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On a Laminated Timoshenko Beam with Nonlinear Structural Damping

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Abstract: In the present work, we study a one-dimensional laminated Timoshenko beam with a single nonlinear structural damping due to interfacial slip. We use the multiplier method and some properties of convex functions to establish an explicit and general decay result. Interestingly, the result is established without any additional internal or boundary damping term and without imposing any restrictive growth assumption on the nonlinear term, provided the wave speeds of the first equations of the system are equal.

Keywords: laminated beams; nonlinear damping; general decay; multiplier method

1. Introduction

In the late 1990s, a model for structure of two identical beams with uniform thickness was developed by Hansen and Spies [1]. This model is called a laminated Timoshenko beam and is given by

$$\begin{aligned} &\rho w_{tt} + G(\psi - w_x)_x = 0, \\ &I_\rho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) = 0, \\ &3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s + 4\beta s_t = 0, \end{aligned} \tag{1}$$

where the terms w, ψ, s represent the transverse displacement, the rotation angle, and the amount of slip along the interface, respectively. The positive parameters $\rho, G, I_\rho, D, \gamma,$ and β are known as the density, shear stiffness, mass moment of inertia, flexural rigidity, adhesive stiffness, and adhesive (structural) damping at the interface, respectively. It is very close to the well-known classical Timoshenko system because the equations of motion modeling the system are derived under the assumption of the Timoshenko beam theory. In addition, the third equation (which contains the structural damping s_t) is coupled with the first two describing the dynamic of the interfacial slip. These structures are highly important and have gained massive popularity in engineering fields.

Stabilization of the system with various internal or boundary damping mechanisms has been the subject of research over the years. Specifically, an increasing interest has been developed to determine the asymptotic behavior of the system which are paramount in classifying the empirical observations of the engineers.

Let us start with the boundary stabilization. To the best of our knowledge, the first known result in this case was established by Wang et al. [2]. They considered (1) together with some mixed homogeneous boundary conditions, introduced a change of variable $\xi = 3s - \psi$, and thereby established an exponential decay result provided $\sqrt{\frac{\rho}{G}} \neq \sqrt{\frac{I_\rho}{D}}$. Tatar [3] established the same exponential result of [2] provided $\rho G < I_\rho$. Mustafa [4] also obtained a similar result under the condition of equality between velocities of wave propagation, that is

$$\frac{\rho}{G} = \frac{I_\rho}{D}. \tag{2}$$

We refer the reader to [5–8] and the references cited therein for some other results on boundary stabilization.

For viscoelastic damping (memory term), we mention the work of Lo and Tatar [9]. They considered system (1) with viscoelastic damping of the form $\int_0^t g(t-r)(3s_{xx} - \psi_{xx})(r)dr$ on the second equation and proved that the resulting system is exponentially stable provided the relaxation function g decays exponentially and (2) is satisfied, in addition to some conditions on the parameter G. Mustafa [10] improved the result in [9] by adopting minimal and general conditions on g. Consequently, he established explicit energy decay result. Some other results can be found in [11–15] and the references cited therein.

The thermal effect (classical or second sound) is another way of stabilizing the laminated beams. Apalara in [16] proved that the heat effect is sufficiently strong enough to stabilize laminated beams exponentially without any other damping term, provided (2) holds. A similar result was obtained by Apalara [17] for the second sound in the presence of a structural damping term.

For frictional damping on the three equations, we mention the work of Raposo [18]. Recently, Alves et al. [19] established an exponential decay for the laminated system with only structural damping, provided the wave velocities are equal. It is imperative to mention that when $s \equiv 0$, then the laminated beams reduced to the standard Timoshenko system, see [17,18]. We refer the reader to [19–24] for some other varios forms of damping mechanisms

In this paper, we consider a laminated Timoshenko beam with only a single source of dissipation in the form of a nonlinear interfacial slip

$$\begin{array}{ll}
\rho w_{tt} + G(\psi - w_x)_x = 0 & \text{in } (0,1) \times (0,\infty), \\
I_\rho(3s_{tt} - \psi_{tt}) - D(3s_{xx} - \psi_{xx}) - G(\psi - w_x) = 0 & \text{in } (0,1) \times (0,\infty), \\
3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - w_x) + 4\gamma s + 4\beta h(s_t) = 0 & \text{in } (0,1) \times (0,\infty), \\
w(x,0) = w_0,\ w_t(x,0) = w_1,\ \psi(x,0) = \psi_0,\ \psi_t(x,0) = \psi_1, & \text{in } (0,1), \\
s(x,0) = s_0,\ s_t(x,0) = s_1, & \text{in } (0,1), \\
w_x(0,t) = \psi(0,t) = s(0,t) = w(1,t) = \psi_x(1,t) = s_x(1,t) = 0, & \text{in } (0,\infty)
\end{array} \tag{3}$$

and discuss the general decay of the energy of the system under suitable assumption on the nonlinear term and coefficients of wave propagation speed. On the nonlinear term h, we assume, as in Lasiecka and Tataru [25], that it satisfies the following hypotheses:

(A1) $h : \mathbb{R} \to \mathbb{R}$ is a non-decreasing C^0-function such that there exist positive constants c_1, c_2, ϵ, and a strictly increasing function $H \in C^1([0, +\infty))$, with $H(0) = 0$, and H is linear or strictly convex C^2-function on $(0, \epsilon]$ such that

$$\begin{cases} y^2 + h^2(y) \le H^{-1}(yh(y)) \text{ for all } |y| \le \epsilon, \\ c_1|y| \le |h(y)| \le c_2|y| \text{ for all } |y| \ge \epsilon. \end{cases}$$

Remark 1.

1. *Hypothesis (A1) implies that* $yh(y) > 0$, *for all* $y \neq 0$.
2. *Lasiecka and Tataru in [25] used the monotonicity and continuity of* h *to establish the existence of* H *as defined in (A1).*

For completeness purpose, we introduce the following spaces

$$H_a^1 = \{\psi : \psi \in H^1(0,1) \ : \psi(0) = 0\}, \quad H_b^1 = \{\psi : \psi \in H^1(0,1) \ : \psi(1) = 0\},$$
$$H_{b\star}^2(0,1) = \{\psi \in H^2(0,1) \ : \psi_x(0) = 0\}, \quad H_{a\star}^2(0,1) = \{\psi \in H^2(0,1) \ : \psi_x(1) = 0\}$$

and state, without proof, the following existence and regularity result:

Proposition 1. *For* $\Phi = (w,u,z,v,s,y)^T;\ u = w_t,\ z = 3s - \psi,\ v = z_t,\ y = s_t$, *and assume that (A1) is satisfied. Then for all* $\Phi_0 \in H_b^1(0,1) \times L^2(0,1) \times \left[H_a^1(0,1) \times L^2(0,1)\right]^2$, *the system* (1) *has a unique global (weak) solution*

$$w \in C(\mathbb{R}_+; H_b^1(0,1)) \cap C^1(\mathbb{R}_+; L^2(0,1)), \quad z, s \in C(\mathbb{R}_+; H_a^1(0,1)) \cap C^1(\mathbb{R}_+; L^2(0,1)).$$

Moreover, if

$$\Phi_0 \in H_{b\star}^2(0,1) \cap H_b^1(0,1) \times H_b^1(0,1) \times \left[H_{a\star}^2(0,1) \cap H_a^1(0,1) \times H_a^1(0,1)\right]^2,$$

then the solution satisfies

$$w \in L^\infty(\mathbb{R}_+; H_{b\star}^2(0,1) \cap H_b^1(0,1)) \cap W^{1,\infty}(\mathbb{R}_+; H_b^1(0,1)) \cap W^{2,\infty}(\mathbb{R}_+; L^2(0,1))$$
$$z, s \in L^\infty(\mathbb{R}_+; H_{a\star}^2(0,1) \cap H_a^1(0,1)) \cap W^{1,\infty}(\mathbb{R}_+; H_a^1(0,1)) \cap W^{2,\infty}(\mathbb{R}_+; L^2(0,1)).$$

Remark 2. *This result can be proved using standard arguments such as nonlinear semi-group method (see [26]) or the Faedo–Galerkin method (see [27]).*

The rest of the paper is organized as follows. In Section 2, we state and prove some essential technical lemmas. We give our stability result in Sections 3. We use c throughout this paper to denote a generic positive constant, which may be different from line to line (even in the same line). Our work gives an adequate answer to the possibility of stabilizing a Timoshenko Laminated beam with single non-linear damping present only in the third equation with some appropriate conditions.

2. Technical Lemmas

This section is devoted to the statements and proofs of some essential technical lemmas, which are highly influential in proving our main result.

Lemma 1. *Let* (w, ψ, s) *be the solution of system* (3) *and assume (A1) holds. Then the energy functional defined by*

$$E(t) = \frac{1}{2}\int_0^1 \left[\rho w_t^2 + I_\rho(3s_t - \psi_t)^2 + D(3s_x - \psi_x)^2 + G(\psi - w_x)^2 + 3I_\rho s_t^2 + 3Ds_x^2 + 4\gamma s^2\right] dx \tag{4}$$

satisfies

$$E'(t) = -4\beta \int_0^1 s_t h(s_t) dx \leq 0. \tag{5}$$

Proof. By multiplying $(3)_1$ by w_t, $(3)_2$ by $(3s_t - \psi_t)$, and $(3)_3$ by s_t, then integrating over $(0,1)$, using integration by parts and the boundary conditions, we obtain

$$\frac{\rho}{2}\frac{d}{dt}\int_0^1 w_t^2 dx = G\int_0^1 (\psi - w_x) w_{tx} dx, \tag{6}$$

$$\frac{I_\rho}{2}\frac{d}{dt}\int_0^1 (3s_t - \psi_t)^2 dx + \frac{D}{2}\frac{d}{dt}\int_0^1 (3s_x - \psi_x)^2 dx = G\int_0^1 (\psi - w_x)(3s_t - \psi_t) dx, \tag{7}$$

$$\frac{3\rho}{2}\frac{d}{dt}\int_0^1 s_t^2 dx + \frac{3\rho}{2}\frac{d}{dt}\int_0^1 s_x^2 dx + 2\frac{d}{dt}\int_0^1 s^2 dx = -3G\int_0^1 s_t(\psi - w_x) dx - 4\beta\int_0^1 s_t h(s_t) dx. \tag{8}$$

The combination of (6) to (8) bearing in mind (4) and the fact that

$$\frac{G}{2}\frac{d}{dt}\int_0^1 (\psi - w_x)^2 dx = G\int_0^1 \psi_t(\psi - w_x) dx - G\int_0^1 w_{tx}(\psi - w_x) dx$$

gives (5). □

Lemma 2. *Let (w, ψ, s) be the solution of system (3). Then the functional*

$$F_1(t) := 3I_\rho \int_0^1 s s_t dx - 3\rho\int_0^1 s\int_0^x w_t(y) dy dx$$

satisfies, for any $\varepsilon_1 > 0$, the estimate

$$F_1'(t) \le -3D\int_0^1 s_x^2 dx - 3\gamma\int_0^1 s^2 dx + \varepsilon_1\int_0^1 w_t^2 dx + c\int_0^1 \left[\left(1 + \frac{1}{\varepsilon_1}\right) s_t^2 + \frac{1}{\epsilon_1} h^2(s_t)\right] dx. \tag{9}$$

Proof. Using the first and third equations in (3), we obtain

$$F_1'(t) := -3D\int_0^1 s_x^2 dx - 4\gamma\int_0^1 s^2 dx + 3I_\rho\int_0^1 s_t^2 dx - 4\beta\int_0^1 s h(s_t) dx - 3\rho\int_0^1 s_t\int_0^x w_t(y) dy dx. \tag{10}$$

Using Young's and Cauchy–Schwarz inequalities, the last two terms in (10), gives

$$-4\beta\int_0^1 s h(s_t) dx \le \gamma\int_0^1 s^2 dx + \frac{4\beta^2}{\gamma}\int_0^1 g^2(s_t) dx, \tag{11}$$

$$-3\rho\int_0^1 s_t\int_0^x w_t(y) dy dx \le \varepsilon_1\int_0^1 w_t^2 dx + \frac{9\rho^2}{4\varepsilon_1}\int_0^1 s_t^2 dx. \tag{12}$$

The substitution of (11) and (12) into (10), gives (6). □

Lemma 3. *Let (w, ψ, s) be the solution of system (3) and assume that $\frac{G}{\rho} = \frac{D}{I_\rho}$. Then the functional*

$$F_2(t) := -3\rho D\int_0^1 w_t s_x dx + 3I_\rho G\int_0^1 (\psi - w_x) s_t dx$$

satisfies, for any $\varepsilon_2 > 0$, the estimate

$$F_2'(t) \le -G^2\int_0^1 (\psi - w_x)^2 dx + \varepsilon_2\int_0^1 (3s_t - \psi_t)^2 dx + c\int_0^1 \left[s_x^2 + \left(1 + \frac{1}{\varepsilon_2}\right) s_t^2 + h^2(s_t)\right] dx. \tag{13}$$

Proof. Direct computations and the fact that $\psi_t = -(3s_t - \psi_t) + 3s_t$, yield

$$F_2'(t) = -3G^2 \int_0^1 (\psi - w_x)^2 dx - 4\gamma G \int_0^1 (\psi - w_x) s dx + 9 I_\rho G \int_0^1 s_t^2 dx - 3 I_\rho G \int_0^1 (3s_t - \psi_t) s_t dx - 4\beta G \int_0^1 (\psi - w_x) h(s_t) dx. \quad (14)$$

Using Young's and Poincaré's inequalities, we obtain

$$-4\gamma G \int_0^1 (\psi - w_x) s dx \leq G^2 \int_0^1 (\psi - w_x)^2 dx + 4\gamma^2 \int_0^1 s_x^2 dx \quad (15)$$

$$-4\beta G \int_0^1 (\psi - w_x) h(s_t) dx \leq G^2 \int_0^1 (\psi - w_x)^2 dx + 4\beta^2 \int_0^1 g^2(s_t) dx \quad (16)$$

$$-3 I_\rho G \int_0^1 (3s_t - \psi_t) s_t dx \leq \varepsilon_2 \int_0^1 (3s_t - s_t)^2 dx + \frac{9 I_\rho^2 G^2}{4\varepsilon_2} \int_0^1 s_t^2 dx. \quad (17)$$

By substituting (15)–(17) into (14), we end up with (13). □

Lemma 4. *Let (w, ψ, s) be the solution of system (3) and assume that $\frac{G}{\rho} = \frac{D}{I_\rho}$. Then the functional*

$$F_3(t) := -\rho D \int_0^1 (3s_x - \psi_x) w_t dx + 3 I_\rho G \int_0^1 (3s - \psi) s_t dx - I_\rho G \int_0^1 (3s_t - \psi_t) w_x dx$$

satisfies, for any $\varepsilon_2 > 0$, the estimate

$$F_3'(t) \leq -\frac{DG}{2} \int_0^1 (3s_x - \psi_x)^2 dx + \varepsilon_2 \int_0^1 (3s_t - \psi_t)^2 dx + c \int_0^1 s_x^2 dx + c \int_0^1 (\psi - w_x)^2 dx + c \int_0^1 \left(1 + \frac{1}{\varepsilon_2}\right) s_t^2 dx + c \int_0^1 h^2(s_t) dx. \quad (18)$$

Proof. Exploiting (3) and integrating by parts, we obtain

$$F_3'(t) = -DG \int_0^1 (3s_x - \psi_x)^2 dx + 3 I_\rho G \int_0^1 (3s_t - \psi_t) s_t dx - 3G^2 \int_0^1 (3s - \psi)(\psi - w_x) dx - 4\gamma G \int_0^1 (3s - \psi) s dx - 4\beta G \int_0^1 (3s - \psi) h(s_t) dx - G^2 \int_0^1 (\psi - w_x) w_x dx.$$

Using the fact that $w_x = -(\psi - w_x) - (3s - \psi) + 3s$, we end up with

$$F_3'(t) = -DG \int_0^1 (3s_x - \psi_x)^2 dx + 3 I_\rho G \int_0^1 s_t (3s_t - \psi_t) dx - 2G^2 \int_0^1 (3s - \psi)(\psi - w_x) dx - 4\gamma G \int_0^1 s(3s - \psi) dx - 4\beta G \int_0^1 s_t (3s - \psi) dx + G^2 \int_0^1 (\psi - w_x)^2 dx - 3G^2 \int_0^1 s(\psi - w_x) dx.$$

The use of Young's and Poincaré inequalities, similar to (15)–(17), yields estimate (18). □

Lemma 5. *Let (w, ψ, s) be the solution of system (3). Then the functional*

$$F_4(t) := -3\rho \int_0^1 w_t w dx - 3 I_\rho \int_0^1 s_t \psi dx$$

satisfies

$$F_4'(t) \leq -3\rho \int_0^1 w_t^2 dx + \frac{I_\rho}{4} \int_0^1 (3s_t - \psi_t)^2 dx + c \int_0^1 \left[(3s_x - \psi_x)^2 + (\psi - w_x)^2 + s_x^2 + s_t^2 + h^2(s_t) \right] dx. \quad (19)$$

Proof. Using (3) and the substitution of $\psi = -(3s - \psi) + 3s$ along with its appropriate derivatives, gives

$$\begin{aligned} F_4'(t) = &-3\rho \int_0^1 w_t^2 dx + 3G \int_0^1 (\psi - w_x)^2 dx + 3I_\rho \int_0^1 (3s_t - \psi_t) s_t dx + 9D \int_0^1 s_x^2 dx - 4\gamma \int_0^1 (3s - \psi) s dx \\ &+ 12\beta \int_0^1 s h(s_t) dx + 12\gamma \int_0^1 s^2 dx - 4\beta \int_0^1 (3s - \psi) h(s_t) dx - 3D \int_0^1 (3s_x - \psi_x) s_x dx. \end{aligned} \tag{20}$$

As in the previous lemmas, (19) follows thanks to Young's and Poincaré's inequalities. □

Lemma 6. *Let (w, ψ, s) be the solution of system (3). Then the functional*

$$F_5(t) := -I_\rho \int_0^1 (3s - \psi)(3s_t - \psi_t) dx$$

satisfies

$$F_5'(t) \le -I_\rho \int_0^1 (3s_t - \psi_t)^2 dx + c \int_0^1 \left[(3s_x - \psi_x)^2 + (\psi - w_x)^2 \right] dx. \tag{21}$$

Proof. Estimate (21) quickly follows thanks to system (3), integration by parts, Young's, and Poincaré inequalities. □

Lemma 7. *Let (w, ψ, s) be the solution of system (3). Then, for $N, N_i (i = 1 \cdots 3) > 0$ sufficiently large, the Lyapunov functional defined by*

$$\mathcal{F}(t) := NE(t) + N_1 F_1(t) + N_2 F_2(t) + N_3 F_3(t) + F_4(t) + F_5(t) \tag{22}$$

satisfies, for some positive constant k,

$$\mathcal{F}'(t) \le -kE(t) + c \int_0^1 (s_t^2 + h^2(s_t)) dx, \quad \forall t \ge 0. \tag{23}$$

Proof. Direct computations using (5), (9), (13), (18), (19), (21), and let

$$\varepsilon_1 = \frac{2\rho}{N_1}, \ \varepsilon_2 = \frac{I_\rho}{4(N_2 + N_3)},$$

give

$$\begin{aligned} \mathcal{F}'(t) \le &-\left[3DN_1 - cN_2 - cN_3 - c\right] \int_0^1 s_x^2 dx - 3\gamma N_1 \int_0^1 s^2 dx - \left[G^2 N_2 - cN_3 - c\right] \int_0^1 (\psi - w_x)^2 dx \\ &- \rho \int_0^1 w_t^2 dx - \left[\frac{DG}{2} N_3 - c\right] \int_0^1 (3s_x - \psi_x)^2 dx + c\left[N_1^2 + N_2 + N_3 + 1\right] \int_0^1 h^2(s_t) dx \\ &- \frac{I_\rho}{2} \int_0^1 (3s_t - \psi_t)^2 dx + c\left[N_1(1 + N_1) + \left(N_2 + (N_2 + N_3)^2\right) + 1\right] \int_0^1 s_t^2 dx. \end{aligned} \tag{24}$$

We now carefully choose the rest of the constants. First, we choose N_3 large so that

$$\frac{DG}{2} N_3 - c > 0.$$

Next, we pick N_2 large so that

$$G^2 N_2 - cN_3 - c > 0.$$

Finally, we select N_1 large so that

$$3DN_1 - cN_2 - cN_3 - c > 0.$$

Thus, we end up with

$$\mathcal{F}'(t) \leq -\alpha \int_0^1 \left[s_x^2 + s^2 + (\psi - w_x)^2 + w_t^2 + (3s_x - \psi_x)^2 + (3s_t - \psi_t)^2\right] dx + c \int_0^1 \left[s_t^2 + h^2(s_t)\right] dx$$

for some $\alpha > 0$. Using the energy functional defined by (4), we obtain (23), for some $k > 0$. □

Remark 3. *Choosing N large, it can easily be shown that $\mathcal{F} \sim E$ in the sense that there exist two positive constants a and b such that*

$$aE(t) \leq \mathcal{F}(t) \leq bE(t), \qquad \forall t \geq 0. \tag{25}$$

3. Stability Result

This section concerns the proof of our stability result, and it states as follows

Theorem 1. *Let (w, ψ, s) be the solution of system (3) and assume (A1) holds. Then there exist positive constants k_1, k_2, k_3, and ϵ_0 such that the solution of (1) satisfies*

$$E(t) \leq k_1 H_1^{-1}(k_2 t + k_3), \quad t \geq 0, \tag{26}$$

where

$$H_1(t) = \int_t^1 \frac{1}{H_0(y)} dy \ \text{ and } \ H_0(t) = tH'(\epsilon_0 t).$$

Proof. Using (23), we consider two cases:

Case I: H is linear. In this case, from (A1), we have

$$c_1'|y| \leq |h(y)| \leq c_2'|y| \quad \forall y \in \mathbb{R},$$

so,

$$h^2(y) \leq c_2' y h(y) \quad \forall y \in \mathbb{R}. \tag{27}$$

Thus, using (5), (23), and (27), we conclude that

$$\mathcal{F}'(t) \leq -kE(t) + c \int_0^1 h(s_t) s_t dx. = -kE(t) - cE'(t) \quad \forall t \in \mathbb{R}^+.$$

So, by exploiting (25), it follows that

$$\mathcal{F}_0(t) := \mathcal{F}(t) + cE(t) \sim E(t). \tag{28}$$

satisfies, for some positive constant λ_1

$$\mathcal{F}_0'(t) \leq -\lambda_1 \mathcal{F}_0(t), \quad \forall t \geq 0.$$

Simple integration and using (28), yield

$$E(t) \leq E(0) e^{-\lambda_1 t}. \tag{29}$$

Case II: H is nonlinear on $[0, \epsilon]$. In this case, as in [25], we choose $0 < \epsilon_1 \leq \epsilon$ such that

$$yh(y) \leq \min\{\epsilon, H(\epsilon)\}, \quad \forall |y| \leq \epsilon_1.$$

Using (A1) and the continuity of h along the fact that $|h(y)| > 0$, for $s \neq 0$, we deduce that

$$\begin{cases} y^2 + h^2(y) \leq \ H^{-1}(yh(y)), \ \ \forall |y| \leq \epsilon_1, \\ c_1|y| \leq |h(y)| \leq c_2|y|, \quad \ \forall |y| \geq \epsilon_1. \end{cases} \tag{30}$$

We now shift our attention to the last integral in (23):

$$\int_0^1 (s_t^2 + h^2(s_t))dx.$$

To estimate this integral, we consider, as in [28], the following partition:

$$I_1 = \{x \in (0,1) : \ |s_t| \leq \epsilon_1\}, \qquad I_2 = \{x \in (0,1) : \ |s_t| > \epsilon_1\}.$$

Thus, with $I(t)$ defined by

$$I(t) = \int_{I_1} s_t h(s_t)dx,$$

we obtain, thanks to Jensen inequality and the fact that H^{-1} is concave

$$H^{-1}(I(t)) \geq c \int_{I_1} H^{-1}\left(s_t h(s_t)\right) dx. \tag{31}$$

Hence, using (30) and (31), we end up with

$$\begin{aligned} \int_0^1 \left(s_t^2 + h^2(s_t)\right) dx &= \int_{I_1} \left(s_t^2 + h^2(s_t)\right) dx + \int_{I_2} \left(s_t^2 + h^2(s_t)\right) dx \\ &\leq \int_{I_1} H^{-1}\left(s_t h(s_t)\right) dx + c\int_{I_2} s_t h(s_t)dx \\ &\leq cH^{-1}(I(t)) - cE'(t). \end{aligned} \tag{32}$$

The substitution of (32) into (23) and using (28), we have

$$\mathcal{F}_0(t) \leq -kE(t) + cH^{-1}(I(t)) \quad \forall t \in \mathbb{R}^+. \tag{33}$$

Now, for $\epsilon_0 < \epsilon$ and $\delta_0 > 0$, using (33) and the following properties of E and H:

$$E' \leq 0, \ H' > 0, \ H'' > 0 \text{ on } (0, \epsilon],$$

we deduce that the functional $\mathcal{F}_1$, defined by

$$\mathcal{F}_1(t) := H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) \mathcal{F}_0(t) + \delta_0 E(t),$$

satisfies, for some $c_1, c_2 > 0$,

$$c_1\mathcal{F}_1(t) \leq E(t) \leq c_2\mathcal{F}_1(t) \tag{34}$$

and

$$\begin{aligned} \mathcal{F}_1(t) :=& \epsilon_0 \frac{E'(t)}{E(0)} H''\left(\epsilon_0 \frac{E(t)}{E(0)}\right) \mathcal{F}_0(t) + H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) \mathcal{F}'(t) + \delta_0 E'(t) \\ \leq & -kE(t)H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + \underbrace{cH'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) H^{-1}\left(I(t)\right)}_{T} + \delta_0 E'(t). \end{aligned} \tag{35}$$

In order to estimate T in (35), we let H^* be the convex conjugate of H defined by

$$H^*(y) = y(H')^{-1}(y) - H[(H')^{-1}(y)] \leq y(H')^{-1}(y), \quad \text{if } y \in (0, H'(\epsilon)], \tag{36}$$

then, using the general Young's inequality

$$AB \leq H^*(A) + H(B), \quad \text{if } A \in (0, H'(\epsilon)], B \in (0, \epsilon]$$

for $A = H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right)$ and $B = H^{-1}(I(t))$, we obtain

$$cH'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) H^{-1}(I(t)) \leq cH^*\left(H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right)\right) + cI(t).$$

By using the energy functional (4) and (36), we end up with

$$cH'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) H^{-1}(I(t)) \leq c\epsilon_0 \frac{E(t)}{E(0)} H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) - cE'(t). \tag{37}$$

Combining (35) and (37), we obtain

$$\begin{aligned}
\mathcal{F}_1(t) \leq & -kE(t)H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + c\epsilon_0\beta \frac{E(t)}{E(0)} H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) - cE'(t) + \delta_0 E'(t) \\
\leq & -(kE(0) - c\epsilon_0) \frac{E(t)}{E(0)} H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) + (\delta_0 - c)E'(t).
\end{aligned}$$

Letting $\epsilon_0 = \frac{k}{2c} E(0)$, $\delta_0 = 2c$, and using the fact that $E'(t) \leq 0$, we get

$$\mathcal{F}_1(t) \leq -a_1 \frac{E(t)}{E(0)} H'\left(\epsilon_0 \frac{E(t)}{E(0)}\right) = -a_1 H_0\left(\frac{E(t)}{E(0)}\right), \tag{38}$$

where $a_1 > 0$ and $H_0(t) = tH'(\epsilon_0 t)$.

So, using the fact that H is strictly convex on $(0, \epsilon]$, we find that $H_0(t),\ H_0'(t) > 0$ on $(0, 1]$. Hence, with $\widetilde{\mathcal{F}}(t) = \frac{c_1 \mathcal{F}_1(t)}{E(0)}$ and using (34) and (38), we have

$$\widetilde{\mathcal{F}}(t) \sim E(t) \tag{39}$$

and, for some $k_2 > 0$,

$$\widetilde{\mathcal{F}}'(t) \leq -k_2 H_0\left(\widetilde{\mathcal{F}}(t)\right). \tag{40}$$

Inequality (40) implies that $\left[H_1\left(\widetilde{\mathcal{F}}(t)\right)\right]' \geq k_2$, where

$$H_1(t) = \int_t^1 \frac{1}{H_0(y)} dy.$$

Thus, by integrating over $[0, t]$, bearing in mind the properties of H_0, and the fact that H_1 is strictly decreasing on $(0, 1]$ we obtain, for some $k_3 > 0$,

$$\widetilde{\mathcal{F}}(t) \leq H_1^{-1}(k_2 t + k_3) \quad \forall t \in \mathbb{R}^+. \tag{41}$$

We complete the proof of Theorem 1 by using (39) and (41). □

4. Concluding Remarks

In this work, as in [19], we show (see (29)) that the structural damping is strong enough to exponentially stabilize the laminated Timoshenko beam system provided the wave speeds of the first two equations of the system are equal. A similar result was recently obtained in [29] when frictional damping is acting on the second equation. For the nonlinear case, the result is more general. For this case (nonlinear), it is an interesting open problem to investigate the case when the wave speeds of the first two equations of the system are not equal.

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