}{H}^1_{loc}(\Omega, \Gamma_1) \cap C_0^{\infty}(\mathbb{R}^n)$, the following integral identity holds*

$$\int_{\Omega} a_{kh}^{ij} \frac{\partial u_j}{\partial x_h} \frac{\partial \varphi_i}{\partial x_k} \, dx + \int_{\Gamma_2} \tau u \, \varphi \, ds = 0. \tag{5}$$

Definition 3. *A solution of the mixed Neumann–Robin problem in Equations (1) and (3) is a vector-valued function $u \in \overset{\circ}{H}^1_{loc}(\Omega)$, such that the integral identity in Equation (5) holds for each vector-valued function $\varphi \in C_0^{\infty}(\mathbb{R}^n)$.*

3. Main Results

Theorem 1. *The mixed Dirichlet–Robin problem in Equations (1) and (2) with the condition $E(u, \Omega) < \infty$ has $n(n+1)/2$ linearly independent solutions if $n \geq 3$, and one linearly independent solution if $n = 2$.*

Theorem 2. *The mixed Dirichlet–Robin problem in Equations (1) and (2) with the condition $E_a(u, \Omega) < \infty$ has:*
   (i) only trivial solution for $n \leq a < \infty$;
   (ii) $n(n-1)/2$ linearly independent solutions for $n - 2 \leq a < n$;
   (iii) $n(n+1)/2$ linearly independent solutions for $-n \leq a < n - 2$; and
   (iv) $k(r, n)$ linearly independent solutions for $-2r - n \leq a < -2r - n + 2$, where

$$k(r, n) = \begin{cases} n \left(\binom{r+n-1}{n-1} + \binom{r+n-2}{n-1} \right), & \text{if} \quad n > 2, \\ 4r + 2, & \text{if} \quad n = 2, \end{cases}$$

and $r > 0$, $k(0, n) = n$; $\binom{r}{s}$ is binomial coefficient from r to s, $\binom{r}{s} = 0$ if $s > r$.

Theorem 3. *The mixed Neumann–Robin problem in Equations (1) and (3) with the condition $E(u, \Omega) < \infty$ has $n(n+1)/2$ linearly independent solutions if $n \geq 2$.*

Theorem 4. *The mixed Neumann–Robin problem in Equations (1) and (3) with the condition $E_a(u, \Omega) < \infty$ has*

(i) $n(n+1)/2$ linearly independent solutions for $-n \leq a < \infty$; and

(ii) $k(r,n)$ linearly independent solutions for $-2r - n \leq a < -2r - n + 2$, $n \geq 2$, where

$$k(r,n) = n\left(\binom{r+n-1}{n-1} + \binom{r+n-2}{n-1}\right), \quad r > 0,$$

$k(0,n) = n$; $\binom{r}{s}$ is binomial coefficient from r to s, $\binom{r}{s} = 0$ if $s > r$.

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