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Article

Numerical and Theoretical Stability Study of a Viscoelastic Plate Equation with Nonlinear Frictional Damping Term and a Logarithmic Source Term

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Abstract: This paper is designed to explore the asymptotic behaviour of a two dimensional visco-elastic plate equation with a logarithmic nonlinearity under the influence of nonlinear frictional damping. Assuming that relaxation function g satisfies $g'(t) \le -\xi(t)\mathbb{G}(g(t))$, we establish an explicit general decay rates without imposing a restrictive growth assumption on the damping term. This general condition allows us to recover the exponential and polynomial rates. Our results improve and extend some existing results in the literature. We preform some numerical experiments to illustrate our theoretical results.

Keywords: plate equation; viscoelasticity; general decay; nonlinear frictional damping; numerical computations

MSC: 35B35; 35L55; 75D05; 74D10; 93D20

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1. Introduction

Denote Ω to be an open bounded domain of $\mathbb{R}^2$ having a smooth boundary $\partial\Omega$. Let $\boldsymbol{n}$ stands for the unit outer normal to $\partial\Omega$. We then consider the following plate model:

$$\begin{cases} u_{tt} + \Delta^2 u + u - \int_0^t g(t-\tau)\Delta^2 u(\tau)d\tau + h(u_t) = ku\ln|u|, & x \in \Omega, \quad t > 0 \\ u(x,t) = \frac{\partial u}{\partial \boldsymbol{n}}(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad t > 0. \end{cases} \tag{1}$$

In this model, the parameter g is assumed to be a positive and decreasing function while k is taken to be small positive real number. After the pionnering work of Dafermos [1], many authors have continue to explore visco-elastic models with various kinds of nonlinearities and damping effects. Lagnese [2] showed that the energy decays to zero as time goes to infinity by the introduction of a dissipative mechanism on the boundary of the system. Besides, Rivera et al. [3] showed that, if the memory kernel decays exponentially as well, then both the first and second order energy related to the solutions of the viscoelastic plate equation decay exponentially. Komornik later in [4] investigated the energy decay while assuming a weak growth assumption. Furthermore, Messaoudi [5] for the following problem,

$$\begin{cases} u_{tt} + \Delta^2 u + |u_t|^{m-2}u_t = |u|^{p-2}u, & \text{in } Q_T = \Omega \times (0,T), \\ u = \frac{\partial u}{\partial n} = 0, & \text{on } \Gamma_T = \partial\Omega \times [0,T), \\ u(x,0) = u_0(x), \; u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{2}$$

developed an existence result and further demonstrated that the solution exists globally if $m \geq p$. However, this solution blows up in finite time provided $m < p$ and the initial energy is negative. Chen and Zhou [6] later improved the result in [5]. The importance of nonlinearity cannot be overemphasized, it occurs naturally in many fields especially in nuclear physics and quantum mechanics [7,8]. In the earliest work of Birula and Mycielski [9], they considered the following problem:

$$\begin{cases} u_{tt} - u_{xx} + u - \lambda u \ln |u|^2 = 0, & (x,t) \in [a,b] \times (0,T), \\ u(a,t) = u(b,t) = 0, \quad t \in (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in [a,b]. \end{cases} \tag{3}$$

The authors demonstrated that in any dimensions, wave equations with this nonlinearity have localized and stable soliton-like solutions. Cazenave and Haraux in [10] established the wellposedness of the associated Cauchy problem of

$$u_{tt} - \Delta u = u \ln |u|^k, \text{ in } \mathbb{R}^3. \tag{4}$$

In the case of one-dimensional, Gorka [11] used some compactness results to obtain the global existence of weak solutions to the initial-boundary value problem of Equation (4). Still on logarithmic nonlinearity, Al-Gharabli and Messaoudi [12] proved the global existence and the exponential decay of solutions of the following plate equation:

$$\begin{cases} u_{tt} + \Delta^2 u + u + h(u_t) = ku \ln |u|, & x \in \Omega, \quad t > 0 \\ u(x,t) = \frac{\partial u}{\partial n}(x,t) = 0, & x \in \partial\Omega, \quad t > 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & t > 0. \end{cases} \tag{5}$$

For more recent works regarding nonlinearity, we refer [13–23]. For the relaxation function, Cavalcanti et al. [24], reported an exponential decay result using relaxation functions which satisfy,

$$-\xi_2 g(t) \geq g'(t) \geq -\xi_1 g(t), \quad t \geq 0.$$

In 2008, Messaoudi [25,26] generalized the decay rates permitting an extended class of relaxation functions. He considered a relaxation function that satisfy

$$-\xi(t)g(t) \geq g'(t), \ t \geq 0, \tag{6}$$

where $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-increasing differentiable function. Afterwards, relaxation functions satisfying

$$-\chi(g(t)) \geq g'(t), \tag{7}$$

with constraints imposed on χ, have been used by several authors. Please, see [27–29]. Al-Gharabli et al. [30] considered the following problem:

$$u_{tt} + \Delta^2 u + u - \int_0^t g(t-\tau)\Delta^2 u(\tau) d\tau + h(u_t) = ku \ln |u|.$$

They proved an existence and decay results of the solutions under the condition that the relaxation function g satisfies

$$-\xi(t)g^p(t) \geq g'(t), \ \forall t \geq 0, \ 1 \leq p < \frac{3}{2}. \tag{8}$$

After extensive studies of the literature on wave models with logarithmic nonlinearities, especially [11,14,29,31], we seek to extend this kind of nonlinear effects to a plate equation. However, we consider general dampings unlike the ones considered in [12,29,30]. It is worth to mention that even-though the logarithmic nonlinearities are not as strong as the polynomial nonlinearities, the method used for establishing the existence and stability results in the case of polynomial nonlinearities cannot be directly adopted. The remaining part of this introduction section contains some basic notations and preliminary results required for this work. Section 2 presents the global and local existence of the solutions to the problem. Our technical Lemmas and decay results are in Sections 3 and 4. Finally, Section 5 contains the numerical results.

Preliminaries

We denote $L^2(\Omega)$ and $H_0^2(\Omega)$ as the usual Lebesgue and Sobolev spaces respectively equipped with their usual scalar products and norms. Throughout this paper, unless specified, c represents a generic constant. The following assumptions are important for this work:

$(\mathbb{A}1)$ $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1 non-increasing function satisfying

$$g(0) > 0, \qquad 1 - \int_0^{+\infty} g(\tau)d\tau = \mu > 0, \tag{9}$$

and there exists a C^1 function $\mathbb{G} : (0,\infty) \to (0,\infty)$ that is linear or is strictly convex and strictly increasing C^2 function on $(0, s_1]$, $s_1 \le g(0)$, with $\mathbb{G}(0) = \mathbb{G}'(0) = 0$, such that

$$-\xi(t)\mathbb{G}(g(t)) \ge g'(t), \quad \forall t > 0, \tag{10}$$

where $\xi(t)$ is a positive non-increasing differentiable function.

$(\mathbb{A}2)$ $h : \mathbb{R} \to \mathbb{R}$ is a nondecreasing C^0 function and there exists a strictly increasing function $h_0 \in C^1(\mathbb{R}^+)$, with $h_0(0) = 0$, and c_1, c_2, λ such that

$$\begin{aligned} &h_0(|\tau|) \le |h(\tau)| \le h_0^{-1}(|\tau|) \quad \text{for all } |\tau| \le \lambda, \\ &c_1|\tau| \le |h(\tau)| \le c_2|\tau| \qquad\quad \text{for all } |\tau| \ge \lambda. \end{aligned} \tag{11}$$

We also assume that H, defined by $H(\tau) = \sqrt{\tau}h_0(\sqrt{\tau})$, is a strictly convex C^2 function on $(0, s_2]$, for some $s_2 > 0$, when h_0 is nonlinear.

$(\mathbb{A}3)$ The constant k in (1) satisfies $0 < k < k_0$, where k_0 is the positive real number satisfying:

$$\sqrt{\frac{2\pi\mu}{k_0 c_p}} = e^{-\frac{3}{2} - \frac{1}{k_0}}, \tag{12}$$

and c_p is the smallest positive number satisfying

$$\|\nabla u\|_2^2 \le c_p \|\Delta u\|_2^2, \quad \forall u \in H_0^2(\Omega),$$

where $\|.\|_2 = \|.\|_{L^2(\Omega)}$.

Remark 1. *If $\mathbb{G}$ is a strictly increasing and strictly convex C^2 function on $(0, s_1]$, with $\mathbb{G}(0) = \mathbb{G}'(0) = 0$, then it has an extension $\overline{\mathbb{G}}$, which is strictly increasing and strictly convex C^2 function on $(0, +\infty)$. For instance, if $\mathbb{G}(s_1) = a, \mathbb{G}'(s_1) = b, \mathbb{G}''(s_1) = C$, we can define $\overline{\mathbb{G}}$, for $t > s_1$, by*

$$\overline{\mathbb{G}}(t) = \frac{C}{2}t^2 + (b - Cs_1)t + \left(a + \frac{C}{2}s_1^2 - bs_1\lambda\right). \tag{13}$$

The energy functional of problem (1) is given by

$$\begin{aligned} \mathbb{E}(t) = \frac{1}{2}\left(\|u_t\|_2^2 + \left(1 - \int_0^t g(\tau)d\tau\right)\|\Delta u\|_2^2 + \frac{k+2}{2}\|u\|_2^2\right) \\ - \frac{1}{2}\int_\Omega u^2 \ln|u|^k dx + \frac{1}{2}(g \circ \Delta u)(t), \end{aligned} \tag{14}$$

where
$$(g \circ \Delta u)(t) = \int_0^t g(t-\tau)||\Delta u(t) - \Delta u(\tau)||_2^2 d\tau.$$

Differentiating (14) and using (1), leads to

$$\mathbb{E}'(t) = \frac{1}{2}(g' \circ \Delta u)(t) - \frac{1}{2}g(t)\|\Delta u\|_2^2 - \int_\Omega u_t h(u_t)dx \le 0. \tag{15}$$

Lemma 1 ([32,33])**.** *Let $u \in H_0^1(\Omega)$ and $b > 0$ be any number. Then*

$$\int_\Omega u^2 \ln|u|dx \le \frac{1}{2}\|u\|_2^2 \ln\|u\|_2^2 + \frac{b^2}{2\pi}\|\nabla u\|_2^2 - (1+\ln b)\|u\|_2^2. \tag{16}$$

Corollary 1. *Let $u \in H_0^2(\Omega)$ and $b > 0$ be any number. Then*

$$\int_\Omega u^2 \ln|u|dx \le \frac{1}{2}\|u\|_2^2 \ln\|u\|_2^2 + \frac{c_p b^2}{2\pi}\|\Delta u\|_2^2 - (1+\ln b)\|u\|_2^2. \tag{17}$$

2. Local and Global Existence

In this section, we present the existence results for problem (1) according to [12,30].

Theorem 1. *Suppose $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$. Then problem (1) has a local weak solution*

$$u \in C([0,T], H_0^2(\Omega)) \cap C^1([0,T], L^2(\Omega)) \cap C^2([0,T], H^{-2}(\Omega)). \tag{18}$$

We define the following functionals for the purpose of the global existence

$$\begin{aligned}\mathbb{J}(u) = \mathbb{J}(u(t)) = \frac{1}{2}\left(\left(1 - \int_0^t g(\tau)d\tau\right)\|\Delta u\|_2^2 + \|u\|_2^2 + (g \circ \Delta u)(t) - \int_\Omega u^2 \ln|u|^k dx\right) \\ + \frac{k}{4}\|u\|_2^2.\end{aligned} \tag{19}$$

$$\mathbb{I}(u) = \mathbb{I}(u(t)) = \left(1 - \int_0^t g(\tau)d\tau\right)\|\Delta u\|_2^2 + \|u\|_2^2 + (g \circ \Delta u)(t) - 3\int_\Omega u^2 \ln|u|^k dx. \tag{20}$$

It follows that

$$\mathbb{J}(t) = \frac{1}{3}\left[\left(1 - \int_0^t g(\tau)d\tau\right)\|\Delta u\|_2^2 + \|u\|_2^2 + (g \circ \Delta u)(t)\right] + \frac{k}{4}\|u\|_2^2 + \frac{1}{6}\mathbb{I}(t), \tag{21}$$

and
$$\mathbb{E}(t) = \mathbb{J}(t) + \frac{1}{2}\|u_t(t)\|_2^2.$$

Lemma 2 ([30])**.** *The inequalities below hold*

$$-kd_0\sqrt{|\Omega|c_*^3}\|\Delta u\|_2^{\frac{3}{2}} \le \int_\Omega u^2 \ln|u|^k dx \le kc_*^3\|\Delta u\|_2^3, \quad \forall u \in H_0^2(\Omega), \tag{22}$$

where $d_0 = \sup_{0<\tau<1}\sqrt{\tau}|\ln\tau|$, $|\Omega|$ is the Lebesgue measure of Ω and c_* is the smallest embedding constant
$$\left(\int_\Omega |u|^3 dx\right)^{\frac{1}{3}} \le c_*\|\Delta u\|_2, \quad \forall u \in H_0^2(\Omega). \tag{23}$$

Lemma 3 ([30])**.** *Let $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$. Suppose that $(\mathbb{A}1)$–$(\mathbb{A}3)$ hold such that*

$$\mathbb{I}(0) > 0 \textit{ and } \sqrt{54}kc_*^3\left(\frac{\mathbb{E}(0)}{\mu}\right)^{\frac{1}{2}} < \mu. \tag{24}$$

Hence,

$$0 < \mathbb{I}(t), \ \forall t \in [0, T). \tag{25}$$

Theorem 2 ([30])**.** *Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ and assume that $(\mathbb{A}1)$–$(\mathbb{A}3)$ and (24) hold. Then the solution of Problem (1) is global and bounded.*

3. Technical Lemmas

We now present some Technical lemmas that are fundamental requirements for our result.

Lemma 4 ([30])**.** *Suppose that g satisfies $(\mathbb{A}1)$. Then, for $u \in H_0^2(\Omega)$,*

$$\int_\Omega \left(\int_0^t g(t-\tau)(u(t) - u(\tau)) d\tau \right)^2 dx \leq c(g \circ \Delta u)(t)$$

and

$$\int_\Omega \left(\int_0^t g'(t-\tau)(u(t) - u(\tau)) d\tau \right)^2 dx \leq -c(g' \circ \Delta u)(t).$$

Lemma 5. *The functional*

$$\Psi_1(t) = \int_\Omega u u_t dx,$$

satisfies (1), provided that $(\mathbb{A}1)$–$(\mathbb{A}3)$ and (24) hold, the following:

$$\Psi_1'(t) \leq ||u_t||_2^2 - \frac{\mu}{2} ||\Delta u||_2^2 - ||u||_2^2 + \int_\Omega u^2 \ln |u|^k dx + c(g \circ \Delta u)(t) + c \int_\Omega h^2(u_t) dx. \tag{26}$$

Proof. In view of Equation (1), we deduce that

$$\begin{aligned} \Psi_1' = ||u_t||_2^2 - ||\Delta u||_2^2 - ||u||_2^2 + \int_\Omega \Delta u \int_0^t g(t-\tau) \Delta u(\tau) d\tau dx \\ + \int_\Omega u^2 \ln |u|^k dx - \int_\Omega u h(u_t) dx. \end{aligned} \tag{27}$$

By using Lemma 4 and Young's inequality, we see that, for any $\rho > 0$,

$$\begin{aligned} &\int_\Omega \Delta u(t) \left(\int_0^t g(t-\tau) \Delta u(\tau) d\tau \right) dx \\ &\leq \left(1 - \mu + \frac{\rho}{2}\right) ||\Delta u||_2^2 + \frac{1}{2\rho}(1-\mu)(g \circ \Delta u)(t). \end{aligned} \tag{28}$$

Similarly, for any $\rho > 0$,

$$\begin{aligned} -\int_\Omega u h(u_t) dx &\leq \frac{\rho}{2} \int_\Omega u^2 dx + \frac{1}{2\rho} \int_\Omega h^2(u_t) dx \\ &\leq c_p \frac{\rho}{2} \int_\Omega |\Delta u|^2 dx + \frac{1}{2\rho} \int_\Omega h^2(u_t) dx. \end{aligned} \tag{29}$$

By choosing $\rho = \frac{\mu}{c_p + 1}$ and combining (27)–(29), we obtain (26). $\square$

Lemma 6. *The functional*

$$\Psi_2(t) = -\int_\Omega u_t \int_0^t g(t-\tau)(u(t) - u(\tau)) d\tau dx,$$

satisfies (1) *provided* ($\mathbb{A}1$)–($\mathbb{A}3$) *and* (24) *hold. Furthermore, for any* $\lambda_0 \in (0,1)$ *and* $\delta > 0$,

$$\Psi_2'(t) \leq \delta||\Delta u||_2^2 + \frac{c}{\delta}(g \circ \Delta u)(t) + \frac{c}{\delta}(-g' \circ \Delta u)(t) + \left(\delta - \int_0^t g(\tau)d\tau\right)\|u_t\|_2^2 + c_{\lambda_0,\delta}(g \circ \Delta u)^{\frac{1}{1+\lambda_0}}(t) + c\int_\Omega h^2(u_t(t))dx. \quad (30)$$

Proof. Direct computations, using (1), yield

$$\begin{aligned}\Psi_2'(t) = &\int_\Omega \Delta u \int_0^t g(t-\tau)(\Delta u(t) - \Delta u(\tau))d\tau dx + \int_\Omega u \int_0^t g(t-\tau)(u(t) - u(\tau))d\tau dx \\ &+ \int_\Omega \int_0^t g(t-\tau)(\Delta u(t) - \Delta u(\tau))d\tau \int_0^t g(t-\tau)\Delta u(\tau)d\tau dx \\ &- \int_\Omega u \ln|u|^k \int_0^t g(t-\tau)(u(t) - u(\tau))d\tau dx - \int_\Omega u_t \int_0^t g'(t-\tau)(u(t) - u(\tau))d\tau dx \\ &- \left(\int_0^t g(\tau)d\tau\right)\int_\Omega u_t^2 dx + \int_\Omega \left(\int_0^t g(t-\tau)(u(t) - u(\tau))d\tau\right)h(u_t)dx.\end{aligned} \quad (31)$$

Estimating the first term in right hand side of (31), we have for any $\delta > 0$,

$$\int_\Omega \Delta u \int_0^t g(t-\tau)(\Delta u(t) - \Delta u(\tau))d\tau dx \leq \frac{\delta}{4}||\Delta u||_2^2 + \frac{c}{\delta}(g \circ \Delta u)(t). \quad (32)$$

Applying Lemma 4, Young's and Poincaré's inequalities, the fifth and second terms in right hand side of (31) give rise to

$$\int_\Omega u \int_0^t g(t-\tau)(u(t) - u(\tau))d\tau dx \leq \frac{\delta}{4}||\Delta u||_2^2 + \frac{c}{\delta}(g \circ \Delta u)(t), \quad (33)$$

and

$$-\int_\Omega u_t \int_0^t g'(t-\tau)(u(t) - u(\tau))d\tau dx \leq \delta||u_t||_2^2 - \frac{c}{\delta}(g' \circ \Delta u)(t). \quad (34)$$

In similar manner, the estimate for the third term is as follows:

$$\begin{aligned}&\int_\Omega \int_0^t g(t-\tau)(\Delta u(t) - \Delta u(\tau))d\tau \int_0^t g(t-\tau)\Delta u(\tau)d\tau dx \\ &\leq \frac{\delta}{4}||\Delta u||_2^2 + c\left(1 + \frac{1}{\delta}\right)(g \circ \Delta u)(t),\end{aligned} \quad (35)$$

and

$$\int_\Omega \left(\int_0^t g(t-\tau)(u(t) - u(\tau))d\tau\right)h(u_t)dx \leq \frac{c}{\delta}(g \circ \nabla u)(t) + \delta\int_\Omega h^2(u_t)dx.$$

Let $\lambda_0 \in (0,1)$ and $\omega(\tau) = \tau^{\lambda_0}(|\ln \tau| - \tau)$. Then ω is continuous on $(0,\infty)$, $\lim_{\tau\to 0}\omega(\tau) = 0$, and $\lim_{\tau\to\infty}\omega(\tau) = 0$. Therefore, ω has a maximum d_{λ_0} on $[0,\infty)$, so the following inequality holds

$$\tau|\ln \tau| \leq \tau^2 + d_{\lambda_0}\tau^{1-\lambda_0},\ \forall \tau > 0. \quad (36)$$

In view of (36) and taking advantage of the embedding of $H_0^2(\Omega)$ in $L^\infty(\Omega)$, we for any $\delta_1 > 0$,

$$\begin{aligned}
&\int_\Omega u \ln|u|^k \int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx \\
&\leq k\int_\Omega \left(u^2 + d_{\lambda_0}|u|^{1-\lambda_0}\right)\left|\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx\right| \\
&\leq c\int_\Omega |u|^2\left|\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\right|dx \\
&+\delta_1\int_\Omega u^2dx + c_{\lambda_0,\delta_1}\int_\Omega\left|\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\right|^{\frac{2}{1+\lambda_0}}dx \\
&\leq c\delta_1||\Delta u||_2^2 + \frac{c}{\delta_1}\int_\Omega\left|\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\right|^2 dx \\
&+c_{\lambda_0,\delta_1}\int_\Omega\left|\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau\right|^{\frac{2}{1+\lambda_0}}dx.
\end{aligned}$$

Taking $\frac{\delta}{4} = c\delta_1$, then applying Lemma 4 and Hölder's inequality, yield

$$\begin{aligned}
\int_\Omega u\ln|u|^k\int_0^t g(t-\tau)(u(t)-u(\tau))d\tau dx \leq \frac{\delta}{4}||\Delta u||_2^2 + \frac{c}{\delta}(g\circ\Delta u)(t) \\
+ c_{\lambda_0,\delta}(g\circ\Delta u)^{\frac{1}{1+\lambda_0}}(t).
\end{aligned} \tag{37}$$

Equation (30) then follows from the last inequality above. □

Lemma 7. *Let $\lambda_0 \in (0,1)$ and assume that $(\mathbb{A}1)$–$(\mathbb{A}3)$ and (24) hold. Then, provided k is small enough, there exist λ_1 and λ_2, two positive constants such that the functional*

$$\mathbb{L}(t) = \mathbb{E}(t) + \lambda_1\Psi_1(t) + \lambda_2\Psi_2(t)$$

satisfies

$$\mathbb{L} \sim \mathbb{E}, \tag{38}$$

and there exists a positive constant d such that

$$\mathbb{L}'(t) \leq -d\mathbb{E}(t) + c(g\circ\Delta u)(t) + c_{\lambda_0}(g\circ\Delta u)^{\frac{1}{1+\lambda_0}}(t) + c\int_\Omega h^2(u_t(t))dx, \quad \forall t\geq t_0. \tag{39}$$

Proof. The proof of (38) is straight forward. To prove the inequality (39) we use the assumptions that the relaxation g is positive and $g(0) > 0$. So, for any $t_0 > 0$,

$$\int_0^t g(\tau)d\tau \geq \int_0^{t_0} g(\tau)d\tau = g_0 > 0,\ \forall t \geq t_0.$$

In view of (15), (26), (30) and the definition of $\mathbb{E}(t)$, then, for $t \geq t_0$ and any $d > 0$, we have

$$\begin{aligned}
\mathbb{L}'(t) \leq& -d\mathbb{E}(t) - \left(\lambda_2(g_0-\delta) - \lambda_1 - \frac{d}{2}\right)\|u_t\|_2^2 \\
&-\left(\frac{\mu}{2}\lambda_1 - \lambda_2\delta - \frac{d}{2}\right)\|\Delta u\|_2^2 - \left(\lambda_1 - \frac{(k+2)d}{4}\right)\|u\|_2^2 \\
&+\left(k\lambda_1 - k\frac{d}{2}\right)\int_\Omega u^2\ln|u|dx + \left(c\lambda_1 + \lambda_2\frac{c}{\delta} + \frac{d}{2}\right)(g\circ\Delta u)(t) \\
&+\left(\frac{1}{2} - \frac{c\lambda_2}{\delta}\right)(g'\circ\Delta u)(t) + \lambda_2 c_{\lambda_0,\delta}(g\circ\Delta u)^{\frac{1}{1+\lambda_0}}(t) + c(\lambda_1+\lambda_2)\int_\Omega h^2(u_t(t))dx.
\end{aligned} \tag{40}$$

Applying the Logarithmic Sobolev inequality, for $d \in (0, 2\lambda_1)$, we get

$$\begin{aligned} \mathbb{L}'(t) \leq & -d\mathbb{E}(t) - \left(\lambda_2(g_0 - \delta) - \lambda_1 - \frac{d}{2}\right)\|u_t\|_2^2 \\ & - \left(\frac{\mu}{2}\lambda_1 - \lambda_2\delta - \frac{d}{2} - k\left(\lambda_1 - \frac{d}{2}\right)\frac{c_p a^2}{2\pi}\right)\|\Delta u\|_2^2 \\ & - \left(\lambda_1 - \frac{d(k+2)}{4} + k\left(\lambda_1 - \frac{d}{2}\right)(1 + \ln a) + k\left(\frac{d}{4} - \frac{\lambda_1}{2}\right)\ln\|u\|_2^2\right)\|u\|_2^2 \\ & + \left(c\lambda_1 + \lambda_2\frac{c}{\delta} + \frac{d}{2}\right)(g \circ \Delta u)(t) + c(\lambda_1 + \lambda_2)\int_\Omega h^2(u_t(t))dx \\ & + \left(\frac{1}{2} - \frac{c\lambda_2}{\delta}\right)(g' \circ \Delta u)(t) + \lambda_2 c_{\lambda_0,\delta}(g \circ \Delta u)^{\frac{1}{1+\lambda_0}}(t). \end{aligned} \tag{41}$$

We then choose δ very small that

$$g_0 - \delta > \frac{1}{2}g_0 \quad \text{and} \quad \delta < \frac{\mu g_0}{16}.$$

Provided δ is fixed, the choice of any two positive constants λ_1 and λ_2 satisfying

$$\frac{g_0}{4}\lambda_2 < \lambda_1 < \frac{g_0}{2}\lambda_2 \tag{42}$$

will make

$$k_1 := \lambda_2(g_0 - \delta) - \lambda_1 > 0 \quad \text{and} \quad k_2 := \frac{\mu}{2}\lambda_1 - \lambda_2\delta > 0.$$

Then, we choose λ_1 and λ_2 very small so that (38) and (42) remain true, and

$$\frac{1}{2} - \frac{c\lambda_2}{\delta} > 0.$$

As a result, we get (38) and

$$\begin{aligned} \mathbb{L}'(t) \leq & -d\mathbb{E}(t) - \left(k_1 - \frac{d}{2}\right)\|u_t\|_2^2 \\ & - \left(k_2 - \frac{d}{2} - k\left(\lambda_1 - \frac{d}{2}\right)\frac{c_p a^2}{2\pi}\right)\|\Delta u\|_2^2 \\ & - \left(\lambda_1 - \frac{d(k+2)}{4} + k\left(\lambda_1 - \frac{d}{2}\right)(1 + \ln a) + k\left(\frac{d}{4} - \frac{\lambda_1}{2}\right)\ln\|u\|_2^2\right)\|u\|_2^2 \\ & + c(g \circ \Delta u)(t) + c_{\lambda_0,\delta}(g \circ \Delta u)^{\frac{1}{1+\lambda_0}}(t) + c(\lambda_1 + \lambda_2)\int_\Omega h^2(u_t(t))dx. \end{aligned} \tag{43}$$

Then, imposing the following condition on a

$$e^{-\frac{3}{2} - \frac{1}{k}} < a < \sqrt{\frac{2\pi\mu}{kc_p}},$$

and selecting d and k small enough so that

$$\alpha_1 = k_1 - \frac{d}{2} > 0, \; \alpha_2 = k_2 - \frac{d}{2} - k\left(\lambda_1 - \frac{d}{2}\right)\frac{c_p a^2}{2\pi} > 0$$

and

$$\alpha_3 = \lambda_1 - \frac{d(k+2)}{4} + k\left(\lambda_1 - \frac{d}{2}\right)(1 + \ln a) + k\left(\frac{d}{4} - \frac{\lambda_1}{2}\right)\ln\|u\|_2^2 > 0,$$

we arrive at the desired result (39). $\square$

Lemma 8 ([34])**.** *Under the assumption* $(\mathbb{A}2)$, *the solution of* (1) *satisfies the inequalities*

$$\int_\Omega h^2(u_t)dx \le c\int_\Omega u_t h(u_t)dx, \qquad \text{if } h_0 \text{ is linear} \tag{44}$$

$$\int_\Omega h^2(u_t)dx \le cH^{-1}(\chi_0(t)) - c\mathbb{E}'(t), \qquad \text{if } h_0 \text{ is nonlinear} \tag{45}$$

where

$$\chi_0(t) := \frac{1}{|\Omega_2|}\int_{\Omega_2} u_t(t)h(u_t(t))dx \le -c\mathbb{E}'(t) \tag{46}$$

and

$$\Omega_2 = \{x \in \Omega : |u_t(t)| \le \lambda_1\}.$$

Lemma 9 ([34])**.** *With the assumption* $(\mathbb{A}1)$, *the following estimate holds:*

$$(g \circ \Delta u)(t) \le \frac{t}{q}\overline{\mathbb{G}}^{-1}\left(\frac{qI(t)}{t\xi(t)}\right), \quad \forall t > 0 \tag{47}$$

where q small enough, $\overline{\mathbb{G}}$ is defined in Remark (1) *and the functional I is defined by*

$$I(t) := (-g' \circ \Delta u)(t) \le -c\mathbb{E}'(t). \tag{48}$$

Remark 2. *Applying* (9), (14), (19), (21) *and* (25), *we get*

$$\mathbb{E}(t) = \mathbb{J}(t) + \frac{1}{2}\|u_t(t)\|_2^2 \ge \mathbb{J}(t) \ge \frac{1}{3}(g \circ \Delta u)(t),$$

then, it follows from (15) *that*

$$(g \circ \Delta u)(t) \le 3\mathbb{E}(t) \le 3\mathbb{E}(0). \tag{49}$$

Furthermore, using (49), *we get*

$$\begin{aligned}(g \circ \Delta u)(t) &= (g \circ \Delta u)^{\frac{\lambda_0}{1+\lambda_0}}(t)(g \circ \Delta u)^{\frac{1}{1+\lambda_0}}(t)\\ &\le c(g \circ \Delta u)^{\frac{1}{1+\lambda_0}}(t).\end{aligned} \tag{50}$$

Remark 3. *In the case of $\mathbb{G}$ is linear and since ξ is nonincreasing, we have*

$$\begin{aligned}\xi(t)(g \circ \Delta u)^{\frac{1}{1+\lambda_0}}(t) &= \left(\xi^{\lambda_0}(t)\xi(t)(g \circ \Delta u)(t)\right)^{\frac{1}{1+\lambda_0}}\\ &\le \left(\xi^{\lambda_0}(0)\xi(t)(g \circ \Delta u)(t)\right)^{\frac{1}{1+\lambda_0}}\\ &\le c(\xi(t)(g \circ \Delta u)(t))^{\frac{1}{1+\lambda_0}}\\ &\le c(-E'(t))^{\frac{1}{(1+\lambda_0)}}.\end{aligned} \tag{51}$$

4. Stability

In this section, we state and prove our stability results.

Theorem 3. *Let* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, $\lambda_0 \in (0,1)$ *and* t_0, t_1 *be two positive constants. If the assumptions* $(\mathbb{A}1)$–$(\mathbb{A}3)$ *and* (24) *hold, then, provided h_0 is linear and k small enough, there exist strictly positive constants c, k_2 and λ_1 such that the solution of* (1) *satisfies,*

$$\mathbb{E}(t) \le c\left(1 + \int_0^t \xi^{1+\lambda_0}(\tau)d\tau\right)^{\frac{-1}{\lambda_0}}, \quad \forall t \ge t_0, \quad \text{if } \mathbb{G} \text{ is linear} \tag{52}$$

and

$$\mathbb{E}(t) \leq ct^{\frac{1}{1+\lambda_0}} K_2^{-1}\left(\frac{k_2}{t^{\frac{1}{1+\lambda_0}} \int_{t_1}^{t} \xi(\tau)d\tau}\right), \quad \forall t \geq t_1, \quad \textit{if } \mathbb{G} \textit{ is nonlinear} \tag{53}$$

where $K_2(\tau) = \tau K'(\lambda_1 \tau)$ *and* $K = \left(\left[\left(\overline{\mathbb{G}}\right)^{-1}\right]^{\frac{1}{1+\lambda_0}}\right)^{-1}$.

Proof of Case 1. $\mathbb{G}$ is linear. We start by multiplying (39) with $\xi(t)$ and applying (2), (10), (15), (44), (47) and (50) to,

$$\xi(t)\mathbb{L}'(t) \leq -d\xi(t)\mathbb{E}(t) + c\left(-\mathbb{E}'(t)\right)^{\frac{1}{(1+\lambda_0)}} - c\mathbb{E}'(t), \quad \forall t \geq t_0. \tag{54}$$

Now, multiply (54) by $\xi^{\lambda_0}(t)\mathbb{E}^{\lambda_0}(t)$, and observe that $\xi' \leq 0$ to obtain

$$\xi^{\lambda_0+1}(t)\mathbb{E}^{\lambda_0}(t)\mathbb{L}'(t) \leq -d\xi^{\lambda_0+1}(t)\mathbb{E}^{\lambda_0+1}(t) + c(\xi\mathbb{E})^{\lambda_0}(t)\left(-\mathbb{E}'(t)\right)^{\frac{1}{\lambda_0+1}} - c\mathbb{E}'(t), \quad \forall t \geq t_0.$$

The use of Young's inequality, with $q = \lambda_0 + 1$ and $q^* = \frac{\lambda_0+1}{\lambda_0}$, yields, for any $\lambda' > 0$,

$$\begin{aligned}
\xi^{\lambda_0+1}(t)\mathbb{E}^{\lambda_0}(t)\mathbb{L}'(t) &\leq -d\xi^{\lambda_0+1}(t)\mathbb{E}^{\lambda_0+1}(t) + c\left(\lambda'\xi^{\lambda_0+1}(t)\mathbb{E}^{\lambda_0+1} - c_{\lambda'}\mathbb{E}'(t)\right) \\
&= -(d - \lambda' c)\xi^{\lambda_0+1}(t)\mathbb{E}^{\lambda_0+1} - c\mathbb{E}'(t), \quad \forall t \geq t_0.
\end{aligned}$$

Choosing $0 < \lambda' < \frac{d}{c}$ and using $\xi' \leq 0$ and $\mathbb{E}' \leq 0$, to get, for $c_1 = d - \lambda' c$,

$$\left(\xi^{\lambda_0+1}\mathbb{E}^{\lambda_0}\mathbb{L}\right)'(t) \leq \xi^{\lambda_0+1}(t)\mathbb{E}^{\lambda_0}(t)\mathbb{L}'(t) \leq -c_1\xi^{\lambda_0+1}(t)E^{\lambda_0+1}(t) - c\mathbb{E}'(t), \quad \forall t \geq t_0,$$

which implies

$$\left(\xi^{\lambda_0+1}\mathbb{E}^{\lambda_0}\mathbb{L} + c\mathbb{E}\right)'(t) \leq -c_1\xi^{\lambda_0+1}(t)\mathbb{E}^{\lambda_0+1}(t), \quad \forall t \geq t_0.$$

Let $\mathbb{L}_1 = \xi^{\gamma+1}\mathbb{E}^{\gamma}\mathbb{L} + c\mathbb{E}$. Then $\mathbb{L}_1 \sim \mathbb{E}$ (thanks to (38)) and

$$\mathbb{E}_1'(t) \leq -c\xi^{\lambda_0+1}(t)\mathbb{E}_1^{\lambda_0+1}(t), \ \forall t \geq t_0.$$

Proceed by integrating over (t_0, t) and using $\mathbb{L}_1 \sim \mathbb{E}$, we obtain (52). □

Proof of Case 2. $\mathbb{G}$ is non-linear. In view of (39), (45), (47) and (50), we obtain, $\forall t \geq t_0$

$$\mathbb{L}'(t) \leq -d\mathbb{E}(t) + ct^{\frac{1}{1+\lambda_0}}\left[\left(\overline{\mathbb{G}}\right)^{-1}\left(\frac{qI(t)}{t\xi(t)}\right)\right]^{\frac{1}{1+\lambda_0}} - c\mathbb{E}'(t). \tag{55}$$

Using the strictly increasing property of $\overline{\mathbb{G}}$ and the fact that $\frac{1}{t} < 1$ whenever $t > 1$, we obtain

$$\left(\overline{\mathbb{G}}\right)^{-1}\left(\frac{q\mathbb{I}(t)}{t\xi(t)}\right) \leq \left(\overline{\mathbb{G}}\right)^{-1}\left(\frac{qI(t)}{t^{\frac{1}{1+\lambda_0}}\xi(t)}\right), \tag{56}$$

and, then, (55) becomes

$$\mathbb{L}'(t) \leq -d\mathbb{E}(t) + ct^{\frac{1}{1+\lambda_0}}\left[\left(\overline{\mathbb{G}}\right)^{-1}\left(\frac{qI(t)}{t^{\frac{1}{1+\lambda_0}}\xi(t)}\right)\right]^{\frac{1}{1+\lambda_0}} - c\mathbb{E}'(t), \quad \forall t \geq t_1, \tag{57}$$

where $t_1 = \max\{t_0, 1\}$. Define $\mathbb{F}_1(t) = \mathbb{L}(t) + c\mathbb{E}(t) \sim \mathbb{E}$, then (57) takes the form

$$\mathbb{F}_1'(t) \leq -d\mathbb{E}(t) + ct^{\frac{1}{1+\lambda_0}} \left[\left(\overline{\mathbb{G}}\right)^{-1} \left(\frac{qI(t)}{t^{\frac{1}{1+\lambda_0}} \xi(t)} \right) \right]^{\frac{1}{1+\lambda_0}}. \tag{58}$$

Set

$$K = \left(\left[\left(\overline{\mathbb{G}}\right)^{-1} \right]^{\frac{1}{1+\lambda_0}} \right)^{-1}, \quad \gamma(t) = \frac{qI(t)}{t^{\frac{1}{1+\lambda_0}} \xi(t)}. \tag{59}$$

It is not difficult to check that $K', K'' > 0$ on $(0, s_1]$. So, (58) reduces to

$$\mathbb{F}_1'(t) \leq -d\mathbb{E}(t) + ct^{\frac{1}{1+\lambda_0}} K^{-1}(\gamma(t)), \quad \forall t \geq t_1. \tag{60}$$

Besides, for $\lambda_0 < s_1$ and using (60) and the fact that $E' \leq 0, K' > 0, K'' > 0$ on $(0, s_1]$, we see that the functional defined by

$$\mathbb{F}_2(t) := K' \left(\frac{\lambda_1}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)} \right) \mathbb{F}_1(t), \quad \forall t \geq t_1,$$

satisfies, for some $\alpha_1, \alpha_2 > 0$ the following:

$$\alpha_1 \mathbb{F}_2(t) \leq \mathbb{E}(t) \leq \alpha_2 \mathbb{F}_2(t), \tag{61}$$

and, $\forall t \geq t_1$,

$$\mathbb{F}_2'(t) \leq -d\mathbb{E}(t) K' \left(\frac{\lambda_1}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)} \right) + ct^{\frac{1}{1+\lambda_0}} K' \left(\frac{\lambda_1}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)} \right) K^{-1}(\gamma(t)). \tag{62}$$

Take K^* to be the convex conjugate of K in the sense of Young [35] (pp. 61–64), then

$$K^*(\tau) = \tau (K')^{-1}(\tau) - K\left[(K')^{-1}(\tau) \right], \quad \text{if } \tau \in (0, K'(s_1)] \tag{63}$$

and K^* satisfies the following generalized version of Young's inequality:

$$K^*(A) + K(B) \geq AB, \quad \text{if } A \in (0, K'(s_1)], \ B \in (0, s_1] \tag{64}$$

and with $A = K' \left(\frac{\lambda_1}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)} \right)$ and $B = K^{-1}(\gamma(t))$, we get

$$\begin{aligned} \mathbb{F}_2'(t) &\leq -d\mathbb{E}(t) K' \left(\frac{\lambda_1}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)} \right) + ct^{\frac{1}{1+\lambda_0}} K^* \left(K' \left(\frac{\lambda_1}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)} \right) \right) \\ &\quad + ct^{\frac{1}{1+\lambda_0}} \gamma(t) \\ &\leq -d\mathbb{E}(t) K' \left(\frac{\lambda_1}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)} \right) + c \frac{\mathbb{E}(t)}{\mathbb{E}(0)} K' \left(\frac{\lambda_1}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)} \right) \\ &\quad + ct^{\frac{1}{1+\lambda_0}} \gamma(t). \end{aligned} \tag{65}$$

Therefore, multiplying (65) by $\xi(t)$ and using (48) and (59) we get, $\forall t \geq t_1$,

$$\xi(t)\mathbb{F}_2'(t) \leq -d\xi(t)\mathbb{E}(t)K'\left(\frac{\lambda_1}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right) + c\lambda_1\xi(t) \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}K'\left(\frac{\lambda_1}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right) - c\mathbb{E}'(t).$$

With the non-increasing property of ξ, we obtain, $\forall t \geq t_1$,

$$(\xi\mathbb{F}_2 + c\mathbb{E})'(t) \leq -d\xi(t)\mathbb{E}(t)K'\left(\frac{\lambda_1}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right) + c\lambda_1\xi(t)\frac{\mathbb{E}(t)}{\mathbb{E}(0)}K'\left(\frac{\lambda_1}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right).$$

Hence, by setting $\mathbb{F}_3 := \xi\mathbb{F}_2 + c\mathbb{E} \sim \mathbb{E}$, we obtain

$$\mathbb{F}_3'(t) \leq -d\xi(t)\mathbb{E}(t)K'\left(\frac{\lambda_1}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right) + c\lambda_1\xi(t) \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}K'\left(\frac{\lambda_1}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right).$$

Then, for a suitable choice of λ_1, we get

$$\mathbb{F}_3'(t) \leq -k\xi(t)\left(\frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right)K'\left(\frac{\lambda_1}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right), \qquad \forall t \geq t_1$$

or

$$k\left(\frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right)K'\left(\frac{\lambda_1}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right)\xi(t) \leq -\mathbb{F}_3'(t), \qquad \forall t \geq t_1. \tag{66}$$

An integration of (66) yields

$$\int_{t_1}^t k\left(\frac{\mathbb{E}(\tau)}{\mathbb{E}(0)}\right)K'\left(\frac{\lambda_1}{s^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(\tau)}{\mathbb{E}(0)}\right)\xi(\tau)d\tau \leq -\int_{t_1}^t \mathbb{F}_3'(\tau)d\tau \leq \mathbb{F}_3(t_1). \tag{67}$$

Drawing on the facts that $K', K'' > 0$ as well as non-increasing property of $\mathbb{E}$, we infer that the map $t \mapsto \mathbb{E}(t)K'\left(\frac{\lambda_1}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right)$ is also non-increasing and a result, we have

$$\begin{aligned} &k\left(\frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right)K'\left(\frac{\lambda_1}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right)\int_{t_1}^t \xi(\tau)d\tau \\ &\leq \int_{t_1}^t k\left(\frac{\mathbb{E}(\tau)}{\mathbb{E}(0)}\right)K'\left(\frac{\lambda_1}{\tau^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(\tau)}{\mathbb{E}(0)}\right)\xi(\tau)d\tau \leq \mathbb{F}_3(t_1), \qquad \forall t \geq t_1. \end{aligned} \tag{68}$$

Multiplying both sides of (68) by $\frac{1}{t^{\frac{1}{1+\lambda_0}}}$, we have

$$k\left(\frac{1}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right)K'\left(\frac{\lambda_1}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right)\int_{t_1}^t \xi(\tau)d\tau \leq \frac{k_2}{t^{\frac{1}{1+\lambda_0}}}, \qquad \forall t \geq t_1. \tag{69}$$

Now, we set $K_2(\tau) = \tau K'(\lambda_1\tau)$ which is strictly increasing, then we obtain,

$$kK_2\left(\frac{1}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right)\int_{t_1}^t \xi(\tau)d\tau \leq \frac{k_2}{t^{\frac{1}{1+\lambda_0}}}, \qquad \forall t \geq t_1. \tag{70}$$

Finally, for two positive constants k_2 and k_3, we obtain

$$\mathbb{E}(t) \leq k_3 t^{\frac{1}{1+\lambda_0}} K_2{}^{-1}\left(\frac{k_2}{t^{\frac{1}{1+\lambda_0}} \int_{t_1}^{t} \xi(\tau)d\tau}\right). \tag{71}$$

□

Theorem 4. *Let $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, $\lambda_0 \in (0,1)$ and $t_0, t_1 > 0$. Assume that $(\mathbb{A}1)$–$(\mathbb{A}3)$ and (24) hold and h_0 is nonlinear. Then, for k small enough, there exist strictly positive constants c_3, c_4, k_2, k_3 and λ_2 such that the solution of (1) satisfies,*

$$\mathbb{E}(t) \leq \mathcal{J}_1^{-1}\left(c_3 \int_{t_0}^{t} \xi(\tau)d\tau + c_4\right), \quad \forall t \geq t_0, \text{ if } \mathbb{G} \text{ is linear,} \tag{72}$$

where $\mathcal{J}_1(t) = \int_t^1 \frac{1}{\mathcal{J}_2(\tau)}d\tau$, $\mathcal{J}_2(t) = t\mathcal{J}'(\lambda_1 t)$ and $\mathcal{J} = \left(t^{\frac{1}{1+\lambda_0}} + H^{-1}\right)^{-1}$,
and

$$\mathbb{E}(t) \leq k_3 t^{\frac{1}{1+\lambda_0}} W_2{}^{-1}\left(\frac{k_2}{t^{\frac{1}{1+\lambda_0}} \int_{t_1}^{t} \xi(\tau)d\tau}\right), \quad \forall t \geq t_1, \text{ if } \mathbb{G} \text{ is non-linear,} \tag{73}$$

where $W_2(t) = tW'(\lambda_2 t)$ and $W = \left(\left[\left(\overline{\mathbb{G}}\right)^{-1}\right]^{\frac{1}{1+\lambda_0}} + \overline{H}^{-1}\right)^{-1}$.

Proof of Case 3. $\mathbb{G}$ is linear. In view of (45), (50) and (51), multiplying (39) by $\xi(t)$ gives

$$\begin{aligned}
\xi(t)\mathbb{L}'(t) &\leq -d\xi(t)\mathbb{E}(t) + c\big(-\mathbb{E}'(t)\big)^{\frac{1}{(1+\lambda_0)}} + c\xi(t)\int_{\Omega} h^2(u_t(t))d\Gamma \\
&\leq -d\xi(t)\mathbb{E}(t) + c\big(-\mathbb{E}'(t)\big)^{\frac{1}{(1+\lambda_0)}} + c\xi(t)H^{-1}(\chi_0(t)).
\end{aligned}$$

Let $\mathbb{F}(t) = t^{1+\lambda_0}$. Then, the last inequality can be written as

$$\xi(t)\mathbb{L}'(t) \leq -d\xi(t)\mathbb{E}(t) + c\mathbb{F}^{-1}(-\mathbb{E}'(t)) + c\xi(t)H^{-1}(\chi_0(t)), \forall t \geq 0, \tag{74}$$

where $\chi_0(t)$ is as defined in (46). As a result, (74) becomes

$$\mathcal{L}'(t) \leq -d\xi(t)\mathbb{E}(t) + c\xi(t)\mathcal{J}^{-1}(\chi_1(t)), \forall t \geq 0, \tag{75}$$

where $\mathcal{L} := \xi\mathbb{L} \sim \mathbb{E}$,

$$\mathcal{J} = \left(F^{-1} + H^{-1}\right)^{-1}, \quad \chi_1(t) = \max\{-\mathbb{E}'(t), \chi_0(t)\}.$$

In fact, one can prove that $\mathcal{J}' > 0, \mathcal{J}'' > 0$ on $(0, s_2]$. For $\lambda_2 < s_2$ and $c_0 > 0$, using (75) and the fact that $\mathbb{E}' \leq 0$, we see that the functional $\mathcal{L}_1$, defined by

$$\mathcal{L}_1(t) := \mathcal{J}'\left(\lambda_2 \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right)\mathcal{L}(t)$$

satisfies, for some $\alpha_3, \alpha_4 > 0$ the following:

$$\alpha_3 \mathcal{L}_1(t) \leq \mathbb{E}(t) \leq \alpha_4 \mathcal{L}_1(t), \tag{76}$$

and

$$\begin{aligned}\mathcal{L}_1'(t) &= \lambda_2 \frac{\mathbb{E}'(t)}{\mathbb{E}(0)} \mathcal{J}''\left(\lambda_2 \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right) \mathcal{L}(t) + \mathcal{J}'\left(\lambda_2 \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right) \mathcal{L}'(t) \\ &\leq -d\mathbb{E}(t) \mathcal{J}'\left(\lambda_2 \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right) + c\xi(t) \mathcal{J}'\left(\lambda_2 \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right) \mathcal{J}^{-1}(\chi_1(t)).\end{aligned} \tag{77}$$

Taking $\mathcal{J}^*$ as the convex conjugate of $\mathcal{J}$ in the sense of Young see [35] (pp. 61–64), then, as in (63) and (64), with $A = \mathcal{J}'\left(\lambda_2 \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right)$ and $B = \mathcal{J}^{-1}(\chi_1(t))$, using (46), we conclude that

$$\begin{aligned}\mathcal{L}_1'(t) &\leq -d\mathbb{E}(t) \mathcal{J}'\left(\lambda_2 \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right) + c\xi(t) \mathcal{J}^*\left(H'\left(\lambda_2 \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right)\right) + c\xi(t)\chi_0(t) \\ &\leq -d\mathbb{E}(t) \mathcal{J}'\left(\lambda_2 \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right) + c\lambda_2 \xi(t) \frac{\mathbb{E}(t)}{\mathbb{E}(0)} \mathcal{J}'\left(\lambda_2 \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right) - c\mathbb{E}'(t).\end{aligned}$$

Therefore, with a suitable choice of λ_2 and c_0, we obtain, for all $t \geq 0$,

$$\mathcal{L}_1'(t) \leq -c\xi(t) \frac{\mathbb{E}'(t)}{\mathbb{E}(0)} \mathcal{J}'\left(\lambda_2 \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right) = -c\xi(t) \mathcal{J}_2\left(\lambda_2 \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right), \tag{78}$$

where $\mathcal{J}_2(t) = t\mathcal{J}'(\lambda_2 t)$. Since $\mathcal{J}_2'(t) = \mathcal{J}'(\lambda_2 t) + \lambda_2 t \mathcal{J}''(\lambda_2 t)$, then, applying the strict convexity of $\mathcal{J}$ on $(0, s_2]$, we find that $\mathcal{J}_2'(t), \mathcal{J}_2(t) > 0$ on $(0, 1]$. Thus, with

$$s_1(t) = \lambda \frac{\alpha_3 \mathcal{L}_1(t)}{\mathbb{E}(0)}, \quad 0 < \lambda < 1,$$

taking in account (76) and (78), we have

$$s_1(t) \sim \mathbb{E}(t) \tag{79}$$

and, for some $c_3 > 0$.

$$s_1'(t) \leq -c_3 \xi(t) \mathcal{J}_2(s_1(t)), \quad \forall t \geq t_0.$$

Then, a straight forward integration gives, for some $c_4 > 0$,

$$s_1(t) \leq \mathcal{J}_1^{-1}\left(c_3 \int_{t_0}^{t} \xi(\tau) d\tau + c_4\right), \quad \forall t \geq t_0, \tag{80}$$

where $\mathcal{J}_1(t) = \int_t^1 \frac{1}{\mathcal{J}_2(\tau)} d\tau$.
□

Proof of Case 4. $\mathbb{G}$ is non-linear. In view of (39), (45), (47) and (50), we obtain

$$\mathbb{L}'(t) \leq -d\mathbb{E}(t) + ct^{\frac{1}{1+\lambda_0}} \left[\left(\overline{\mathbb{G}}\right)^{-1} \left(\frac{qI_1(t)}{t\xi(t)} \right) \right]^{\frac{1}{1+\lambda_0}} + cH^{-1}(\chi_0(t)) - c\mathbb{E}'(t). \tag{81}$$

Applying the strictly increasing and strictly convex properties of $\overline{H}$ and $\overline{\mathbb{G}}$, setting

$$\theta = \left(\frac{1}{t}\right)^{\frac{1}{1+\lambda_0}} < 1, \quad \forall t > 1,$$

and using

$$\overline{H}(\theta z) \leq \theta \overline{H}(z), \; 0 \leq \theta \leq 1 \text{ and } z \in (0, s_2], \tag{82}$$

we obtain

$$\overline{H}^{-1}(\chi_0(t)) \leq t^{\frac{1}{1+\lambda_0}} \overline{H}^{-1}\left(\frac{\chi_0(t)}{t^{\frac{1}{1+\lambda_0}}}\right), \quad \forall t > 1,$$

and
$$\left(\overline{\mathbb{G}}\right)^{-1}\left(\frac{qI_1(t)}{t\xi(t)}\right) \leq \left(\overline{\mathbb{G}}\right)^{-1}\left(\frac{qI_1(t)}{t^{\frac{1}{1+\lambda_0}}\xi(t)}\right), \quad \forall t > 1,$$

hence (81) becomes

$$\begin{aligned} \mathbb{L}'(t) \leq -d\mathbb{E}(t) + ct^{\frac{1}{1+\lambda_0}}\left[\left(\overline{\mathbb{G}}\right)^{-1}\left(\frac{qI_1(t)}{t^{\frac{1}{1+\lambda_0}}\xi(t)}\right)\right]^{\frac{1}{1+\lambda_0}} \\ + ct^{\frac{1}{1+\lambda_0}}\overline{H}^{-1}\left(\frac{\chi_0(t)}{t^{\frac{1}{1+\lambda_0}}}\right) - c\mathbb{E}'(t), \qquad \forall t \geq t_1, \end{aligned} \tag{83}$$

where $t_1 = \max\{t_0, 1\}$. Let $\mathcal{F}(t) = \mathbb{L}(t) + c\mathbb{E}(t) \sim \mathbb{E}$, then (83) takes the form

$$\mathcal{F}'(t) \leq -d\mathbb{E}(t) + ct^{\frac{1}{1+\lambda_0}}\left[\left(\overline{\mathbb{G}}\right)^{-1}\left(\frac{qI_1(t)}{t^{\frac{1}{1+\lambda_0}}\xi(t)}\right)\right]^{\frac{1}{1+\lambda_0}} + ct^{\frac{1}{1+\lambda_0}}\overline{H}^{-1}\left(\frac{\chi_0(t)}{t^{\frac{1}{1+\lambda_0}}}\right). \tag{84}$$

$$\text{Let } s_0 = \min\{s_1, s_2\}, \chi(t) = \max\{\frac{qI_1(t)}{t^{\frac{1}{1+\lambda_0}}\xi(t)}, \frac{\chi_0(t)}{t^{\frac{1}{1+\lambda_0}}}\}, \tag{85}$$

and
$$W = \left(\left[\left(\overline{\mathbb{G}}\right)^{-1}\right]^{\frac{1}{1+\lambda_0}} + \overline{H}^{-1}\right)^{-1}.$$

Making use of the strictly increasing and strictly convex properties of $\overline{H}$ and $\overline{\mathbb{G}}$ imply that $W' > 0$ and $W'' > 0$. So, (84) reduces to

$$\mathcal{F}'(t) \leq -d\mathbb{E}(t) + ct^{\frac{1}{1+\lambda_0}}W^{-1}(\chi(t)), \quad \forall t \geq t_1. \tag{86}$$

We also see that, with $\lambda_2 < s_0$ and applying (81) and the fact that $\mathbb{E}' \leq 0, W' > 0, W'' > 0$ on $(0, s_0]$, the functional $\mathcal{F}_1$, defined by

$$\mathcal{F}_1(t) := W'\left(\frac{\lambda_2}{(t-t_1)^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right)L_1(t), \quad \forall t \geq t_2,$$

satisfies, for some $\alpha_5, \alpha_6 > 0$, the following:

$$\alpha_5\mathcal{F}_1(t) \leq \mathbb{E}(t) \leq \alpha_6\mathcal{F}_1(t), \tag{87}$$

and, for all $t \geq t_1$,

$$\mathcal{F}_1'(t) \leq -d\mathbb{E}(t)W'\left(\frac{\lambda_2}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right) + ct^{\frac{1}{1+\lambda_0}}W'\left(\frac{\lambda_2}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right)W^{-1}(\chi(t)). \tag{88}$$

Taking W^* as the convex conjugate of W in the sense of Young see [35] (pp. 61–64), we get

$$W^*(s) = s(W')^{-1}(s) - W\left[(W')^{-1}(s)\right], \quad \text{if } s \in (0, W'(s_0)]. \tag{89}$$

Infact, W^* also satisfies the following generalized Young inequality:

$$AB \leq W^*(A) + W(B), \quad \text{if } A \in (0, W'(s_0)],\ B \in (0, s_0]. \tag{90}$$

Now, with $A = W'\left(\frac{\lambda_2}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right)$ and $B = W^{-1}(\chi(t))$, we arrive at

$$\begin{aligned}\mathcal{F}_1'(t) &\leq -d\mathbb{E}(t)W'\left(\frac{\lambda_2}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right) + ct^{\frac{1}{1+\lambda_0}}W^*\left(W'\left(\frac{\lambda_2}{t^{\frac{1}{1+\varepsilon_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right)\right) \\ &+ ct^{\frac{1}{1+\lambda_0}}\chi(t) \\ &\leq -d\mathbb{E}(t)W'\left(\frac{\lambda_2}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right) + c\epsilon_2\frac{\mathbb{E}(t)}{\mathbb{E}(0)}W'\left(\frac{\lambda_2}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right) \\ &+ ct^{\frac{1}{1+\lambda_0}}\chi(t).\end{aligned} \tag{91}$$

Hence, multiplying (91) by $\xi(t)$ and using (46), (48), (85) give

$$\begin{aligned}\xi(t)\mathcal{F}_1'(t) &\leq -d\xi(t)\mathbb{E}(t)W'\left(\frac{\lambda_2}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right) \\ &+ c\lambda_2\xi(t) \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}W'\left(\frac{\lambda_2}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right) - c\mathbb{E}'(t), \quad \forall t \geq t_1.\end{aligned}$$

Applying the non-increasing property of ξ, we obtain, for all $t \geq t_1$,

$$\begin{aligned}(\xi\mathcal{F}_1 + cE)'(t) &\leq -d\xi(t)\mathbb{E}(t)W'\left(\frac{\lambda_2}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right) \\ &+ c\lambda_2\xi(t)\frac{\mathbb{E}(t)}{\mathbb{E}(0)}W'\left(\frac{\lambda_2}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right)\end{aligned}$$

Therefore, by setting $\mathcal{F}_2 := \xi\mathcal{F}_1 + c\mathbb{E} \sim \mathbb{E}$, we get

$$\mathcal{F}_2'(t) \leq -d\xi(t)\mathbb{E}(t)W'\left(\frac{\lambda_2}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right) + c\lambda_2\xi(t) \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}W'\left(\frac{\lambda_2}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right).$$

So, for a suitable choice of λ_2,

$$\mathcal{F}_2'(t) \leq -c\xi(t)\left(\frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right)W'\left(\frac{\lambda_2}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right), \qquad \forall t \geq t_1$$

or

$$c\left(\frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right)W'\left(\frac{\lambda_2}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right)\xi(t) \leq -\mathcal{F}_2'(t), \qquad \forall t \geq t_1. \tag{92}$$

A direct integration of (92) yields

$$\int_{t_1}^{t} c\left(\frac{E(s)}{\mathbb{E}(0)}\right)W'\left(\frac{\lambda_2}{s^{\frac{1}{1+\lambda_0}}} \cdot \frac{E(s)}{\mathbb{E}(0)}\right)\xi(\tau)d\tau \leq -\int_{t_1}^{t}\mathcal{F}_2'(s)d\tau \leq \mathcal{F}_2(t_1). \tag{93}$$

Since $W', W'' > 0$ $\mathbb{E}$ is non-increasing, we deduce that the map $t \mapsto \mathbb{E}(t)W'\left(\frac{\lambda_2}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right)$ is non-increasing and as a result, we have

$$\begin{aligned}&c\left(\frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right)W'\left(\frac{\lambda_2}{t^{\frac{1}{1+\lambda_0}}} \cdot \frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right)\int_{t_1}^{t}\xi(\tau)d\tau \\ &\leq \int_{t_1}^{t} c\left(\frac{E(s)}{\mathbb{E}(0)}\right)W'\left(\frac{\lambda_2}{s^{\frac{1}{1+\lambda_0}}} \cdot \frac{E(s)}{\mathbb{E}(0)}\right)\xi(\tau)d\tau \leq \mathcal{F}_2(t_1), \qquad \forall t \geq t_1.\end{aligned} \tag{94}$$

Multiplying both sides of (94) by $\frac{1}{t^{\frac{1}{1+\lambda_0}}}$, we have

$$c\left(\frac{1}{t^{\frac{1}{1+\lambda_0}}}\cdot\frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right)W'\left(\frac{\lambda_2}{t^{\frac{1}{1+\lambda_0}}}\cdot\frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right)\int_{t_1}^{t}\xi(\tau)d\tau \leq \frac{k_2}{t^{\frac{1}{1+\lambda_0}}}, \qquad \forall t \geq t_1. \tag{95}$$

Next, we set $W_2(\tau) = \tau W'(\lambda_2\tau)$. Since it is strictly increasing, we obtain,

$$cW_2\left(\frac{1}{t^{\frac{1}{1+\lambda_0}}}\cdot\frac{\mathbb{E}(t)}{\mathbb{E}(0)}\right)\int_{t_1}^{t}\xi(\tau)d\tau \leq \frac{k_2}{t^{\frac{1}{1+\lambda_0}}}, \qquad \forall t \geq t_1. \tag{96}$$

Finally, for two positive constants k_2 and k_3, we obtain

$$\mathbb{E}(t) \leq k_3 t^{\frac{1}{1+\lambda_0}} W_2^{-1}\left(\frac{k_2}{t^{\frac{1}{1+\lambda_0}}\int_{t_1}^{t}\xi(\tau)d\tau}\right). \tag{97}$$

This finishes the proof. □

Example 1. *We now provide some examples to demonstrate our results.*

1. *Firstly, consider the case when h_0 and $\mathbb{G}$ are both linear.*
   Take $g(t) = ae^{-b(1+t)}$, where $b > 0$ and $a > 0$. Then $g'(t) = -\xi(t)\mathbb{G}(g(t))$ where $\mathbb{G}(t) = t$ and $\xi(t) = b$. For the frictional nonlinearity, assume that $h_0(t) = ct$. So, $H(t) = \sqrt{t}h_0(\sqrt{t}) = ct$. Hence, it follows from (52) that

$$\mathbb{E}(t) \leq \frac{c_1}{(1+t)^{\frac{1}{\lambda_0}}}. \tag{98}$$

2. *Secondly, we consider the case when h_0 is linear and $\mathbb{G}$ is non-linear.*
   We let $g(t) = \frac{a}{(1+t)^q}$, where $q > 1+\lambda_0$ and a is chosen so that (9) holds. Then

$$g'(t) = -b\mathbb{G}(g(t)), \quad \textit{with} \quad \mathbb{G}(\tau) = \tau^{\frac{q+1}{q}},$$

   where b is a fixed constant. Here, we take $h_0(t) = ct$, and $H(t) = \sqrt{t}h_0(\sqrt{t}) = ct$, as the frictional nonlinearity. Since $K(\tau) = \tau^{\frac{(\lambda_0+1)(q+1)}{q}}$. Then, (53) gives, $\forall t \geq t_1$

$$\mathbb{E}(t) \leq \frac{c}{t^{\frac{q-1-\lambda_0}{(1+\lambda_0)^2(q+1)}}}. \tag{99}$$

3. *Thirdly, when h_0 is non-linear and $\mathbb{G}$ is linear.*
   We take $g(t) = ae^{-b(1+t)}$, where $b > 0$ and $a > 0$ is small enough so that (9) is satisfied. Then $g'(t) = -\xi(t)\mathbb{G}(g(t))$ where $\mathbb{G}(t) = t$ and $\xi(t) = b$. Also, assume that $h_0(t) = ct^2$, where $H(t) = \sqrt{t}h_0(\sqrt{t}) = ct^{\frac{3}{2}}$. Then, after taking $\lambda_0 = \frac{1}{2}$, we have

$$F = t^{\frac{3}{2}}$$

   and

$$\mathcal{J}(t) = ct^{\frac{3}{2}}.$$

   Therefore, applying (72), we obtain

$$\mathbb{E}(t) \leq \frac{c}{(1+t)^2}. \tag{100}$$

4. *Lastly, we consider the case when h_0 and $\mathbb{G}$ are non-linear.*
   Let $g(t) = \frac{a}{(1+t)^4}$, where a is chosen so that hypothesis (9) remains true. Then

$$g'(t) = -b\mathbb{G}(g(t)), \quad \textit{with} \quad \mathbb{G}(\tau) = \tau^{\frac{5}{4}},$$

where b is a fixed constant. In this case, we let $h_0(t) = ct^2$ and $H(t) = ct^{\frac{3}{2}}$ Hence with $\lambda_0 = \frac{1}{5}$, we obtain

$$W = \left(\left[\left(\overline{\mathbb{G}} \right)^{-1} \right]^{\frac{1}{1+\lambda_0}} + \overline{H}^{-1} \right)^{-1} = c\tau^{\frac{3}{2}},$$

and

$$W_2(\tau) = c\tau^{\frac{3}{2}}.$$

Therefore, applying (73), we obtain, $\forall t \geq t_1$

$$\mathbb{E}(t) \leq \frac{c}{t^{\frac{7}{18}}}.$$

5. Numerical Results

In this section, we perform some numerical experiments to illustrate the theoretical results in Theorems 1 and 2. For this purpose, we discretize the system (9) using a finite difference method (FDM) in both time and space with second-order in time and forth-order in space for the time-space domain $[0, L] \times [0, T] = [0, 1] \times [0, 8]$. The spatial interval $(0, 1)$ is divided into 50 subintervals, where the time interval $(0, T)$ is divided into 4×10^4 subintervals with a time step $\Delta t = \frac{T}{N}$.

The homogeneous Dirichlet boundary condition of the problem (9) is given and the normal derivative is equal to zero at the boundary using the following initial conditions:

- $u(x, 0) = x^2(1 - x)^2$
- $u_t(x, 0) = 0$

We compare the energy decay and the solution of problem (9) through four numerical tests based on the function h and the kernel function g.

- Test 1: We consider $h(u_t) = u_t$ and $g(t) = e^{-t}$.
- Test 2: We consider $h(u_t) = u_t|u_t|$ and $g(t) = e^{-t}$.
- Test 3: We consider $h(u_t) = u_t$ and $g(t) = \frac{1}{(t+1)^2}$
- Test 4: We consider $h(u_t) = u_t|u_t|$ and $g(t) = \frac{1}{(t+1)^2}$.

In Figures 1–4 we show the cross section cuts of the approximate solution u at $x = 0.3$, $x = 0.5$, $x = 0.6$, and $x = 0.8$ for Test 1, Test 2, Test 3, and Test 4, Respectively. In Figures 5–8 we sketch the corresponding energy functional (14). Also, we sketch the decay behavior of the whole wave over the time interval $[0, 8]$ in Figures 9–12 for Test 1 to Test 4, Respectively.

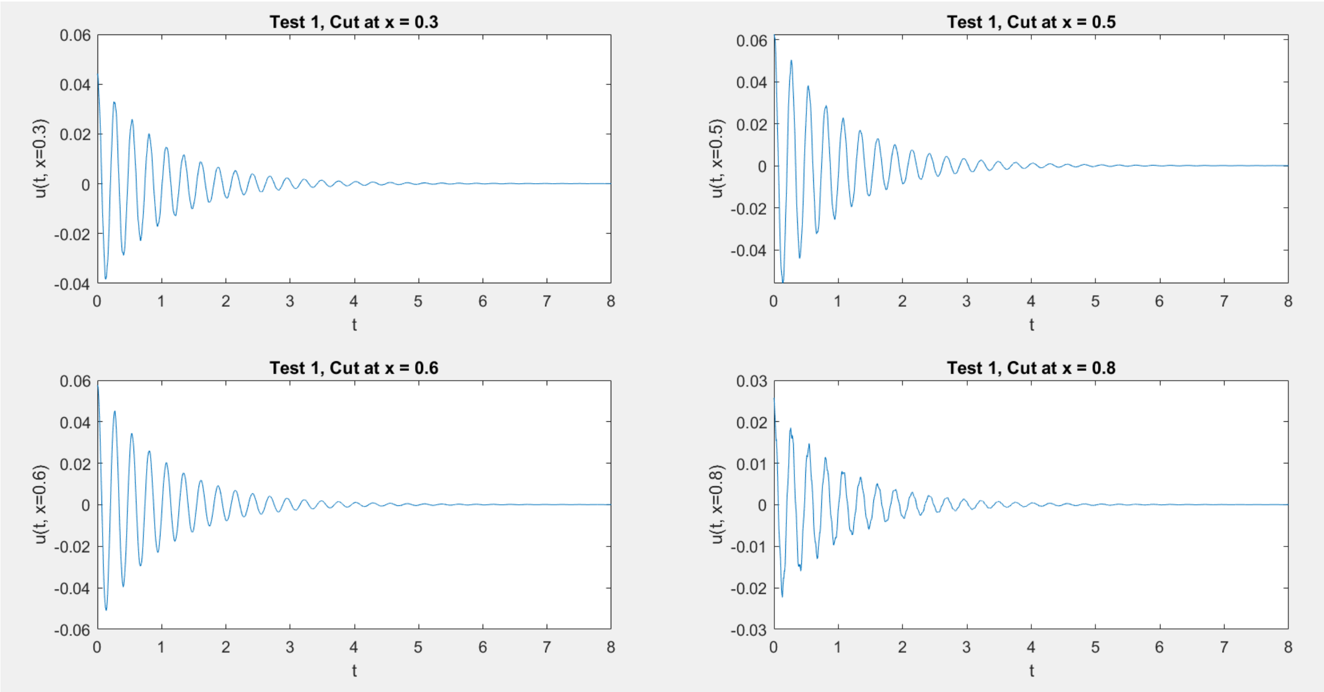

Figure 1. Test 1: The solution. $u(t)$ at fixed values of x.

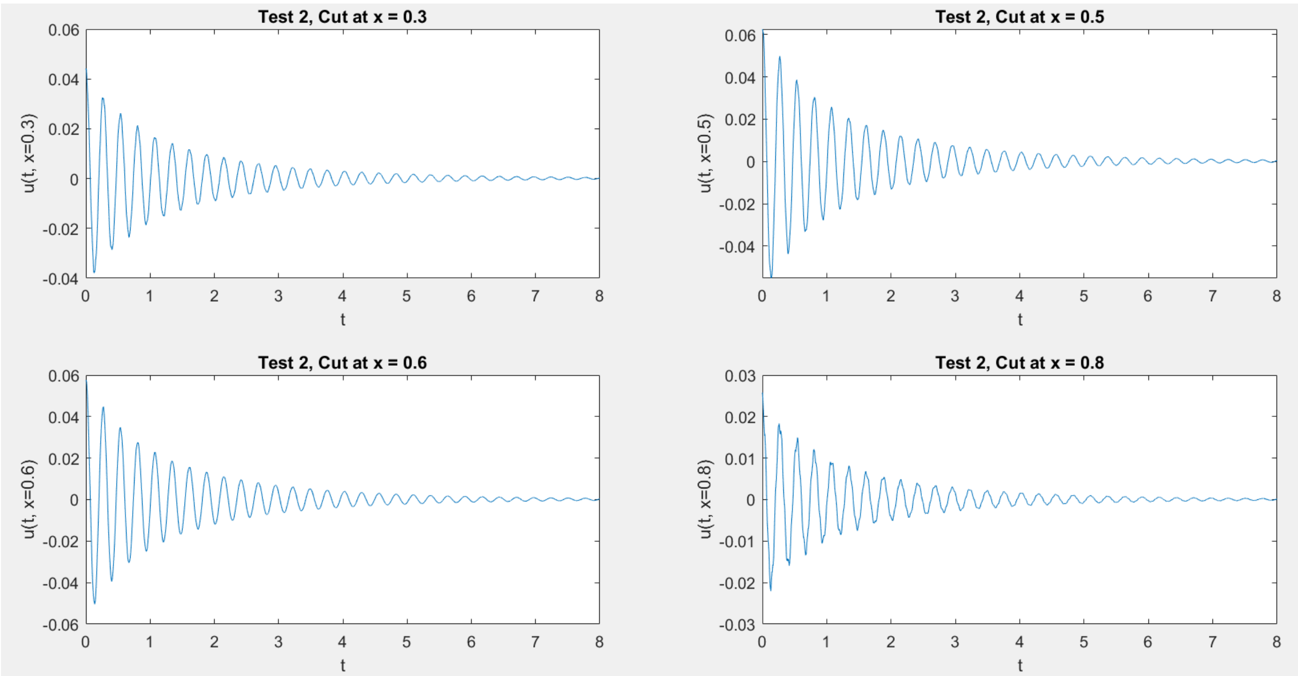

Figure 2. Test 2: The solution $u(t)$ at fixed values of x.

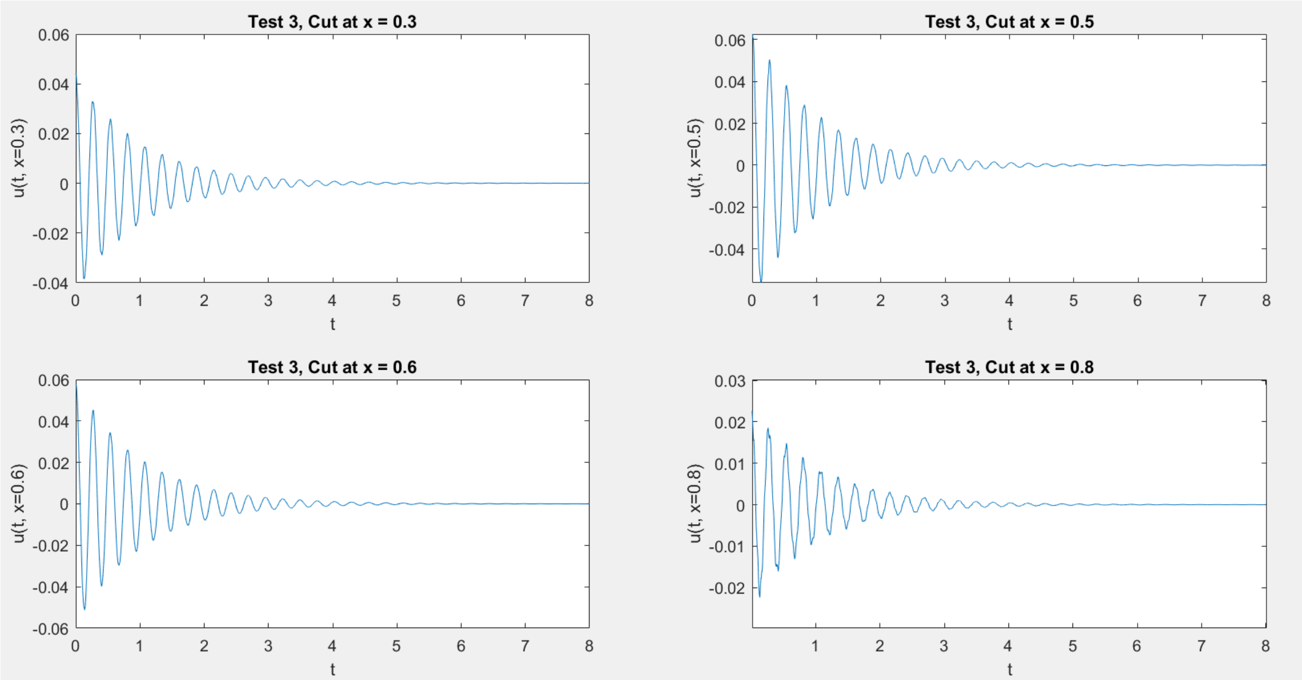

Figure 3. Test 3: The solution $u(t)$ at fixed values of x.

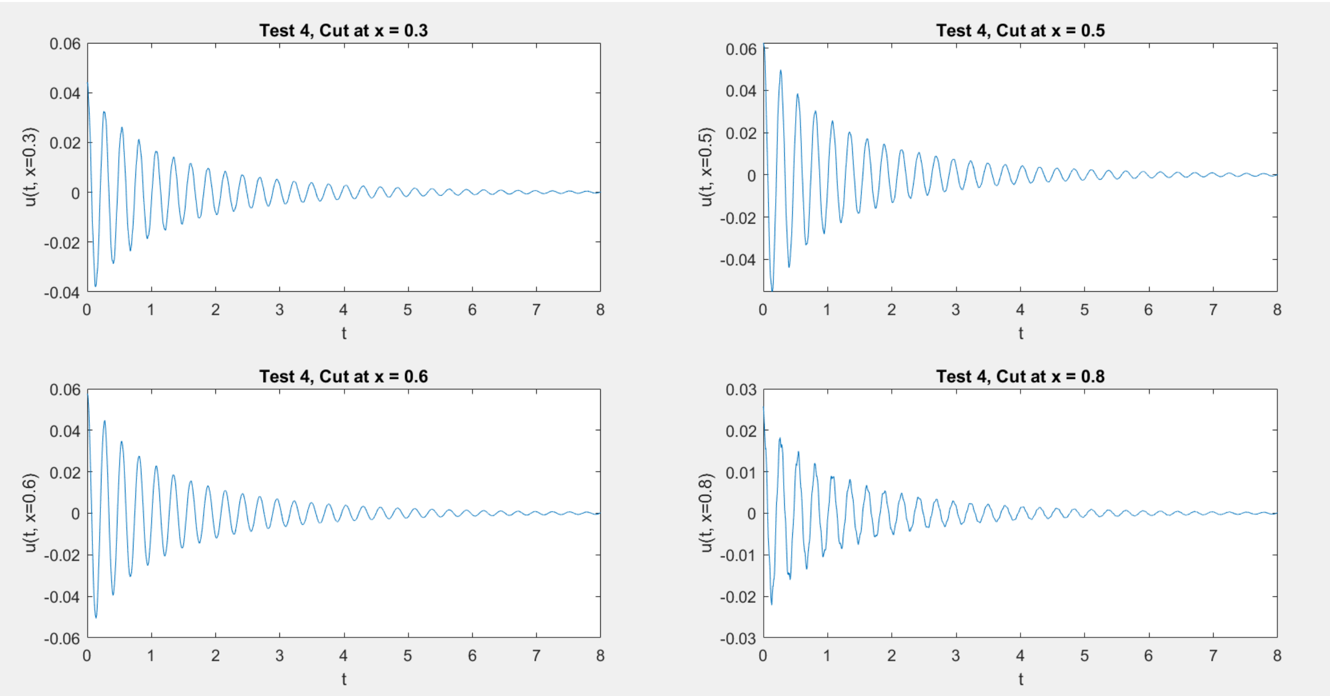

Figure 4. Test 4: The solution $u(t)$ at fixed values of x.

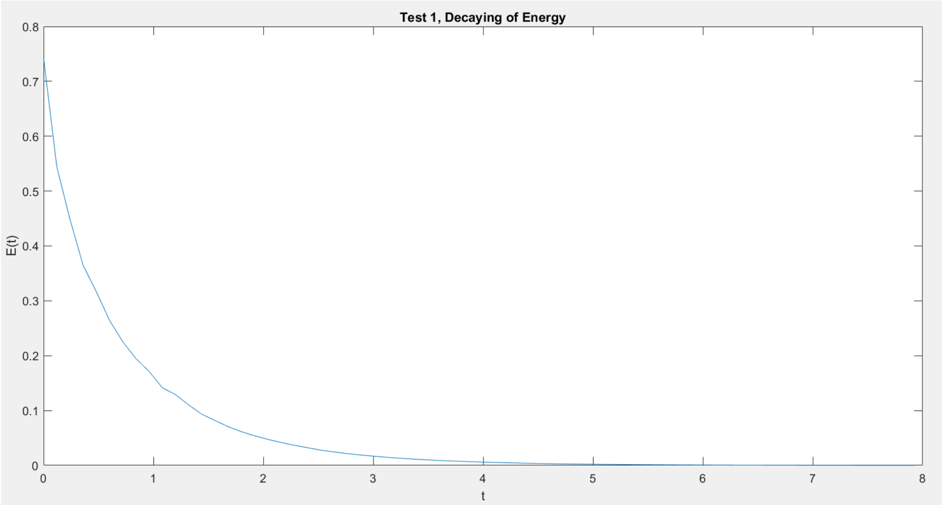

Figure 5. Test 1: The energy decay.

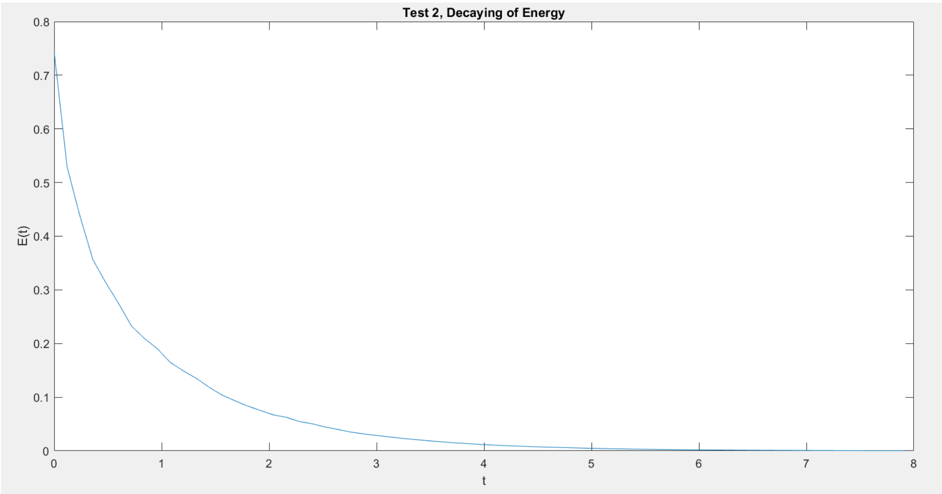

Figure 6. Test 2: The energy decay.

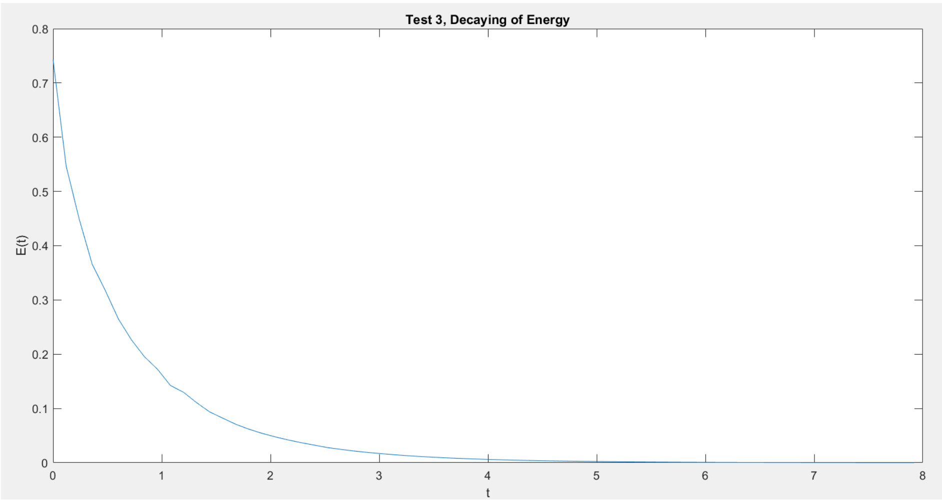

Figure 7. Test 3: The energy decay.

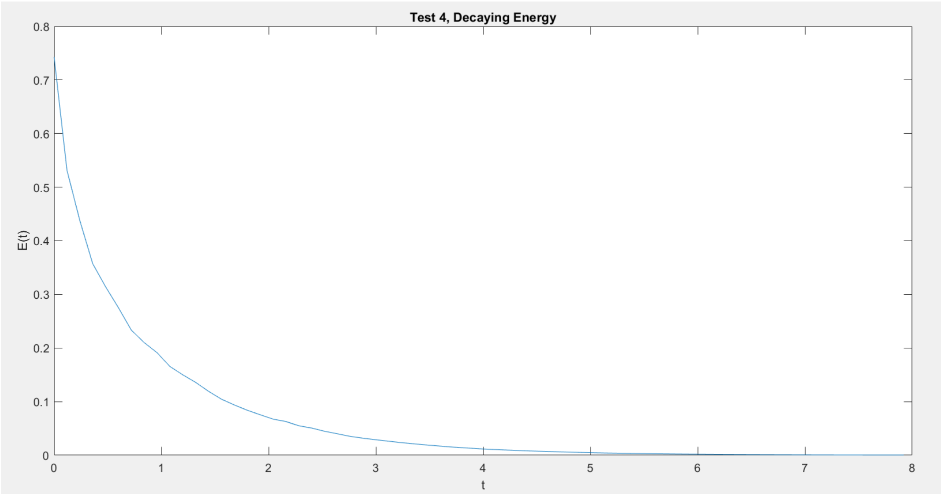

Figure 8. Test 4: The energy decay.

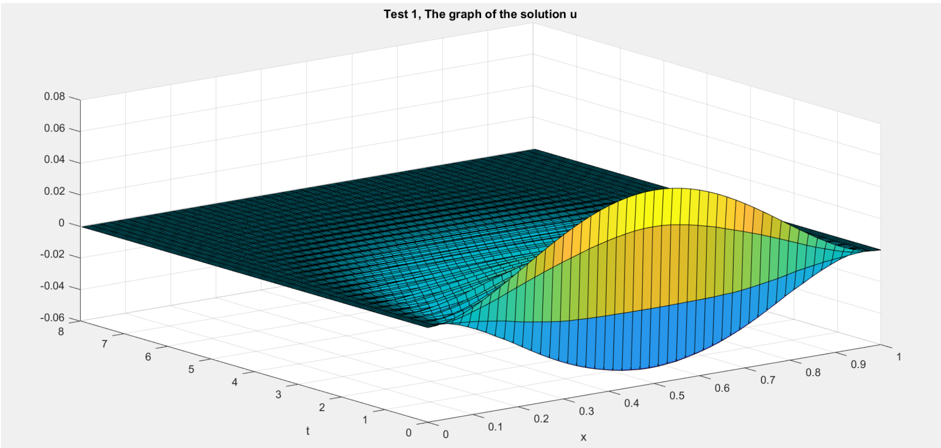

Figure 9. Test 1: The solution $u(x,t)$.

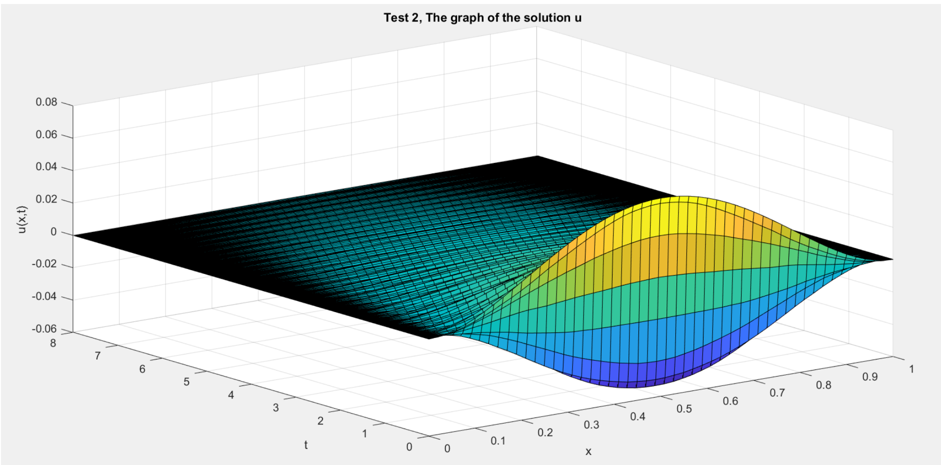

Figure 10. Test 2: The solution $u(x,t)$.

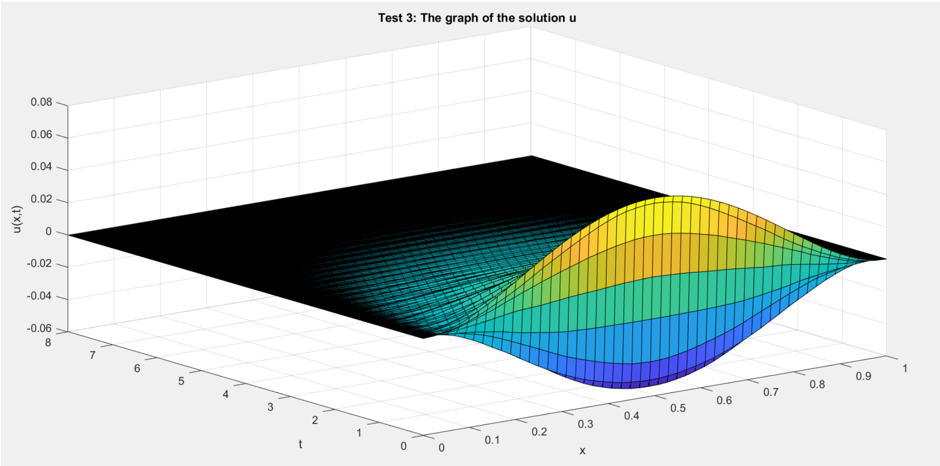

Figure 11. Test 3: The solution $u(x,t)$.

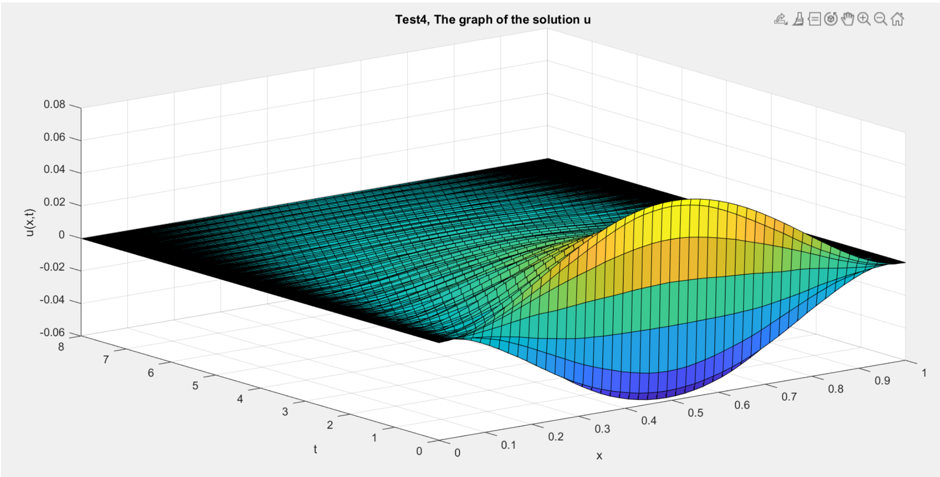

Figure 12. Test 4: The solution $u(x,t)$.

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