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Article

On a Modified Weighted Exponential Distribution with Applications

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Abstract: Practitioners in all applied domains value simple and adaptable lifetime distributions. They make it possible to create statistical models that are relatively easy to manage. A novel simple lifetime distribution with two parameters is proposed in this article. It is based on a parametric mixture of the exponential and weighted exponential distributions, with a mixture weight depending on a parameter of the involved distribution; no extra parameter is added in this mixture operation. It can also be viewed as a special generalized mixture of two exponential distributions. This decision is based on sound mathematical and physical reasoning; the weight modification allows us to combine some joint properties of the exponential and weighted exponential distribution, which are known as complementary in several modeling aspects. As a result, the proposed distribution may have a decreasing or unimodal probability density function and possess the demanded increasing hazard rate property. Other properties are studied, such as the moments, Bonferroni and Lorenz curves, Rényi entropy, stress-strength reliability, and mean residual life function. Subsequently, a part is devoted to the associated model, which demonstrates how it can be used in a real-world statistical scenario involving data. In this regard, we demonstrate how the estimated model performs well using five different estimation methods and simulated data. The analysis of two data sets demonstrates these excellent results. The new model is compared to the weighted exponential, Weibull, gamma, and generalized exponential models for performance. The obtained comparison results are overwhelmingly in favor of the proposed model according to some standard criteria.

Keywords: exponential distribution; weighted exponential distribution; moments; Rényi entropy; stress-strength parameter; simulation; data analysis

MSC: 60E05, 62E15, 62F10

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1. Introduction

1.1. State of Art

The exponential distribution is one of the most popular and useful lifetime distributions for modeling and analysis. Its simple probability density and distribution functions help to derive various mathematical results in closed forms.

Several extensions of this distribution were also studied in the statistical literature. For example, the weighted exponential (WE), generalized exponential, gamma, and Weibull distributions are different extensions of the exponential distribution. In this article, we will introduce a modified version of the WE distribution. Gupta and Kundu [1] have introduced the WE distribution by using the Azzalini [2] method. The WE distribution can also be obtained as the sum of two independent but non-identical exponential distributions. Mathematically, the WE distribution has the following cumulative distribution function (cdf), and probability density function (pdf):

$$F(x;\alpha,\lambda) = \frac{1+\alpha}{\alpha}\left[1 - e^{-\lambda x} - \frac{1}{1+\alpha}\left(1 - e^{-\lambda x(1+\alpha)}\right)\right], \quad x > 0, \tag{1}$$

and

$$f(x;\alpha,\lambda) = \frac{1+\alpha}{\alpha}\lambda e^{-\lambda x}\left(1 - e^{-\lambda\alpha x}\right), \quad x > 0,$$

respectively. Here, $\alpha > 0$ and $\lambda > 0$ are parameters of the distribution. The pdf of the WE distribution is unimodal (contrary to the pdf of the exponential distribution) and the corresponding hazard rate function (hrf) is increasing for all values of α. It also possesses various likelihood ratio properties. Also, all its moments can be calculated explicitly. It follows that the related mean, variance, skewness, kurtosis, coefficient of variation, etc. can be computed easily. The technical details can be found in Gupta and Kundu [1] and Das and Kundu [3]. On the practical side, the WE distribution is suitable for modelling lifetime data when wear-out or ageing is present, providing a real alternative to the exponential distribution for this aim. The success of this weighted version of the exponential distribution has inspired a generation of researchers and practitioners for more in this direction.

Thus, several authors have conducted works in the theory of WE distribution, including the development of useful extensions, i.e., Roy and Adnan [4] proposed the wrapped WE distribution along with its basic characteristics, Hussian [5] developed the weighted inverted exponential distribution, Dey et al. [6], Oguntunde [7] and Kharazmi et al. [8] introduced different structures of generalized WE distributions and Alqallaf et al. [9] discussed several methods of estimation to estimate the parameters of the WE distribution. Later, Das and Kundu [3] presented various reliability properties of the original WE distribution given by Gupta and Kundu [1], Perveen et al. [10] considered size-biased double WE distribution, and Mahdavi and Jabbari [11] explained an extended exponential distribution with the method of Azzalini [2]. Recently, Oguntunde et al. [12] proposed a two-parameter inverted WE distribution, Mallick et al. [13] supplemented a new bounded distribution using the structure of the WE distribution and Bakouch et al. [14] proposed a new kind of weighted exponential distribution with the feature of logarithmic weight function. Further discussions on the WE distribution can be found in [15,16], and the references therein.

1.2. Contributions

The starting point of our investigation is the following remark: It can be seen that the cdf of the WE distribution, given in (1), is not reduced to the cdf of the exponential distribution for any choice of α (except the limit case). Thus, the WE distribution does not involve the most popular and useful exponential distribution. To overcome this drawback, we consider a new cdf based on a simple mixture of the WE and exponential distributions, with the mixture weight defined with α. It aims to privilege the WE distribution for large values of α, and the exponential distribution for small values of α. The proposed distribution is based on a mixture scheme involving the exponential and WE distributions, as described below:

Let $F_*(x;\lambda) = 1 - e^{-\lambda x}$, $\lambda, x > 0$ be the cdf of the former exponential distribution with parameter λ. Then, we consider the following cdf, obtained by the mixture operation:

$$\begin{aligned} F_o(x;\alpha,\lambda) &= \frac{1}{\alpha+1}F_*(x;\lambda) + \frac{\alpha}{1+\alpha}F(x;\alpha,\lambda) \\ &= 1 - \frac{1}{\alpha+1}e^{-\lambda x}\left(\alpha + 2 - e^{-\lambda\alpha x}\right), \end{aligned} \tag{2}$$

where $x > 0, \alpha \geq 0, \lambda > 0$, and $F(x;\alpha,\lambda)$ is the cdf of the WE distribution as defined in (1). The distribution corresponding to the cdf $F_o(x;\alpha,\lambda)$ can be called the modified WE (MWE) distribution. The main novelty in the construction of the MWE distribution remains in the considered distribution mixture, with a weight depending on the distribution parameter α only. Thus, it approaches the WE distribution when α is large, with the same scale parameter as α, or approaches the basic exponential distribution when α is small; the case $\alpha = 0$ is allowed, and we rediscover the exponential distribution with parameter λ; we recall that the exponential distribution is not a special case of the WE distribution. The moderate values of α operate as a balance between the WE and exponential distributions. To our knowledge, the MWE distribution is the first one to have such a specific compromise distributional structure.

In addition to that, the arguments in favor of the MWE distribution are listed as:

(i) The cdf can be written as $F_o(x;\alpha,\lambda) = aF(x;\lambda) + bF(x;\lambda(\alpha+1))$, where $a = (\alpha+2)/(\alpha+1) > 0$ and $b = -1/(\alpha+1) < 0$, meaning that the MWE distribution also belongs to the family of generalized mixture of two exponential distributions, following the spirit of the distribution proposed by [16],
(ii) The cdf is quite simple to manage and consequently, the MWE distribution can be studied in an-depth manner on all the theoretical and practical aspects,
(iii) Thanks to the parameter α, the related pdf can be decreasing or unimodal, and the related hrf can be constant or increasing as proven later,
(iv) In some concrete scenarios, the MWE model can be more efficient in data fitting than the exponential or WE models, among other lifetime models.

All these points will be discussed in detail in this article. In the first part, we deeply investigate the properties of the proposed distribution. The main functions are expressed, such as the cumulative distribution, probability density, hazard rate, cumulative hazard rate, quantile, and survival functions. The shape behavior of the probability density and hazard rate functions is highlighted. The moments and moment-generating functions are calculated in an explicit form. The Bonferroni and Lorenz curves are discussed. We perform an entropy analysis, with the determination of the Rényi entropy. The stress-strength reliability is expressed under general and precise configurations. The mean residual life function is also studied and commented on. The remaining part is devoted to the associated model, showing how it can be applied in a real-world statistical scenario dealing with data. In this regard, by employing five different estimation methods and considering simulated data, we show how the obtained estimates performed well. The real-data analysis section is devoted to the concrete applications of the model with the use of famous data sets. The performance of the new model is compared with that of the WE, Weibull, gamma, and generalized exponential models. The obtained comparison results are quite favorable to the proposed model.

1.3. Paper Organization

The following sections make up the article. Section 2 provides the mathematical background behind the MWE distribution, along with some notable statistical properties. Section 3 is devoted to the parameter estimation of the proposed model. A simulation study is performed in Section 4. An application is given in Section 5. Concluding remarks are formulated in Section 6.

2. Statistical Properties

The mathematical background of the MWE distribution, as well as several important statistical features, are presented in this section.

2.1. Quantile and Survival Functions

Inverting $F_o(x;\alpha,\lambda)$ yields the quantile function of the MWE distribution. By denoting it as $Q_o(y;\alpha,\lambda)$, it thus satisfies the following nonlinear equation:

$$e^{-\lambda Q_o(y;\alpha,\lambda)}\left(\alpha+2-e^{-\lambda\alpha Q_o(y;\alpha,\lambda)}\right) = (\alpha+1)(1-y), \quad y \in (0,1).$$

It is evident that this function has no closed-form, but numerical work can be done to determine some punctual values. The other characteristic of the MWE distribution is the survival function (sf) obtained as

$$S_o(x;\alpha,\lambda) = \frac{1}{\alpha+1}e^{-\lambda x}\left(\alpha+2-e^{-\lambda\alpha x}\right), \quad x>0.$$

2.2. Shapes of the Probability Density and Hazard Rate Functions

The expression of the pdf of the MWE distribution is given as

$$f_o(x;\alpha,\lambda) = \lambda e^{-\lambda x}\left(\frac{\alpha+2}{\alpha+1}-e^{-\lambda\alpha x}\right), \quad x>0.$$

Note that, for $x>0$, we can write it as

$$f_o(x;\alpha,\lambda) = af_*(x;\lambda)+bf_*(x;\lambda(\alpha+1)), \quad a=\frac{\alpha+2}{\alpha+1}, \quad b=-\frac{1}{\alpha+1}. \tag{3}$$

where $f_*(x;\lambda)=\lambda e^{-\lambda x}$. Since the exponential distribution is well mastered in all the theoretical and practical aspects, this formula will be useful in the next.

We have $f_o(0;\alpha,\lambda)=\lambda/(\alpha+1)$ and $\lim_{x\to+\infty} f_o(x;\alpha,\lambda)=0$, for all the values of the parameters. The shape behavior of this pdf is summarized in the following proposition.

Proposition 1. *The two following cases can be distinguished:*

- *if $\alpha\in[0,(\sqrt{5}-1)/2)$, then $f_o(x;\alpha,\lambda)$ is decreasing.*
- *if $\alpha\geq(\sqrt{5}-1)/2$, $f_o(x;\alpha,\lambda)$ is unimodal, with the mode:*

$$x_* = \frac{1}{\alpha\lambda}\log\left[\frac{(\alpha+1)^2}{\alpha+2}\right].$$

Proof. The proof follows the standard function analysis lines; it consists of studying the derivative of $f_o(x;\alpha,\lambda)$. We have

$$\frac{d}{dx}f_o(x;\alpha,\lambda) = \frac{\lambda^2}{\alpha+1}e^{-\lambda(\alpha+1)x}[(\alpha+1)^2-(\alpha+2)e^{\lambda\alpha x}].$$

Then,

- if $\alpha\in[0,(\sqrt{5}-1)/2)$, then

$$\frac{d}{dx}f_o(x;\alpha,\lambda) \leq \frac{\lambda^2}{\alpha+1}e^{-\lambda(\alpha+1)x}[(\alpha+1)^2-(\alpha+2)]<0,$$

  which implies that $f_o(x;\alpha,\lambda)$ is decreasing; the maximal point is obtained at $x=0$.
- In the case, $\alpha\geq(\sqrt{5}-1)/2$, we have $(\alpha+1)^2\geq(\alpha+2)$, and one value of x vanished $df_o(x;\alpha,\lambda)/dx$; it is given by $x=x_*$. For $x<x_*$, we have $df_o(x;\alpha,\lambda)/dx>0$ and for $x>x_*$, $df_o(x;\alpha,\lambda)/dx<0$, implying that $x=x_*$ is a maximal point; it corresponds to the mode of the MWE distribution.

This ends the proof of Proposition 1. □

In order to complete the results in Proposition 1, note that $(\sqrt{5}-1)/2\approx 0.618034$, and $f_o(x_*;\alpha,\lambda)=\alpha\lambda(\alpha+2)^{1/\alpha+1}/(\alpha+1)^{2(1/\alpha+1)}$.

Here, we see a difference with the WE distribution; the pdf of the MWE distribution can be decreasing, whereas the pdf of the WE distribution is always bell-shaped. So the case $\alpha\in[0,(\sqrt{5}-1)/2)$ offers more flexibility in this regard. This behavior of the pdf can be seen in Figure 1.

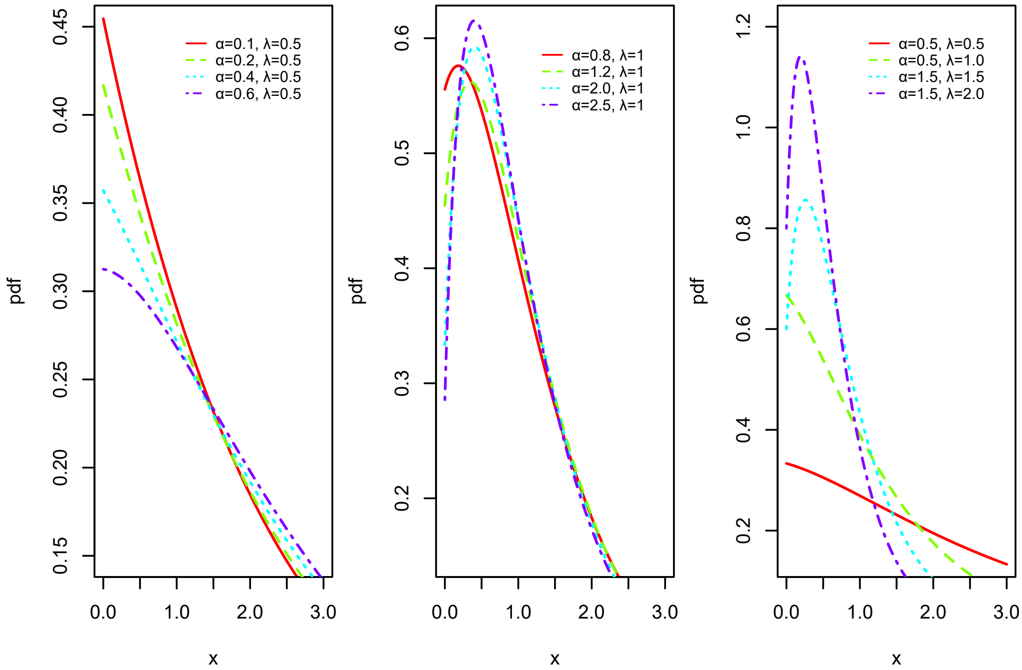


Figure 1. Graphics of the pdf of the MWE distribution.

We would like to point out that this figure was created using the R software, and that all subsequent graphical and numerical works will be provided as supplementary material.

Now, a hazard rate analysis is performed. To begin, the cumulative hrf (chrf) is obtained as

$$H_o(x;\alpha,\lambda) = \log(\alpha+1) + \lambda x - \log(\alpha+2-e^{-\lambda\alpha x}), \quad x > 0.$$

The hrf follows:

$$h_o(x;\alpha,\lambda) = \lambda - \lambda\alpha\frac{e^{-\lambda\alpha x}}{\alpha+2-e^{-\lambda\alpha x}}, \quad x > 0.$$

We have $h_o(0;\alpha,\lambda) = \lambda/(\alpha+1)$ and $\lim_{x\to+\infty} h_o(x;\alpha,\lambda) = \lambda$. In addition, on can remark that $h_o(x;\alpha,\lambda) < h_*(x;\lambda)$, where $h_*(x;\lambda) = \lambda$ is the hrf of the exponential distribution with parameter $\lambda > 0$. We have a stochastic ordering in this sense. The shape behavior of this hrf is investigated in the following proposition.

Proposition 2. *The hrf $h_o(x;\alpha,\lambda)$ is increasing and concave.*

Proof. The proof follows typical function analysis lines, with the derivative of $h_o(x;\alpha,\lambda)$ being studied. For $\alpha > 0$, we have

$$\frac{d}{dx}h_o(x;\alpha,\lambda) = \alpha^2(\alpha+2)\lambda^2\frac{e^{\lambda\alpha x}}{\left[e^{\lambda\alpha x}(\alpha+2)-1\right]^2} > 0,$$

which implies that the hrf is increasing. Therefore, we have $h_o(x;\alpha,\lambda) \in [\lambda/(\alpha+1),\lambda)$ for all values of x and the parameters.
Moreover, for $\alpha > 0$, we have

$$\frac{d^2}{dx^2}h_o(x;\alpha,\lambda) = -\alpha^3(\alpha+2)\lambda^3 e^{\lambda\alpha x}\frac{e^{\lambda\alpha x}(\alpha+2)+1}{\left[e^{\lambda\alpha x}(\alpha+2)-1\right]^3} < 0.$$

This implies that $h_o(x;\alpha,\lambda)$ is concave for $\alpha > 0$. Proposition 2 is proved. □

Thus, based on Proposition 2, the MWE distribution has the increasing failure rate property for $\alpha > 0$ (the case $\alpha = 0$ corresponding to the exponential distribution is excluded), which is a demanded property for statistical analysis of lifetime phenomena. Figure 2 illustrates these curvature properties by considering various values of α and λ.

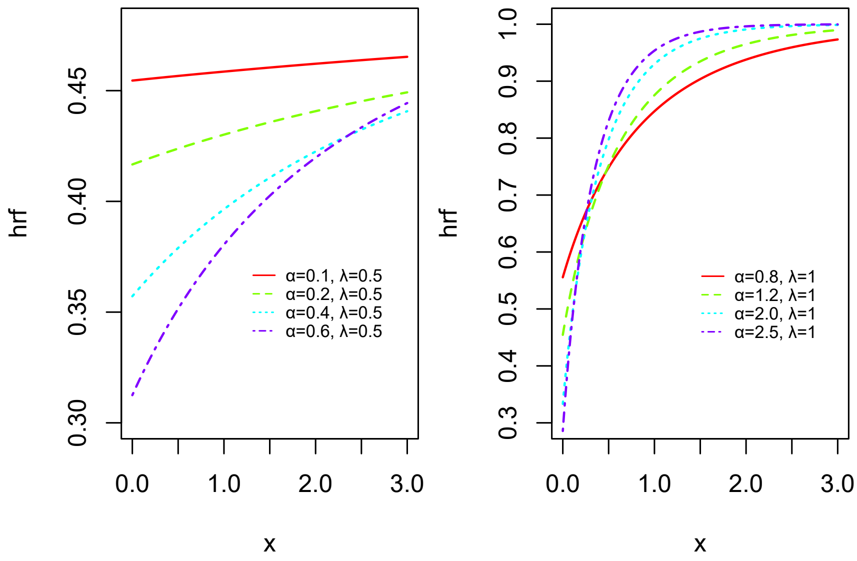


Figure 2. Graphics of the hrf of the MWE distribution.

2.3. Moments and Moment Generating Function

The moments of the MWE distribution are expressed in the next proposition.

Proposition 3. *For any positive integer r, the r^{th} raw moment of a random variable X with the MWE distribution is given by*

$$m_r = E(X^r) = \frac{r!}{\lambda^r(\alpha+1)}\left(\alpha + 2 - \frac{1}{(\alpha+1)^r}\right).$$

Proof. Owing to the general mixture expressions given in (3) and the expression of the moments of the exponential distribution, the r^{th} raw moment of the MWE distribution is

$$\begin{aligned} m_r &= \int_0^{+\infty} x^r f_o(x;\alpha,\lambda)dx = a\int_0^{+\infty} x^r f_*(x;\lambda)dx + b\int_0^{+\infty} x^r f_*(x;\lambda(\alpha+1))dx \\ &= a\frac{r!}{\lambda^r} + b\frac{r!}{\lambda^r(\alpha+1)^r} = \frac{r!}{\lambda^r(\alpha+1)}\left(\alpha+2-\frac{1}{(\alpha+1)^r}\right). \end{aligned}$$

The stated result is obtained □

In particular, based on Proposition 3, we have

$$m_1 = \frac{\alpha(\alpha+3)+1}{\lambda(\alpha+1)^2}, \quad m_2 = \frac{2(\alpha^3+4\alpha^2+5\alpha+1)}{\lambda^2(\alpha+1)^3}$$

$$m_3 = \frac{6(\alpha^4+5\alpha^3+9\alpha^2+7\alpha+1)}{\lambda^3(\alpha+1)^4}, \quad m_4 = \frac{24(\alpha^5+6\alpha^4+14\alpha^3+16\alpha^2+9\alpha+1)}{\lambda^4(\alpha+1)^5}.$$

In particular, we can remark that, for all the values of α and λ, we have $m_1 \geq 1/\lambda$. The variance of the MWE distribution is

$$V = \frac{\alpha^4+4\alpha^3+7\alpha^2+6\alpha+1}{\lambda^2(\alpha+1)^4}.$$

It is worth noting that, for all the values of α and λ, we have $V \geq 1/\lambda^2$.

The moment-generating function encodes all the moments of the distribution. It is expressed for the MWE distribution in the following result.

Proposition 4. *The moment generating function of a random variable X with the MWE distribution is given by, for any $t < \lambda$,*

$$M(t) = E(e^{tX}) = \frac{\lambda[\lambda(\alpha+1)^2 - t]}{(\alpha+1)(\lambda-t)(\lambda(\alpha+1)-t)}.$$

Proof. For $t < \lambda$, the moment generating function is obtained as

$$\begin{aligned} M(t) &= \int_0^{+\infty} e^{tx} f_o(x;\alpha,\lambda)dx = a\int_0^{+\infty} e^{tx} f_*(x;\lambda)dx + b\int_0^{+\infty} e^{tx} f_*(x;\lambda(\alpha+1))dx \\ &= a\frac{\lambda}{\lambda-t} + b\frac{\lambda(\alpha+1)}{\lambda(\alpha+1)-t} = \frac{\alpha+2}{\alpha+1}\frac{\lambda}{\lambda-t} - \frac{\lambda}{\lambda(\alpha+1)-t} \\ &= \frac{\lambda[\lambda(\alpha+1)^2 - t]}{(\alpha+1)(\lambda-t)(\lambda(\alpha+1)-t)}. \end{aligned}$$

The stated result is obtained. □

Owing to Proposition 4, we see that the moment generating function is quite manageable. In particular, we have

$$\log[M(t)] = \log\lambda + \log[\lambda(\alpha+1)^2 - t] - \log(\alpha+1) - \log(\lambda-t) - \log[\lambda(\alpha+1)-t],$$

from which we deduce the r^{th} cumulant of the MWE distribution as follows:

$$\kappa_r = (r-1)!\frac{1}{\lambda^r}\left[1 + \frac{1}{(\alpha+1)^r} - \frac{1}{(\alpha+1)^{2r}}\right].$$

One can check that $\kappa_1 = m_1^*$ and $\kappa_2 = V$. From κ_3, we get the third central moments:

$$\kappa_3 = \frac{2(\alpha^6 + 6\alpha^5 + 15\alpha^4 + 21\alpha^3 + 18\alpha^2 + 9\alpha + 1)}{\lambda^3(\alpha+1)^6}.$$

Based on these cumulants, the moment skewness and kurtosis are defined by

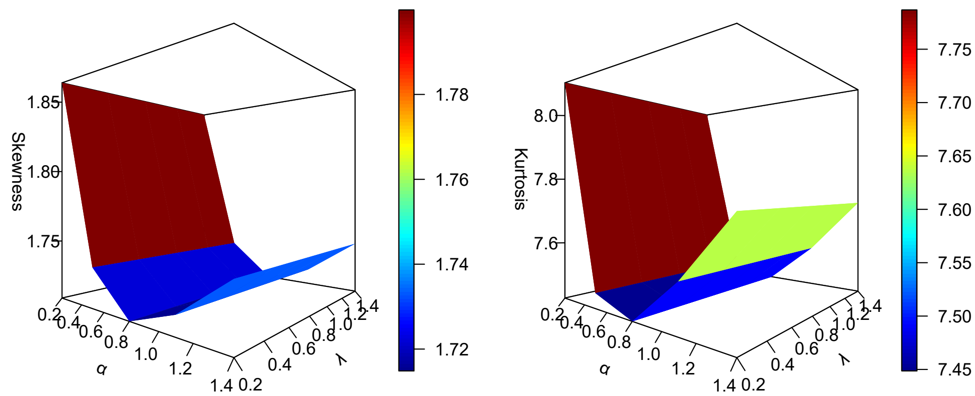
$$\gamma_1 = \frac{\kappa_3}{V^{3/2}}, \quad \beta_2 = \frac{\kappa_4 + 3\kappa_2^2}{V^2},$$

respectively. Table 1 lists some moment measures of the MWE distribution, whose variability in values is also illustrative of the moment flexibility of the MWE distribution.

Table 1. Numerical values of the mean, variance, skewness and kurtosis of the MWE distribution.

λ	α	m_1	V	γ_1	β_2
0.5000	0.2000	2.2778	4.8488	1.8639	8.1028
0.5000	0.4000	2.4082	4.9996	1.7628	7.5801
0.5000	0.6000	2.4688	4.9521	1.7198	7.4207
0.5000	0.8000	2.4938	4.8535	1.7089	7.4265
0.5000	1.0000	2.5000	4.7500	1.7146	7.5042
0.5000	1.2000	2.4959	4.6557	1.7283	7.6100
0.5000	1.4000	2.4861	4.5739	1.7454	7.7230
α	λ	m_1	V	γ_1	β_2
1.2000	0.4000	3.1198	7.2745	1.7283	7.6100
1.2000	0.6000	2.0799	3.2331	1.7283	7.6100
1.2000	0.8000	1.5599	1.8186	1.7283	7.6100
1.2000	1.0000	1.2479	1.1639	1.7283	7.6100
1.2000	1.2000	1.0399	0.8083	1.7283	7.6100
1.2000	1.4000	0.8914	0.5938	1.7283	7.6100
1.2000	1.6000	0.7800	0.4547	1.7283	7.6100

A graphical approach to the variation of the moment skewness and kurtosis is given in Figure 3.



(**a**) (**b**)

Figure 3. Graphics of the (**a**) skewness and (**b**) kurtosis of the MWE distribution.

From Table 1 and Figure 3, it is visible that the proposed distribution is positively skewed and leptokurtic distribution for the given parametric values. In addition, the moment skewness and kurtosis are non-monotonic with respect to α.

Using similar developments, the incomplete moment of the MWE distribution can be expressed. This is formulated in the following result.

Proposition 5. *For any integer r and $y \geq 0$, the rth incomplete moment of a random variable X with the MWE distribution taken at y is given by*

$$m_r^*(y) = \int_0^y x^r f_o(x;\alpha,\lambda)dx = \frac{\alpha+2}{\lambda^r(\alpha+1)}\gamma(r+1,\lambda y) - \frac{1}{\lambda^r(\alpha+1)^{r+1}}\gamma(r+1,\lambda(\alpha+1)y),$$

where $\gamma(a,x)$ refers to the standard lower incomplete gamma function.

One can use it to express the Bonferroni and Lorenz curves, as well as the mean residual life function, among other quantities of interest.

2.4. Bonferroni and Lorenz Curves

The Bonferroni and the Lorenz curves, as well as the Bonferroni and the Gini indices, have many applications in the field of reliability, insurance, medicine, demography, and economics. By adopting the notations of the MWE distribution, the Bonferroni and the Lorenz curves are defined by

$$b_0(p) = \frac{1}{pm_1}\int_0^q x f_o(x;\alpha,\lambda)dx, \quad l_0(p) = \frac{1}{\mu}\int_0^q x f_o(x;\alpha,\lambda)dx,$$

respectively, where $q = Q_o(p;\alpha,\lambda)$. It is worth noting that the integral terms refer to the first incomplete moment of the MWE distribution. In addition, the Bonferroni and the Gini indices are defined by

$$\mathrm{BI} = 1 - \int_0^1 b_0(p)dp, \quad \mathrm{GI} = 1 - 2\int_0^1 l_0(p)dp,$$

respectively. A comprehensive explanation about these indices can be seen in Giorgi and Nadrajah [17] for different parametric families. For the MWE distribution, based on Proposition 5, the Bonferroni and Lorenz curves are given as follows:

$$b_0(p) = \frac{(\alpha^2+3\alpha+1) - (\alpha+2)(\alpha+1)(1+\lambda q)e^{-\lambda q} + (1+\lambda(\alpha+1)q)e^{-\lambda(\alpha+1)q}}{p(\alpha(\alpha+3)+1)}$$

and

$$l_0(p) = \frac{(\alpha^2+3\alpha+1) - (\alpha+2)(\alpha+1)(1+\lambda q)e^{-\lambda q} + (1+\lambda(\alpha+1)q)e^{-\lambda(\alpha+1)q}}{(\alpha(\alpha+3)+1)},$$

respectively.

2.5. Rényi Entropy

The entropy of a distribution (or a random variable) is a measure of the variation of the uncertainty. The concept of entropy is useful in the area of physics, probability, statistics, communication theory, and economics. There are different types of entropy in the distribution theory. In this part, we focus on one of the most useful entropy measures: the Rényi entropy. For the historical and current results on entropy, we refer to the survey by Amigo et al. [18].

In the setting of the MWE distribution, the Rényi entropy is defined as

$$R_\eta = \frac{1}{1-\eta}\log\left(\int_0^{+\infty}(f_o(x;\alpha,\lambda))^\eta dx\right),$$

where $\eta > 0, \eta \neq 1$. The following proposition expresses the Rényi entropy of the MWE distribution.

Proposition 6. *The Rényi entropy of the MWE distribution can be expanded as*

$$R_\eta = -\log\lambda + \frac{1}{1-\eta}\log\left(\sum_{r=0}^{+\infty}\frac{(-1)^r}{\eta+\alpha r}\binom{\eta}{r}\left(\frac{\alpha+2}{\alpha+1}\right)^{\eta-r}\right).$$

Proof. By noticing that $[(\alpha+1)/(\alpha+2)]e^{-\lambda\alpha x} \in (0,1)$, the generalized binomial theorem gives

$$\begin{aligned}\int_0^{+\infty}(f_o(x;\alpha,\lambda))^\eta dx &= \lambda^\eta\int_0^{+\infty}e^{-\lambda\eta x}\left(\frac{\alpha+2}{\alpha+1}-e^{-\lambda\alpha x}\right)^\eta dx\\ &= \lambda^\eta\int_0^{+\infty}\sum_{r=0}^{+\infty}(-1)^r\binom{\eta}{r}\left(\frac{\alpha+2}{\alpha+1}\right)^{\eta-r}e^{-\lambda(\eta+\alpha r)x}dx\\ &= \lambda^\eta\sum_{r=0}^{+\infty}(-1)^r\binom{\eta}{r}\left(\frac{\alpha+2}{\alpha+1}\right)^{\eta-r}\int_0^{+\infty}e^{-\lambda(\eta+\alpha r)x}dx\\ &= \lambda^{\eta-1}\sum_{r=0}^{+\infty}\frac{(-1)^r}{\eta+\alpha r}\binom{\eta}{r}\left(\frac{\alpha+2}{\alpha+1}\right)^{\eta-r}.\end{aligned}$$

By taking the logarithmic function of this expression, and multiplying by $1/(1-\eta)$, we prove Proposition 6. □

Owing to Proposition 6, one may use the following approximation for computational purposes:

$$R_\eta \approx -\log\lambda+\frac{1}{1-\eta}\log\left(\sum_{r=0}^{U}\frac{(-1)^r}{\eta+\alpha r}\binom{\eta}{r}\left(\frac{\alpha+2}{\alpha+1}\right)^{\eta-r}\right),$$

where U denotes any large integer. This provides an alternative approach to the numerical computation of the integral term in the original definition of R_η.

2.6. Reliability Characteristics of the MWE Distribution

The stress-strength reliability of two systems is a metric for comparing their lifetimes. Here, we derive the stress-strength reliability $R = P(X_1 > X_2)$, where X_1 and X_2 are independent random variables distributed with the MWE distribution with possible different parameters.

Proposition 7. *The stress-strength reliability $R = P(X_1 > X_2)$, where X_1 and X_2 are independent random variables distributed with the MWE distribution with parameters (α_1,λ_1) and (α_2,λ_2), respectively, is given by*

$$\begin{aligned}R = 1-\frac{\lambda_1}{\alpha_2+1}\Bigg[&\frac{(\alpha_1+2)(\alpha_2+2)}{(\alpha_1+1)(\lambda_1+\lambda_2)}-\frac{\alpha_2+2}{\lambda_1(\alpha_1+1)+\lambda_2}-\frac{\alpha_1+2}{(\alpha_1+1)(\lambda_1+\lambda_2(1+\alpha_2))}\\ &+\frac{1}{\lambda_1(\alpha_1+1)+\lambda_2(1+\alpha_2)}\Bigg].\end{aligned}$$

Proof. The proof uses the independence of X_1 and X_2, the definition of the MWE distribution, and several integral developments; we have

$$\begin{aligned}R &= \int_0^{+\infty}f_o(x;\alpha_1,\lambda_1)F_o(x;\alpha_2,\lambda_2)dx\\ &= \int_0^{+\infty}\lambda_1e^{-\lambda_1x}\left(\frac{\alpha_1+2}{\alpha_1+1}-e^{-\lambda_1\alpha_1x}\right)\left\{1-\frac{1}{\alpha_2+1}e^{-\lambda_2x}\left(\alpha_2+2-e^{-\lambda_2\alpha_2x}\right)\right\}dx\\ &= 1-\frac{\lambda_1}{\alpha_2+1}\Bigg[\frac{(\alpha_1+2)(\alpha_2+2)}{(\alpha_1+1)(\lambda_1+\lambda_2)}-\frac{\alpha_2+2}{\lambda_1(\alpha_1+1)+\lambda_2}-\frac{\alpha_1+2}{(\alpha_1+1)(\lambda_1+\lambda_2(1+\alpha_2))}\\ &\qquad+\frac{1}{\lambda_1(\alpha_1+1)+\lambda_2(1+\alpha_2)}\Bigg].\end{aligned}$$

This ends the proof. □

In particular, by applying Proposition 7, if $\alpha_1 = \alpha_2 = \alpha$, then R is reduced to

$$R = 1 - \frac{\lambda_1}{\alpha+1}\left(\frac{(\alpha+2)^2+1}{(\alpha+1)(\lambda_1+\lambda_2)} - \frac{\alpha+2}{\lambda_1(\alpha+1)+\lambda_2} - \frac{\alpha+2}{(\alpha+1)(\lambda_1+\lambda_2(1+\alpha))}\right).$$

Furthermore, if $\lambda_1 = \lambda_2 = \lambda$, then

$$R = 1 - \frac{1}{\alpha_2+1}\left(\frac{(\alpha_1+2)(\alpha_2+2)}{2(\alpha_1+1)} - \frac{\alpha_2+2}{\alpha_1+2} - \frac{\alpha_1+2}{(\alpha_1+1)(\alpha_2+2)} + \frac{1}{\alpha_1+\alpha_2+2}\right).$$

One can notice that, in this case, R does not depend on λ.

2.7. Mean Residual Life Function

The mean residual life function is useful in life testing situations. It is a function of time s which describes the expected additional lifetime given that a component has survived until time s. In the context of this study, for a random variable X with the MWE distribution, it is immediately given by

$$m_s = E(X - s \mid X \geq s) = \frac{1}{1 - F_o(s;\alpha,\lambda)}\int_s^{+\infty}(1 - F_o(x;\alpha,\lambda))dx$$
$$= \frac{(\alpha+1)(\alpha+2) - e^{-\lambda\alpha s}}{\lambda(\alpha+1)\left(\alpha+2-e^{-\lambda\alpha s}\right)}$$

Figure 4 represents the plot of this function for varying values of α and λ.

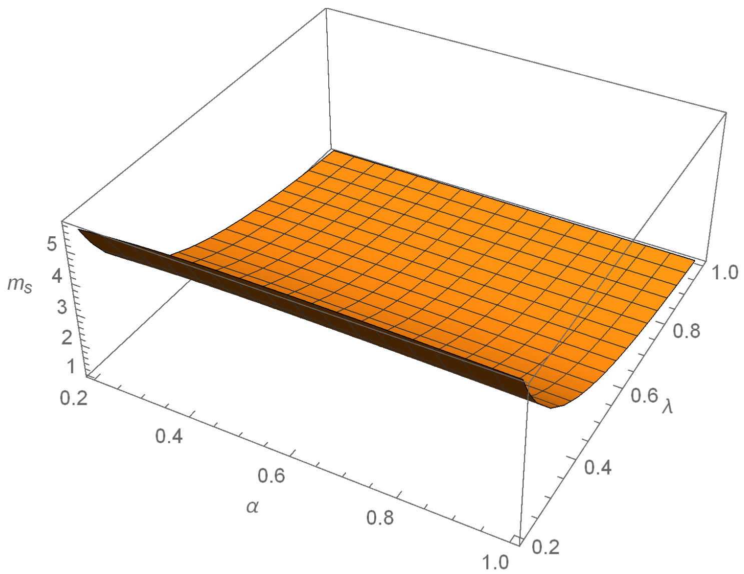


Figure 4. Graphic of the mean residual life function of the MWE distribution.

From Figure 4, we observe that, for the fixed values of α, the values of mean residual life decrease with the increasing values of λ. Similarly, for the fixed values of λ, the values of mean residual life decrease with the increasing values of α. So, we can conclude that the mean residual life function is decreasing in nature.

3. Parameters Estimation

In this section, we consider the MWE model, and focus on the parameter estimation of α and λ, assuming they are unknown. We consider several well-referenced methods to provide the estimates, and the performance of these methods is analyzed through a simulation study.

3.1. Maximum Likelihood Estimates

Let $(x_1, x_2, \ldots, x_n)$ be a random sample taken from the MWE distribution with parameters α and λ. The log-likelihood function is given by

$$l(\alpha, \lambda) = \sum_{i=1}^{n} \log(f_o(x_i; \alpha, \lambda)) = n \log \lambda - \lambda \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \log\left(\frac{\alpha+2}{\alpha+1} - e^{-\lambda\alpha x_i}\right).$$

The maximum likelihood estimates (MLEs) of parameters are defined by

$$(\hat{\alpha}, \hat{\lambda}) = \operatorname{argmax}_{(\alpha,\lambda)\in(0,+\infty)^2}\, l(\alpha, \lambda).$$

They can be obtained by solving the following nonlinear equations according to α and λ:

$$\frac{\partial l(\alpha, \lambda)}{\partial \alpha} = \sum_{i=1}^{n} \frac{\lambda(\alpha+1)^2 x_i e^{-\lambda\alpha x_i} - 1}{(\alpha+1)(\alpha+2) - (\alpha+1)^2 e^{-\lambda\alpha x_i}} = 0 \quad (4)$$

and

$$\frac{\partial l(\alpha, \lambda)}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{\alpha(\alpha+1) x_i e^{-\lambda\alpha x_i}}{\alpha + 2 - (\alpha+1) e^{-\lambda\alpha x_i}} = 0. \quad (5)$$

Since we do not have any explicit form of the MLEs, therefore, we need to use numerical methods to get estimates of parameters. After solving Equations (4) and (5), we get a precise numerical evaluation of $\hat{\alpha}$ and $\hat{\lambda}$.

3.2. Method of Moments Estimates

Let $(x_1, x_2, \ldots, x_n)$ be a random sample taken from the MWE distribution with parameters α and λ. Let $\mathcal{M}_r = (1/n) \sum_{i=1}^{n} x_i^r$. The method of moments estimates (MOMEs) of α and λ can be obtained by equating the first two raw moments of with sample moments, leading to solve the two following equations according to α and λ:

$$\frac{\alpha(\alpha+3)+1}{\lambda(\alpha+1)^2} - \mathcal{M}_1 = 0 \quad (6)$$

and

$$\frac{2(\alpha^3 + 4\alpha^2 + 5\alpha + 1)}{\lambda^2(\alpha+1)^3} - \mathcal{M}_2 = 0. \quad (7)$$

Thus, the MOMEs of α and λ can be obtained by solving (6) and (7) numerically.

3.3. Least Squares and Weighted Least Squares Estimates

The method of least squares was proposed by Swain et al. [19]. In this method, we minimize the distance between the vector of uniformized order statistics and the corresponding vector of expected values (for more details, see [20]).

Let $(x_1, x_2, \ldots, x_n)$ be a random sample taken from the MWE distribution with parameters α and λ. Also, let $x_{(1)}, x_{(2)}, \ldots, x_{(n)}$ be the order values in increasing order of $x_1, x_2, \ldots, x_n$.

Least Squares Estimates: The least square function is defined by

$$\mathcal{S}(\alpha,\lambda)=\sum_{i=1}^{n}\left(F_o\left(x_{(i)};\alpha,\lambda\right)-\frac{i}{n+1}\right)^2. \tag{8}$$

The least square estimates (LSEs) of α and λ are defined by

$$(\hat{\alpha},\hat{\lambda})=\mathrm{argmin}_{(\alpha,\lambda)\in(0,+\infty)^2}\,\mathcal{S}(\alpha,\lambda).$$

Now, using the cdf given in (2), and (8), we can derive two nonlinear equations after partially differentiating with respect to unknown parameters. The solution of these nonlinear equations can be computed by using the Monte Carlo simulation.

Weighted Least Squares Estimates: The weighted least square function is defined by

$$\mathcal{W}(\alpha,\lambda)=\sum_{i=1}^{n}\eta_i\left(F_o\left(x_{(i)};\alpha,\lambda\right)-\frac{i}{n+1}\right)^2,$$

where $\eta_i=(n+1)^2(n+2)/[i(n-i+1)]$. The weighted least square estimates (WLSEs) of α and λ are defined by

$$(\hat{\alpha},\hat{\lambda})=\mathrm{argmin}_{(\alpha,\lambda)\in(0,+\infty)^2}\,\mathcal{W}(\alpha,\lambda).$$

Therefore, the WLSEs of α and λ can be obtained by using a similar procedure to the LSEs.

3.4. Cramér-von Mises Estimates

The Cramér-von Mises method is also similar to the previously mentioned two methods. The Cramér-von Mises function is defined by

$$C(\alpha,\lambda)=\frac{1}{12n}+\sum_{i=1}^{n}\left(F_o\left(x_{(i)};\alpha,\lambda\right)-\frac{2i-1}{2n}\right)^2.$$

The Cramér-von Mises estimates (CMEs) of α and λ are defined by

$$(\hat{\alpha},\hat{\lambda})=\mathrm{argmin}_{(\alpha,\lambda)\in(0,+\infty)^2}\,C(\alpha,\lambda).$$

Hence, the CMEs of α and λ can be obtained by using a similar procedure to the WLSEs or LSEs.

4. Simulation

In this section, we perform a simulation study to evaluate the efficiency of the estimates of the MWE model parameters as described in Section 3. From the technical viewpoint, we used the Monte Carlo algorithm and applied Newton's method and the BFGS (Broyden-Fletcher-Goldfarb-Shanno) algorithm, established by Broyden [21], Fletcher [22], Goldfarb [23], and Shanno [24] in R software. Biases and MSEs based on 5000 replicates of the parameters of the proposed model are reported in Tables 2–4. Different sample sizes ($n=50,100,200,500$) and different parameter combinations are used to better understand the behavior of the estimates.

Table 2. Biases and MSEs of the estimates for $\alpha = 0.5$ and $\lambda = 1$.

n	Estimate	Bias		MSE	
		$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$
50	MLE	0.6411	−0.0459	6.4825	0.0195
	MOME	0.7038	−0.0356	0.9878	0.0217
	OLSE	0.3002	−0.0956	1.0082	0.0272
	WLSE	0.3090	−0.0840	1.2376	0.0246
	CME	0.4610	−0.0800	1.3098	0.0244
100	MLE	0.2742	−0.0446	0.7519	0.0109
	MOME	0.6530	−0.0367	0.8751	0.0124
	OLSE	0.1904	−0.0828	0.4913	0.0172
	WLSE	0.1782	−0.0712	0.5446	0.0147
	CME	0.2772	−0.0727	0.6083	0.0152
200	MLE	0.1165	−0.0409	0.1834	0.0059
	MOME	0.5855	−0.0330	0.7346	0.0058
	OLSE	0.1391	−0.0674	0.2210	0.0099
	WLSE	0.1177	−0.0573	0.1701	0.0080
	CME	0.1794	−0.0619	0.2406	0.0090
500	MLE	0.0620	−0.0365	0.0317	0.0028
	MOME	0.5134	−0.0274	0.5407	0.0023
	OLSE	0.0981	−0.0527	0.0693	0.0047
	WLSE	0.0822	−0.0457	0.0532	0.0037
	CME	0.1136	−0.0506	0.0744	0.0044

Table 3. Biases and MSEs of the estimates for $\alpha = 1$ and $\lambda = 1$.

n	Estimate	Bias		MSE	
		$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$
50	MLE	0.7146	−0.0072	4.7462	0.0201
	MOME	0.1304	−0.0103	0.5513	0.0270
	OLSE	0.1174	−0.0410	1.9271	0.0243
	WLSE	0.2331	−0.0352	2.9491	0.0223
	CME	0.3445	−0.0250	2.2686	0.0227
100	MLE	0.4174	−0.0078	3.5071	0.0116
	MOME	−0.0152	−0.0182	0.5008	0.0158
	OLSE	0.0858	−0.0207	1.5188	0.0137
	WLSE	0.1388	−0.0166	2.0104	0.0126
	CME	0.2204	−0.0114	1.6787	0.0130
200	MLE	0.0649	−0.0017	1.5542	0.0062
	MOME	−0.1548	−0.0145	0.4094	0.0086
	OLSE	−0.0483	−0.0052	0.9840	0.0072
	WLSE	−0.0602	−0.0028	1.0602	0.0064
	CME	0.0158	−0.0002	0.9988	0.0069
500	MLE	−0.2141	0.0089	0.3936	0.0022
	MOME	−0.3341	−0.0051	0.3066	0.0030
	OLSE	−0.2070	0.0160	0.2761	0.0026
	WLSE	−0.2370	0.0141	0.2548	0.0023
	CME	−0.1802	0.0179	0.2866	0.0026

Table 4. Biases and MSEs of the estimates for $\alpha = 1.5$ and $\lambda = 1$.

n	Estimate	Bias		MSE	
		$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\lambda}$
50	MLE	0.2723	0.0012	4.7454	0.0162
	MOME	−0.0325	0.0342	0.3603	0.0200
	OLSE	0.0888	−0.0405	2.2201	0.0202
	WLSE	0.2376	−0.0312	2.8245	0.0182
	CME	0.2884	−0.0303	2.2121	0.0192
100	MLE	−0.0163	0.0078	2.0841	0.0083
	MOME	−0.0247	0.0291	0.3181	0.0096
	OLSE	0.1303	−0.0244	1.7747	0.0103
	WLSE	0.2232	−0.0166	2.0862	0.0091
	CME	0.2683	−0.0205	1.8484	0.0099
200	MLE	−0.2227	0.0148	0.6364	0.0042
	MOME	0.0318	0.0267	0.2444	0.0046
	OLSE	0.1333	−0.0099	1.3872	0.0050
	WLSE	0.1396	−0.0039	1.3626	0.0044
	CME	0.2233	−0.0092	1.4512	0.0049
500	MLE	−0.3446	0.0196	0.2372	0.0017
	MOME	0.1420	0.0222	0.1582	0.0018
	OLSE	0.0786	0.0008	0.8774	0.0021
	WLSE	−0.0197	0.0063	0.4987	0.0018
	CME	0.1225	0.0006	0.9003	0.0021

The results of Tables 2–4 disclose that the range of biases and MSEs of all the parameters is very small, indicating the stable nature of the MWE distribution. In some cases, we observe that if we increase the sample size, we get fewer MSEs for all estimates. This indicates that these competing estimates have performed well in terms of bias and MSEs for large sample sizes. Hence, simulation results found empirical evidence of the stability of our estimates.

5. Real Data Analysis

To see the applications in real-life scenarios of the MWE model, we use two famous real data sets. To achieve this aim, we determine the MLEs and their standard errors of the MWE model with other competing models. To find the best model, we consider standard performance validation criteria such as log-likelihood, Akaike's Information Criterion (AIC), and a few goodness-of-fit tests statistics, which are the Kolmogorov-Smirnov (KS), Cramér-von Mises (CVM) and Anderson-Darling (AD). The p-values related to the KS, CVM, and AD are also considered. They are denoted as p(KS), p(CVM) and p(AD), respectively. The best model is the one with the smallest AIC, KS, CVM, and AD, and the greatest p-values. For the definitions and more information on these criteria, see [25].

The data are fitted to the MWE, WE, Weibull (W), gamma (G), and generalized exponential (GE) distributions (see [26] for the GE distribution). The pdfs of these distributions are given as follows:

$$f_{WE}(x;\alpha,\lambda) = \frac{\alpha+1}{\alpha}\lambda e^{-\alpha\lambda x}(1-e^{-\alpha\lambda x}),\ x,\alpha,\lambda > 0,$$

$$f_W(x;\mu,\sigma) = \frac{\mu}{\sigma}\left(\frac{x}{\sigma}\right)^{\mu-1} e^{-\left(\frac{x}{\sigma}\right)^{\mu}},\ x,\mu,\sigma > 0,$$

$$f_G(x;\mu,\sigma) = \frac{1}{\Gamma(\mu)}\frac{1}{\sigma^{\mu}} x^{\mu-1} e^{-\frac{x}{\sigma}},\ x,\mu,\sigma > 0,$$

where $\Gamma(\mu)$ denotes the standard gamma function, and

$$f_{GE}(x;\alpha,\lambda) = \alpha\lambda(1-e^{-\alpha x})^{\alpha-1}e^{-\lambda x}, \; x,\alpha,\lambda > 0.$$

The following are the details of the two real data sets and statistical analysis:

Data set 1: The failure times data can be found in [27]. The values are: 0.12, 0.43, 0.92, 1.14, 1.24, 1.61, 1.93, 2.38, 4.51, 5.09, 6.79, 7.64, 8.45, 11.9, 11.94, 13.01, 13.25, 14.32, 17.47, 18.1, 18.66, 19.23, 24.39, 25.01, 26.41, 26.8, 27.75, 29.69, 29.84, 31.65, 32.64, 35, 40.7, 42.34, 43.05, 43.4, 44.36, 45.4, 48.14, 49.1, 49.44, 51.17, 58.62, 60.29, 72.13, 72.22, 72.25, 72.29, 85.2, 89.52.

Table 5 reports the MLEs and their standard errors of the fitted models for the data set 1.

Table 5. MLEs (standard errors) for the data set 1.

Model	MLEs (Standard Errors)
MWE	$\hat{\alpha} = 0.4069$ (0.3763), $\hat{\lambda} = 0.0399$ (0.0056)
W	$\hat{\mu} = 1.0149$ (0.1210), $\hat{\sigma} = 30.3358$ (4.4144)
G	$\hat{\mu} = 0.9267$ (0.1621, $\hat{\sigma} = 32.5640$ (7.4381)
GE	$\hat{\alpha} = 0.9086$ (0.1622), $\hat{\lambda} = 0.0312$ (0.0058)
WE	$\hat{\alpha} = 0.0097$ (0.6436), $\hat{\lambda} = 0.0660$ (0.0198)

In order to complete Table 5, we also provide the estimates of the parameters via different estimation methods in Table 6.

Table 6. The parameter estimates for the data set 1.

	MOME	OLSE	WLSE	CME
$\hat{\alpha}$	1.8324	0.2399	0.3517	0.3289
$\hat{\lambda}$	0.0417	0.0344	0.0373	0.0358

Table 7 reports the values of the selection criteria statistics for the data set 1.

Table 7. Selection criteria statistics of the models for the data set 1.

Model	AIC	KS	*p*(KS)	CVM	*p*(CVM)	AD	*p*(AD)
MWE	443.8088	0.0955	0.7161	0.1015	0.5796	0.8336	0.4568
W	444.6980	0.1113	0.5290	0.1269	0.4698	0.8907	0.4195
G	444.5201	0.1226	0.4074	0.1477	0.3976	0.8702	0.4325
GE	444.4178	0.1243	0.3903	0.1520	0.3844	0.8719	0.4314
WE	468.9382	0.1544	0.1658	0.2851	0.1489	4.6374	0.0043

The results in Table 7 suggest that the MWE model is the best model among other competing lifetime models. Therefore, the MWE model may be a good alternative to other lifetime models having similar statistical properties. Figure 5 displays the histogram and fitted pdfs, and Figure 6 presents the empirical and estimated cdfs plots.

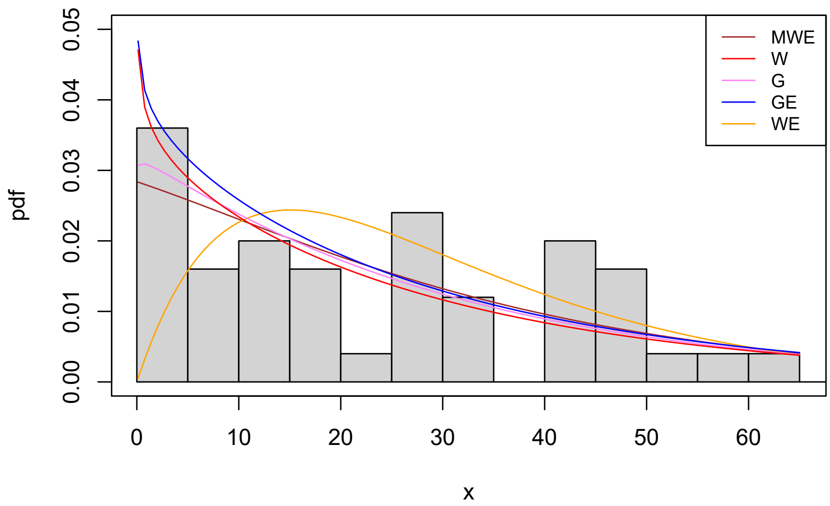


Figure 5. The histogram with the fitted pdfs for the data set 1.

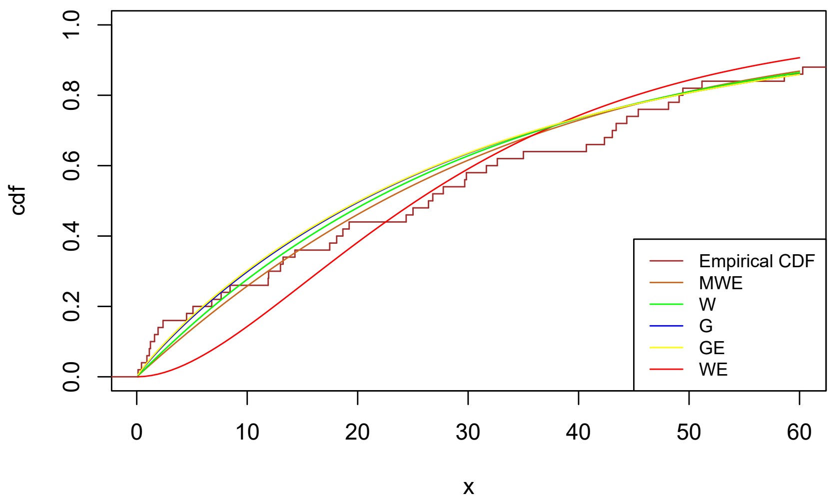


Figure 6. The empirical cdf with the fitted cdfs for the data set 1.

From Figures 5 and 6, we can see that the WE model has well-captured the main features of the data.

Data set 2: We also consider the bladder cancer data set by Aldeni et al. [28]. The values are: 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 1.46, 18.10, 11.79, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 13.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 12.07, 6.76, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

Table 8 reports the MLEs and their standard errors of the fitted models for the data set 2.

Table 8. MLEs (standard errors) for the data set 2.

Distribution	MLEs (Standard Errors)
MWE	$\hat{\alpha} = 6.8800$ (5.6074), $\hat{\lambda} = 0.1180$ (0.0126)
W	$\hat{\mu} = 1.0528$ (0.0680), $\hat{\sigma} = 9.6581$ (0.8574)
G	$\hat{\mu} = 1.1782$ (0.1315), $\hat{\sigma} = 8.0157$ (1.1076)
GE	$\hat{\alpha} = 1.2227$ (0.1493), $\hat{\lambda} = 0.1204$ (0.0135)
WE	$\hat{\alpha} = 13.1550$ (10.6474), $\hat{\lambda} = 0.1134$ (0.0115)

Table 8 is supplemented by Table 9, which contains parameter estimations based on several estimation methods.

Table 9. The parameter estimates for the data set 2.

	MOME	OLSE	WLSE	CME
$\hat{\alpha}$	0.0003	5.5633	6.4080	5.0124
$\hat{\lambda}$	0.1003	0.1294	0.1263	0.1312

Table 10 reports $-2\log L$, AIC, and three different goodness-of-fit tests statistics for the data set 2 and the results show that the proposed model is having the lowest AIC and $-2\log L$ values of all the models mentioned. In terms of goodness-of-fit tests statistics, the WE model is a bit better.

Table 10. Selection criteria statistics for the data set 2.

Model	$-2\log L$	AIC	KS	p(KS)	CVM	p(CVM)	AD	p(AD)
MWE	827.2080	831.2080	0.0619	0.7100	0.0842	0.6689	0.5012	0.7453
W	830.1968	834.1968	0.0663	0.6272	0.1380	0.4286	0.8743	0.4302
G	828.7471	832.7471	0.0692	0.5722	0.1178	0.5049	0.6849	0.5713
GE	828.1806	832.1806	0.0684	0.5877	0.1100	0.5389	0.6275	0.6221
WE	828.3536	832.3536	0.0594	0.7579	0.0757	0.7182	0.4816	0.7653

Figures 7 and 8 present the histogram with the fitted pdfs and empirical and theoretical cdfs plots of the fitted distributions for the data set 2, respectively.

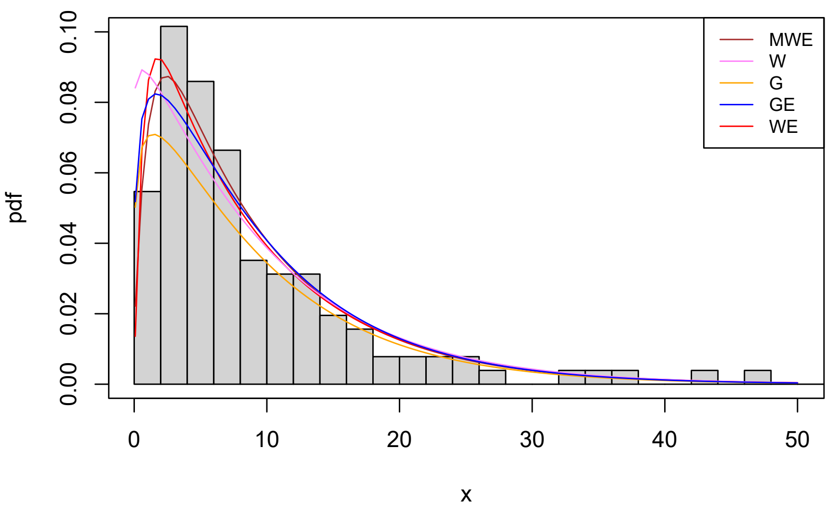


Figure 7. The histogram with the fitted pdfs for the data set 2.

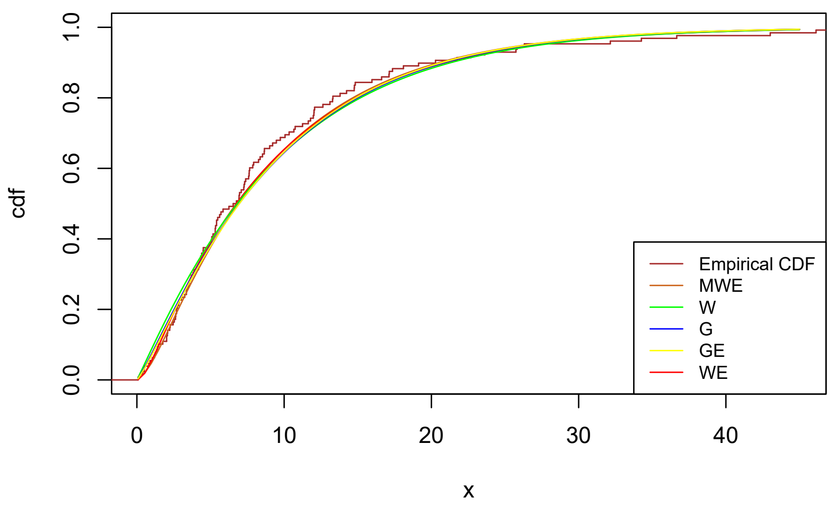


Figure 8. The empirical cdf with the fitted cdf for the data set 2.

The nice fits of the estimated functions of the WE model can be observed.

6. Concluding Remarks

Because of its simplicity and mathematical feasibility, the exponential distribution is used in the majority of distributional advancements, making it the most frequently used lifetime model in reliability theory. In this article, we have innovated by proposing a novel simple lifetime distribution with two parameters derived from a special mixture of the exponential and weighted exponential distributions. Among its qualities, it is simple on the mathematical side, may have a decreasing or unimodal probability density function, and possesses the demanded increasing hazard rate property. We have expressed the moments, Bonferroni and Lorenz curves, Rényi entropy, stress-strength reliability, and mean residual life function. The corresponding model is then given its own part, which shows how it might be applied in a real-world statistical scenario using data. In this regard, we have used five alternative estimation approaches and simulated data to show how well the estimated model performs. These good results are demonstrated by a study of some well-known datasets. The performance of the novel model is compared to that of the weighted exponential, Weibull, gamma, and generalized exponential models. The results of the comparison show that the proposed model is superior for some criteria. Bivariate extensions, discrete extensions, and other regression models are all conceivable future improvements to the new model. These directions necessitate additional research, which we will defer for now.

Supplementary Materials: All graphical and numerical works are created using the R software; R codes are available online at https://www.mdpi.com/article/10.3390/mca27010017/s1.

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