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A Survey on the Hausdorff Dimension of Intersections †

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Abstract: Let A and B be Borel subsets of the Euclidean n-space with $\dim A + \dim B > n$. This is a survey on the following question: what can we say about the Hausdorff dimension of the intersections $A \cap (g(B) + z)$ for generic orthogonal transformations g and translations by z?

Keywords: Hausdorff dimension; intersection; projection; energy integral; Fourier transform

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1. Introduction

The books [1,2] contain most of the required background information and the proofs of some of the results discussed below.

Let $\mathcal{L}^n$ stand for the Lebesgue measure on the Euclidean n-space $\mathbb{R}^n$ and let dim stand for the Hausdorff dimension and $\mathcal{H}^s$ for the s-dimensional Hausdorff measure. For $A \subset \mathbb{R}^n$, denote by $\mathcal{M}(A)$ the set of Borel measures μ with $0 < \mu(A) < \infty$ and with the compact support $\operatorname{spt}\mu \subset A$.

We let $O(n)$ denote the orthogonal group of $\mathbb{R}^n$ and θ_n its Haar probability measure. The main fact needed about the measure θ_n is the inequality

$$\theta_n(\{g \in O(n) : |x - g(z)| < r\}) \lesssim (r/|z|)^{n-1} \text{ for } x, z \in \mathbb{R}^n, r > 0. \tag{1}$$

This is quite easy, and is in fact trivial in the plane.

Let A and B be Borel subsets of $\mathbb{R}^n$ with Hausdorff dimensions $s = \dim A$ and $t = \dim B$. What can we say about the Hausdorff dimensions of the intersections of A and typical rigid motions of B—more precisely, of $\dim A \cap (g(B) + z)$ for almost all $g \in O(n)$ and for $z \in \mathbb{R}^n$ in a set of positive Lebesgue measure? Optimally, one could hope that this dimension is given by the larger of the numbers $s + t - n$ and 0, which happens when smooth surfaces meet in a general position.

The problem on the upper bound is much easier than on the lower bound. Let

$$V_z = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : x = y + z\},\ z \in \mathbb{R}^n, \tag{2}$$

be the z translate of the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$, and let π be the projection $\pi(x, y) = x$. Then,

$$A \cap (g(B) + z) = \pi((A \times g(B)) \cap V_z), \tag{3}$$

and it follows from a Fubini-type inequality for the Hausdorff dimension [1] (Theorem 7.7) that for any $g \in O(n)$,

$$\dim A \cap (g(B) + z) \leq \dim(A \times B) - n \text{ for almost all } z \in \mathbb{R}^n, \tag{4}$$

provided $\dim(A \times B) \geq n$. We have always $\dim(A \times B) \geq \dim A + \dim B$ and the equation $\dim(A \times B) = \dim A + \dim B$ holds if, for example, $0 < \mathcal{H}^s(A) < \infty, 0 < \mathcal{H}^t(B) < \infty$, and one of the sets has positive lower density, say

$$\theta^s_*(A, x) = \liminf_{r \to 0} r^{-s} \mathcal{H}^s(A \cap B(x, r)) > 0 \text{ for } \mathcal{H}^s \text{ almost all } x \in A. \tag{5}$$

Even the weaker condition that the Hausdorff and packing dimensions of A agree suffices; see [1], pp. 115–116. Then, we have

$$\dim A \cap (g(B) + z) \leq \dim A + \dim B - n \text{ for almost all } z \in \mathbb{R}^n, \tag{6}$$

provided $\dim A + \dim B \geq n$. Without some extra condition, this inequality fails seriously: for any $0 \leq s \leq n$, there exists a Borel set $A \subset \mathbb{R}^n$ of dimension s such that $\dim A \cap f(A) = s$ for all similarity maps f of $\mathbb{R}^n$. This was proven by Falconer in [3]; see also Example 13.19 in [1] and the further references given there.

We have the lower bound for the dimension of intersections if we use larger transformation groups—for example, similarities.

Theorem 1. *Let A and B be Borel subsets of $\mathbb{R}^n$ with $\dim A + \dim B > n$. Then, for every $\varepsilon > 0$,*

$$\mathcal{L}^n(\{z \in \mathbb{R}^n : \dim A \cap (rg(B) + z) \geq \dim A + \dim B - n - \varepsilon\}) > 0,$$

for almost all $g \in O(n)$ and almost all $r > 0$.

If A and B have positive and finite Hausdorff measures, ε is not needed. This theorem was proven in the 1980s independently by Kahane [4] and in [5]. More generally, Kahane proved that the similarities can be replaced by any closed subgroup of the general linear group of $\mathbb{R}^n$ that is transitive outside the origin. He gave applications to multiple points of stochastic processes.

There are many special cases where the equality $\dim A \cap (g(B) + z) = \dim A + \dim B - n$ holds for almost all g and for z in a set of positive measure. The case where one of the sets is a plane, initiated by Marstrand in [6], has been studied a lot; see discussions in [1] (Chapter 10) and [2] (Chapter 6), and [7] for a more recent result. More generally, one of the sets can be rectifiable; see [5].

The main open problem is as follows: what conditions on the Hausdorff dimensions or measures of A and B guarantee that for θ_n almost all $g \in O(n)$,

$$\mathcal{L}^n(\{z \in \mathbb{R}^n : \dim A \cap (g(B) + z) \geq \dim A + \dim B - n\}) > 0, \tag{7}$$

or perhaps for all $\varepsilon > 0$,

$$\mathcal{L}^n(\{z \in \mathbb{R}^n : \dim A \cap (g(B) + z) \geq \dim A + \dim B - n - \varepsilon\}) > 0? \tag{8}$$

If one of the sets is a Salem set, i.e., it supports a measure with an optimal Fourier decay allowed by its Hausdorff dimension, then (8) holds without dimensional restrictions; see [8]. I expect (8) to be true for all Borel subsets A and B of $\mathbb{R}^n$.

Below, I shall discuss some partial results on this question. I shall also say something about the exceptional sets of transformations.

In this survey, I shall concentrate on the Hausdorff dimension and general Borel sets. For remarks and references about related results on other dimensions, see [1] (Section 13.20) and [2] (Section 7.3). There is a rich body of literature on various questions about intersections of dynamically generated and related sets. For recent results and further references, see [9–11]. For probabilistic sets, see [12] and its references.

I would like to thank the referees for their useful comments.

2. Projections and Plane Intersections

This topic can be thought of as a study of the integral-geometric properties of fractal sets and the Hausdorff dimension. Let us briefly review some of the basic related results on projections and plane sections. This was started by Marstrand in [6] in the plane. His main results in general dimensions are the following. Let $G(n,m)$ be the Grassmannian of linear m-dimensional subspaces of $\mathbb{R}^n$ and $P_V : \mathbb{R}^n \to V$ the orthogonal projection onto $V \in G(n,m)$. Let also $\gamma_{n,m}$ be the orthogonally invariant Borel probability measure on $G(n,m)$.

Theorem 2. *Let $A \subset \mathbb{R}^n$ be a Borel set. Then, for almost all $V \in G(n,m)$,*

$$\dim P_V(A) = \dim A \text{ if } \dim A \le m, \tag{9}$$

and

$$\mathcal{H}^m(P_V(A)) > 0 \text{ if } \dim A > m. \tag{10}$$

Theorem 3. *Let $n - m \le s \le n$ and let $A \subset \mathbb{R}^n$ be $\mathcal{H}^s$ measurable with $0 < \mathcal{H}^s(A) < \infty$. Then, for almost all $V \in G(n,m)$,*

$$\mathcal{H}^{n-m}(\{u \in V^\perp : \dim(A \cap (V+u)) = s+m-n\}) > 0, \tag{11}$$

and for almost all $V \in G(n,n-m)$ and for $\mathcal{H}^s$ almost all $x \in A$,

$$\dim(A \cap (V+x)) = s+m-n. \tag{12}$$

One can sharpen these results by deriving estimates on the Hausdorff dimension of the exceptional sets of the planes V. For the first part of Theorem 2, this was first done by Kaufman in [13] in the plane, and then in [14,15] in higher dimensions. For the second part of Theorem 2, the exceptional set estimates were proven by Falconer in [16]. Thus we have, recall that $\dim G(n,m) = m(n-m)$.

Theorem 4. *Let $A \subset \mathbb{R}^n$ be a Borel set with $s = \dim A$. Then,*

$$\dim\{V \in G(n,m) : P_V(A) < \dim A\} \le s - m + m(n-m) \text{ if } s \le m, \tag{13}$$

and

$$\dim\{V \in G(n,m) : \mathcal{H}^m(P_V(A)) = 0\}) \le m - s + m(n-m) \text{ if } s > m, \tag{14}$$

These inequalities are sharp by the examples in [14] (and their modifications), but the proof for (13) also gives the upper bound $t - m + m(n-m)$ if $\dim A$ on the left-hand side is replaced by $t, 0 \le t \le \dim A$. Then, for $t < \dim A$, this is not always sharp; see the discussion in [2] (Section 5.4).

For the plane sections, Orponen proved in [17]—see also [2] (Theorem 6.7)—the exceptional set estimate (which of course is sharp, as (14) is).

Theorem 5. *Let $n - m \le s \le n$ and let $A \subset \mathbb{R}^n$ be $\mathcal{H}^s$ measurable with $0 < \mathcal{H}^s(A) < \infty$. Then, there is a Borel set $E \subset G(n,m)$ such that*

$$\dim E \le n - m - s + m(n-m)$$

and for $V \in G(n,m) \setminus E$,

$$\mathcal{H}^{n-m}(\{u \in V^\perp : \dim(A \cap (V+u)) = s+m-n\}) > 0. \tag{15}$$

We can also ask for exceptional set estimates corresponding to (12). We proved with Orponen [7] the following:

Theorem 6. *Let $n - m \le s \le n$ and let $A \subset \mathbb{R}^n$ be $\mathcal{H}^s$ measurable with $0 < \mathcal{H}^s(A) < \infty$. Then, the set B of points $x \in \mathbb{R}^n$ with*

$$\gamma_{n,m}(\{V \in G(n,m) : \dim A \cap (V + x) = s + m - n\}) = 0$$

has dimension $\dim B \le n - m$.

Very likely, the bound $n - m$ is not sharp. When $m = 1$, probably, the sharp bound should be $2(n-1) - s$ in accordance with Orponen's sharp result for radial projections in [18].

Another open question is whether there could be some sort of non-trivial estimate for the dimension of the exceptional pairs (x, V).

3. Some Words about the Methods

The methods in all cases use Frostman measures. Suppose that the Hausdorff measures $\mathcal{H}^s(A)$ and $\mathcal{H}^t(B)$ are positive. Then, there are $\mu \in \mathcal{M}(A)$ and $\nu \in \mathcal{M}(B)$ such that $\mu(B(x,r)) \le r^s$ and $\nu(B(x,r)) \le r^t$ for $x \in \mathbb{R}^n, r > 0$. In particular, for $0 < s < \dim A$ and $0 < t < \dim B$, there are $\mu \in \mathcal{M}(A)$ and $\nu \in \mathcal{M}(B)$ such that $I_s(\mu) < \infty$ and $I_t(\nu) < \infty$, where the s energy $I_s(\mu)$ is defined by

$$I_s(\mu) = \iint |x - y|^{-s} \, d\mu x \, d\mu y.$$

Then, the goal is to find intersection measures $\lambda_{g,z} \in \mathcal{M}(A \cap (g(B) + z))$ such that

$$\operatorname{spt} \lambda_{g,z} \subset \operatorname{spt} \mu \cap (g(\operatorname{spt} \nu) + z), \tag{16}$$

$$\int \lambda_{g,z}(\mathbb{R}^n) \, d\mathcal{L}^n z = \mu(\mathbb{R}^n)\nu(\mathbb{R}^n) \text{ for } \theta_n \text{ almost all } g \in O(n), \tag{17}$$

$$\iint I_{s+t-n}(\lambda_{g,z}) \, d\mathcal{L}^n z \, d\theta_n g \lesssim I_s(\mu) I_t(\nu). \tag{18}$$

This would give (8).

There are two closely related methods to produce these measures. The first, used in [5], is based on (3): the intersections $A \cap (g(B) + z)$ can be realized as level sets of the projections S_g:

$$S_g(x,y) = x - g(y), \; x, y \in \mathbb{R}^n, \tag{19}$$

$$A \cap (g(B) + z) = \pi((A \times g(B)) \cap S_g^{-1}\{z\}), \; \pi(x,y) = x. \tag{20}$$

Notice that the map S_g is essentially the orthogonal projection onto the n-plane $\{(x, -g(x)) : x \in \mathbb{R}^n\}$.

Thus, one slices (disintegrates) $\mu \times g_{\#}\nu$ ($g_{\#}\nu$ is the push-forward) with the planes $V_z = \{(x,y) : x = y + z\}, z \in \mathbb{R}^n$. For this to work, one needs to know that

$$S_{g\#}(\mu \times \nu) \ll \mathcal{L}^n \text{ for } \theta_n \text{ almost all } g \in O(n). \tag{21}$$

This is usually proven by establishing the L^2 estimate

$$\iint S_{g\#}(\mu \times \nu)(x)^2 \, dx \, d\theta_n g \lesssim 1, \tag{22}$$

which, by Plancherel's formula, is equivalent to

$$\iint \mathcal{F}(S_{g\#}(\mu \times \nu))(x)^2 \, dx \, d\theta_n g \lesssim 1, \tag{23}$$

where $\mathcal{F}$ stands for the Fourier transform.

The second method, used in [4], is based on convolution approximation. Letting $\psi_\varepsilon, \varepsilon > 0$, be a standard approximate identity, set $\nu_\varepsilon = \psi_\varepsilon * \nu$ and

$$\nu_{g,z,\varepsilon}(x) = \nu_\varepsilon(g^{-1}(x-z)),\ x \in \mathbb{R}^n. \tag{24}$$

Then, the plan is to show that when $\varepsilon \to 0$, the measures $\nu_{g,z,\varepsilon}\mu$ converge weakly to the desired intersection measures.

No Fourier transform is needed to prove Theorem 1, but the proofs of all theorems discussed below, except Theorems 10 and 11, rely on the Fourier transform defined by

$$\widehat{\mu}(x) = \int e^{-2\pi i x\cdot y}\, d\mu y,\ x \in \mathbb{R}^n.$$

The basic reason for its usefulness in this connection is the formula

$$I_s(\mu) = \iint |x-y|^{-s}\, d\mu x\, d\mu y = c(n,s)\int |\widehat{\mu}(x)|^2 |x|^{s-n}\, dx, \tag{25}$$

which is a consequence of Parseval's formula and the fact that the distributional Fourier transform of the Riesz kernel $k_s, k_s(x) = |x|^{-s}$, is a constant multiple of k_{n-s}.

The decay of the spherical averages,

$$\sigma(\mu)(r) = r^{1-n}\int_{S(r)} |\widehat{\mu}(x)|^2\, d\sigma_r^{n-1}x, r > 0,$$

of $\mu \in \mathcal{M}(\mathbb{R}^n)$, where σ_r^{n-1} is the surface measure on the sphere $S(r) = \{x \in \mathbb{R}^n : |x| = r\}$, often plays an important role. By integration in polar coordinates, if $\sigma(\mu)(r) \lesssim r^{-t}$ for $r > 0$ and for some $t > 0$, then $I_s(\mu) < \infty$ for $0 < s < t$. Hence, the best decay that we can hope for under the finite s energy assumption (or the Frostman assumption $\mu(B(x,r)) \le r^s$) is r^{-s}. This is true when $s \le (n-1)/2$—see [2] (Lemma 3.5)—but false otherwise.

The decay estimates for $\sigma(\mu)(r)$ have been studied by many people; a discussion can be found in [2] (Chapter 15). The best-known estimates, due to Wolff [19] when $n = 2$ (the proof can also be found in [2] (Chapter 16)) and to Du and Zhang [20] in the general case, are the following. Let $\mu \in \mathcal{M}(\mathbb{R}^n)$ with $\mu(B(x,r)) \le r^s$ for $x \in \mathbb{R}^n, r > 0$. Then, for all $\varepsilon > 0, r > 1$,

$$\sigma(\mu)(r) \lesssim \begin{cases} r^{-(n-1)s/n+\varepsilon} \text{ for all } 0 < s < n, \\ r^{-(n-1)/2+\varepsilon} \text{ if } (n-1)/2 \le s \le n/2, \\ r^{-s+\varepsilon} \text{ if } 0 < s \le (n-1)/2. \end{cases} \tag{26}$$

The essential case for the first estimate is $s > n/2$; otherwise, the second and third are better. Up to ε, these estimates are sharp when $n = 2$. When $n \ge 3$, the sharp bounds are not known for all s; see [21] for a discussion and the most recent examples. As mentioned above, the last bound is always sharp.

4. The First Theorem

If one of the sets has a dimension greater than $(n+1)/2$, we have the following theorem. It was proven in [22]; see also [1] (Theorem 13.11) or [2] (Theorem 7.4).

Theorem 7. *Let s and t be positive numbers with $s + t > n$ and $s > (n+1)/2$. Let A and B be Borel subsets of $\mathbb{R}^n$ with $\mathcal{H}^s(A) > 0$ and $\mathcal{H}^t(B) > 0$. Then,*

$$\mathcal{L}^n(\{z \in \mathbb{R}^n : \dim A \cap (g(B)+z) \ge \dim A + \dim B - n\}) > 0 \tag{27}$$

for almost all $g \in O(n)$.

The proof is based on the slicing method. The key estimate is

$$\mu \times \mu(\{(x,y) : r \leq |x-y| \leq r+\delta\}) \lesssim I_s(\mu)\delta r^{s-1} \quad (28)$$

if $\mu \in \mathcal{M}(\mathbb{R}^n), 0 < \delta \leq r$ and $(n+1)/2 \leq s < n$. This is combined with the inequality (1).

The inequality (28) is obtained with the help of the Fourier transform, and that is the only place in the proof of Theorem 7 where the Fourier transform is needed.

One problem of extending Theorem 7 below the dimension bound $(n+1)/2$ is that the estimate (28) then fails, at least in the plane by [2] (Example 4.9) and in $\mathbb{R}^3$ by [23].

In Section 7, we discuss estimates on the exceptional sets of orthogonal transformations. The proof of Theorem 13 gives another proof for Theorem 7 but under the stronger assumption $s+t > n+1$. On the other hand, Theorem 12 below holds with the assumption $s+(n-1)t/n > n$ but under the additional condition of positive lower density. Of course, $s+(n-1)t/n > n$ is sometimes stronger and sometimes weaker than $s > (n+1)/2, s+t > n$. For example, consider these in the plane. When $s = t$, the first one says $s > 4/3$ and the second one $s > 3/2$. On the other hand, when s is slightly larger than $3/2$, the first requires t to be at least 1, but the second allows $t = 1/2$.

Theorem 7 says nothing in $\mathbb{R}^1$, and there is nothing to say: in [5], I constructed compact sets $A, B \subset \mathbb{R}$ such that $\dim A = \dim B = 1$ and $A \cap (B+z)$ contains at most one point for any $z \in \mathbb{R}$. With $A, B \subset \mathbb{R}$ as above, the n-fold Cartesian products $A \times \cdots \times A \subset \mathbb{R}^n$ and $B \times \cdots \times B \subset \mathbb{R}^n$ yield the corresponding examples in $\mathbb{R}^n$—that is, simply, with translations, we obtain nothing in general.

Donoven and Falconer proved in [24] an analogue of Theorem 7 for the isometries of the Cantor space. They did not need any dimensional restrictions. They used martingales to construct the desired random measures with finite energy integrals on the intersections.

5. The Projections S_g

We now discuss further the projections S_g; recall (19). They are particular cases of restricted projections, which recently have been studied extensively; see [2] (Section 5.4), and [25,26] and the references given there. Restricted means that we are considering a lower-dimensional subspace of the Grassmannian $G(2n,n)$. For the full Grassmannian, we have Marstrand's projection Theorem 2.

As mentioned above, to prove Theorem 7, one first needs to know (21) when $s+t > n$ and $s > (n+1)/2$ and μ and ν have finite s and t energies. A simple proof using spherical averages is given in [2] (Lemma 7.1). This immediately yields the weaker result: with the assumptions of Theorem 7, for almost all $g \in O(n)$,

$$\mathcal{L}^n(\{z \in \mathbb{R}^n : A \cap (g(B)+z) \neq \emptyset\}) > 0, \quad (29)$$

because (29) is equivalent to $\mathcal{L}^n(S_g(A \times B)) > 0$. Even for this, I do not know if the assumption $s > (n+1)/2$ is needed.

Let us first look at general Borel subsets of $\mathbb{R}^{2n}$.

Theorem 8. *Let $A \subset \mathbb{R}^{2n}$ be a Borel set. If $\dim A > n+1$, then $\mathcal{L}^n(S_g(A)) > 0$ for θ_n almost all $g \in O(n)$.*

This was proven in [26]. The paper also contains dimension estimates for $S_g(A)$ when $\dim A \leq n+1$ and estimates on the dimension of exceptional sets of transformations g. In particular, if $n \leq \dim A \leq n+1$, then

$$\dim S_g(A) \geq \dim A - 1 \text{ for } \theta_n \text{ almost all } g \in O(n). \quad (30)$$

The bound $n+1$ in Theorem 8 is sharp. This was shown by Harris in [27]. First, (30) is sharp. The example for $\dim A = n$ is simply the diagonal $D = \{(x,x) : x \in \mathbb{R}^n\}$. To see this, suppose that $g \in O(n)$ is such that $\det g = (-1)^{n+1}$, which is satisfied by half of the orthogonal transformations. Then, by some linear algebra, g has a fixed point, whence the kernel of $x \mapsto S_g(x,x)$ is non-trivial, so $\dim S_g(D) \le n-1$. Taking the Cartesian product of D with a one-dimensional set of zero $\mathcal{H}^1$ measure, we obtain A with $\dim A = n+1$ and $\mathcal{L}^n(S_g(A)) = 0$, which proves the sharpness.

However, this only gives an example A of dimension $n+1$ for which $\mathcal{L}^n(S_g(A)) = 0$ for $g \in O(n)$ with measure $1/2$. Is there a counter-example that works for almost all $g \in O(n)$?

Here are the basic ingredients of the proof of Theorem 8. They were inspired by Oberlin's paper [28].

Let $0 < n+1 < s < \dim A$ and $\mu \in \mathcal{M}(A)$ with $I_s(\mu) < \infty$, and let $\mu_g \in \mathcal{M}(S_g(A))$ be the push-forward of μ under S_g. The Fourier transform of μ_g is given by

$$\widehat{\mu_g}(\xi) = \widehat{\mu}(\xi, -g^{-1}(\xi)).$$

By fairly standard arguments, using also the inequality (1), one can then show that for $R > 1$,

$$\iint_{R \le |\xi| \le 2R} |\widehat{\mu}(\xi, -g^{-1}(\xi))|^2 \, d\xi \, d\theta_n g \lesssim R^{n+1-s}. \tag{31}$$

This is summed over the dyadic annuli, $R = 2^k, k = 1,2,\ldots$. The sum converges since $s > n+1$. Hence, for θ_n almost all $g \in O(n)$, μ_g is absolutely continuous with L^2 density, and so $\mathcal{L}^n(S_g(A)) > 0$.

For product sets, we can improve this, which is essential for the applications to intersections.

Theorem 9. *Let $A, B \subset \mathbb{R}^n$ be Borel sets. If $\dim A + (n-1)\dim B/n > n$ or $\dim A + \dim B > n$ and $\dim A > (n+1)/2$, then $\mathcal{L}^n(S_g(A \times B)) > 0$ for θ_n almost all $g \in O(n)$.*

The case $\dim A > (n+1)/2$ is a special case of Theorem 7; recall (29). The proof of the case $\dim A + (n-1)\dim B/n > n$ is based on the spherical averages and the first estimate of (26). Here is a sketch.

Let $0 < s < \dim A, 0 < t < \dim B$ and $\varepsilon > 0$ such that $s + (n-1)t/n - \varepsilon > n$, and let $\mu \in \mathcal{M}(A), \nu \in \mathcal{M}(B)$ with $\mu(B(x,r)) \le r^s, \nu(B(x,r)) \le r^t$ for $x \in \mathbb{R}^n, r > 0$. Let $\lambda_g = S_{g\#}(\mu \times \nu) \in \mathcal{M}(S_g(A \times B))$. Then, $\widehat{\lambda_g}(\xi) = \widehat{\mu}(\xi)\widehat{\nu}(-g^{-1}(\xi))$. By (26), we have

$$\begin{aligned} &\iint |\widehat{\lambda_g}(\xi)|^2 \, d\xi \, d\theta g = \int |\widehat{\mu}(\xi)|^2 \sigma(\nu)(|\xi|) \, d\xi \\ &\lesssim \int |\widehat{\mu}(\xi)|^2 |\xi|^{-(n-1)t/n+\varepsilon} \, d\xi = c I_{n-(n-1)t/n+\varepsilon}(\mu) \lesssim I_s(\mu) < \infty. \end{aligned} \tag{32}$$

This gives Theorem 9.

In fact, for some results on the intersections below, we again need absolute continuity as in (21). In the case $s + (n-1)t > n$, we need the quantitative estimate: if $s + (n-1)t > n$, $\mu, \nu \in \mathcal{M}(\mathbb{R}^n)$ and $\mu(B(x,r)) \le r^s, \nu(B(x,r)) \le r^t$ for $x \in \mathbb{R}^n, r > 0$, then

$$\iint S_{g\#}(\mu \times \nu)(x)^2 \, dx \, d\theta_n g \lesssim 1, \tag{33}$$

with the implicit constant independent of μ and ν. The arguments described above give this too.

6. Level Sets and Intersections

The estimate (33) can be used to derive information on the Hausdorff dimension of the level sets of S_g, and hence, by (20), of intersections. The following results were proven in [29]. We shall first discuss a more general version of this principle: a quantitative projection theorem leads to estimates of the Hausdorff dimension of level sets. This is also how, in [1] (Chapter 10), the proof for Marstrand's section Theorem 3 runs.

We consider the following general setting. Let $P_\lambda : \mathbb{R}^n \to \mathbb{R}^m, \lambda \in \Lambda$, be orthogonal projections, where Λ is a compact metric space. Suppose that $\lambda \mapsto P_\lambda x$ is continuous for every $x \in \mathbb{R}^n$. Let also ω be a finite non-zero Borel measure on Λ. We denote by $D(\mu, \cdot)$ the Radon–Nikodym derivative of a measure μ on $\mathbb{R}^m$.

Theorem 10. *Let $s > m$. Suppose that there exists a positive number C such that $P_{\lambda\sharp}\mu \ll \mathcal{L}^m$ for ω almost all $\lambda \in \Lambda$ and*

$$\iint D(P_{\lambda\sharp}\mu, u)^2\, d\mathcal{L}^m u\, d\omega\lambda < C \tag{34}$$

whenever $\mu \in \mathcal{M}(B^n(0,1))$ is such that $\mu(B(x,r)) \le r^s$ for $x \in \mathbb{R}^n, r > 0$.

If $A \subset \mathbb{R}^n$ is $\mathcal{H}^s$ measurable, $0 < \mathcal{H}^s(A) < \infty$ and $\theta^s_*(A,x) > 0$ (recall (5)) for $\mathcal{H}^s$ almost all $x \in A$, then for ω almost all $\lambda \in \Lambda$,

$$\mathcal{L}^m(\{u \in \mathbb{R}^m : \dim P_\lambda^{-1}\{u\} \cap A = s - m\}) > 0. \tag{35}$$

For an application to intersections, we shall need the following product set version of Theorem 10. There, $P_\lambda : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^m, \lambda \in \Lambda, m < n + p$, are orthogonal projections with the same assumptions as before.

Theorem 11. *Let $s, t > 0$ with $s + t > m$. Suppose that there exists a positive number C such that $P_{\lambda\sharp}(\mu \times \nu) \ll \mathcal{L}^m$ for ω almost all $\lambda \in \Lambda$ and*

$$\iint D(P_{\lambda\sharp}(\mu \times \nu), u)^2\, d\mathcal{L}^m u\, d\omega\lambda < C \tag{36}$$

whenever $\mu \in \mathcal{M}(B^n(0,1)), \nu \in \mathcal{M}(B^p(0,1))$ are such that $\mu(B(x,r)) \le r^s$ for $x \in \mathbb{R}^n, r > 0$, and $\nu(B(y,r)) \le r^t$ for $y \in \mathbb{R}^p, r > 0$.

If $A \subset \mathbb{R}^n$ is $\mathcal{H}^s$ measurable, $0 < \mathcal{H}^s(A) < \infty$, $B \subset \mathbb{R}^p$ is $\mathcal{H}^t$ measurable, $0 < \mathcal{H}^t(B) < \infty$, $\theta^s_*(A,x) > 0$ for $\mathcal{H}^s$ almost all $x \in A$, and $\theta^t_*(B,y) > 0$ for $\mathcal{H}^t$ almost all $y \in B$, then for ω almost all $\lambda \in \Lambda$,

$$\mathcal{L}^m(\{u \in \mathbb{R}^m : \dim P_\lambda^{-1}\{u\} \cap (A \times B) = s + t - m\}) > 0. \tag{37}$$

I do not know if the assumptions on positive lower density are needed.

I give a few words about the proof of Theorem 10. First, notice that $D(P_{\lambda\sharp}(\mu), u)$ is given by
$$D(P_{\lambda\sharp}\mu, u) = \lim_{\delta \to 0} \mathcal{L}^m(B(0,1))^{-1}\delta^{-m}\mu(\{y : |P_\lambda(y) - u| \le \delta\}).$$

Let μ be the restriction of $\mathcal{H}^s$ to a subset of A so that μ satisfies the Frostman s condition. Then, (34) is applied to the measures

$$\mu_{a,r,\delta} = r^{-s}T_{a,r\sharp}(\mu_\delta \llcorner B(a,r)) \in \mathcal{M}(B(0,1)), a \in \mathbb{R}^n, r > 0, \delta > 0,$$

where $\mu_\delta(B) = \delta^{-n}\int_B \mu(B(x,r))\, d\mathcal{L}^n x$, $T_{a,r}(x) = (x-a)/r$ is the blow-up map and $\mu_\delta \llcorner B(a,r)$ is the restriction of μ_δ to $B(a,r)$. This leads for almost all $x \in A, \lambda \in \Lambda$, to

$$\lim_{r \to 0} \liminf_{\delta \to 0} r^{-t}\delta^{-m}\mu(\{y \in B(x,r) : |P_\lambda(y - x)| \le \delta\}) = 0, \tag{38}$$

which is a Frostman-type condition along the level sets of the P_λ. With some further work, it leads to (35). The proof of Theorem 11 is similar.

Theorem 11, together with the quantitative version of Theorem 9 and with (20), can be applied to the projections S_g to obtain the following result on the Hausdorff dimension of intersections.

Theorem 12. *Let $s, t > 0$ with $s + (n-1)t/n > n$ and let $A \subset \mathbb{R}^n$ be $\mathcal{H}^s$ measurable with $0 < \mathcal{H}^s(A) < \infty$, and let $B \subset \mathbb{R}^n$ be $\mathcal{H}^t$ measurable with $0 < \mathcal{H}^t(B) < \infty$. Suppose that*

$\theta^s_*(A,x) > 0$ for $\mathcal{H}^s$ almost all $x \in A$ and $\theta^t_*(B,y) > 0$ for $\mathcal{H}^t$ almost all $y \in B$. Then, for θ_n almost all $g \in O(n)$,

$$\mathcal{L}^n(\{z \in \mathbb{R}^n : \dim A \cap (g(B)+z) = s+t-n\}) > 0. \tag{39}$$

Again, I do not know if the positive lower density assumptions are needed for the lower bound $s+t-n$. As mentioned before, they are needed for the upper bound.

7. Exceptional Set Estimates

Recall the exceptional set estimates for orthogonal projections and for intersections with planes from Section 2. Now, we discuss some similar results from [30] for intersections.

First, we have an exceptional set estimate related to Theorem 7. However, we need a slightly stronger assumption: the sum of the dimensions is required to be larger than $n+1$, rather than only one of the sets having a dimension larger than $(n+1)/2$. Recall that the dimension of $O(n)$ is $n(n-1)/2$.

Theorem 13. *Let s and t be positive numbers with $s+t > n+1$. Let A and B be Borel subsets of $\mathbb{R}^n$ with $\mathcal{H}^s(A) > 0$ and $\mathcal{H}^t(B) > 0$. Then, there is $E \subset O(n)$ such that*

$$\dim E \leq n(n-1)/2 - (s+t-(n+1))$$

and for $g \in O(n) \setminus E$,

$$\mathcal{L}^n(\{z \in \mathbb{R}^n : \dim A \cap (g(B)+z) \geq s+t-n\}) > 0. \tag{40}$$

The proof is based on the Fourier transform and the convolution approximation mentioned in Section 3. Instead of θ_n, one uses a Frostman measure θ on the exceptional set E: if $\alpha > (n-1)(n-2)/2$ is such that $\theta(B(g,r)) \leq r^\alpha$ for all $g \in O(n)$ and $r > 0$, then for $x, z \in \mathbb{R}^n \setminus \{0\}, r > 0$,

$$\theta(\{g : |x - g(z)| < r\}) \lesssim (r/|z|)^{\alpha-(n-1)(n-2)/2}. \tag{41}$$

This replaces the inequality (1).

In the case where one of the sets has a small dimension, we have the following improvement of Theorem 13.

Theorem 14. *Let A and B be Borel subsets of $\mathbb{R}^n$ and suppose that $\dim A \leq (n-1)/2$. If $0 < u < \dim A + \dim B - n$, then there is $E \subset O(n)$ with*

$$\dim E \leq n(n-1)/2 - u$$

such that for $g \in O(n) \setminus E$,

$$\mathcal{L}^n(\{z \in \mathbb{R}^n : \dim A \cap (g(B)+z) \geq u\}) > 0. \tag{42}$$

The last decay estimate in (26) of spherical averages is essential for the proof. The reason that the assumption $\dim A \leq (n-1)/2$ leads to a better result is that this estimate in (26) is stronger than the others. For $\dim A > (n-1)/2$, the inequalities (26) would only give weaker results with u replaced by a smaller number; see [30] (Section 4).

If one of the sets supports a measure with sufficiently fast decay of the averages $\sigma(\mu)(r)$, we can improve the estimate of Theorem 13. Then, the results even hold without any rotations, provided that the dimensions are large enough. In particular, we have the following result in the event that one of the sets is a Salem set. By definition, A is a Salem set if, for every $0 < s < \dim A$, there is $\mu \in \mathcal{M}(A)$ such that $|\widehat{\mu}(x)|^2 \lesssim |x|^{-s}$. A discussion on Salem sets can be found, for example, in [2], Section 3.6.

Theorem 15. *Let A and B be Borel subsets of $\mathbb{R}^n$ and suppose that A is a Salem set. Suppose that $0 < u < \dim A + \dim B - n$.*

(*a*) *If $\dim A + \dim B > 2n - 1$, then*

$$\mathcal{L}^n(\{z \in \mathbb{R}^n : \dim A \cap (B + z) \geq u\}) > 0. \tag{43}$$

(*b*) *If $\dim A + \dim B \leq 2n - 1$, then there is $E \subset O(n)$ with*

$$\dim E \leq n(n-1)/2 - u$$

such that for $g \in O(n) \setminus E$,

$$\mathcal{L}^n(\{z \in \mathbb{R}^n : \dim A \cap (g(B) + z) \geq u\}) > 0. \tag{44}$$

Could this hold for general sets, perhaps in the form that $\dim E = 0$, if $\dim A + \dim B > 2n - 1$? It follows from Theorem 5 that this is true if one of the sets is a plane. In $\mathbb{R}^2$, a slightly stronger question reads as follows: if $s + t > 2$ and A and B are Borel subsets of $\mathbb{R}^2$ with $\mathcal{H}^s(A) > 0$ and $\mathcal{H}^t(B) > 0$, is there $E \subset O(2)$ with $\dim E = 0$, if $s + t \geq 3$, and $\dim E \leq 3 - s - t$, if $s + t \leq 3$, such that for $g \in O(2) \setminus E$,

$$\mathcal{L}^2(\{z \in \mathbb{R}^2 : \dim A \cap (g(B) + z) \geq s + t - 2\}) > 0?$$

8. Some Relations to the Distance Set Problem

There are some connections of this topic to Falconer's distance set problem. For a general discussion and references, see, for example, [2]. Falconer showed in [31] that, for a Borel set $A \subset \mathbb{R}^n$, the distance set $\{|x - y| : x, y \in A\}$ has a positive Lebesgue measure if $\dim A > (n+1)/2$. We had the same condition in Theorem 7. Moreover, for distance sets, it is expected that $\dim A > n/2$ should be enough.

When $n = 2$, Wolff [19] improved $3/2$ to $4/3$ using (26). Observe that when $\dim A = \dim B$, the assumption $\dim A + \dim B/2 > 2$ in Theorem 12 becomes $\dim A > 4/3$ and it is the same as Wolff's. This is no coincidence: both results use Wolff's estimate (26).

The proofs of distance set results often involve the distance measure $\delta(\mu)$ of a measure μ defined by

$$\delta(\mu)(B) = \mu \times \mu(\{(x, y) : |x - y| \in B\}),\ B \subset \mathbb{R}.$$

The crucial estimate (28) means that $\delta(\mu)$ is absolutely continuous with bounded density if $I_{(n+1)/2}(\mu) < \infty$. Hence, it yields Falconer's result. As mentioned before, we cannot hope to obtain bounded density when $s < (n+1)/2$, at least when $n = 2$ or 3. In many of the later improvements, one verifies absolute continuity with L^2 density. For example, Wolff showed that $\delta(\mu) \in L^2(\mathbb{R})$, if $I_s(\mu) < \infty$ for some $s > 4/3$. To do this, he used decay estimates for the spherical averages $\sigma(\mu)(r)$ and proved (26) for $n = 2$. The proofs of the most recent, and so far the best known, distance set results in [20,32–34] involve using deep harmonic analysis techniques; restriction and decoupling. In the plane, the result of [33] says that the distance set of A has a positive Lebesgue measure if $\dim A > 5/4$. See Shmerkin's survey [35] for the distance set and related problems.

Distance measures are related to the projections S_g by the following:

$$\iint D(S_{g\#}(\mu \times \nu))(z)^2 \, d\mathcal{L}^n z \, d\theta_n g = c \int \delta(\mu)(t)\delta(\nu)(t)t^{1-n} \, dt, \tag{45}$$

at least if μ and ν are smooth functions with compact support; see [26] (Section 5.2).

Since, by an example in [33], when $n = 2$, for any $s < 4/3$, $I_s(\mu) < \infty$ is not enough for $\delta(\mu)$ to be in L^2, probably, because of (45), it is not enough for $S_{g\#}(\mu \times \mu)$ to be in L^2. However, in [33], it was shown that if $I_s(\mu) < \infty$ for some $s > 5/4$, there is a complex valued modification of μ with good L^2 behavior. In even higher dimensions, similar results were proven in [34] with $n/2 + 1/4$ in place of $5/4$. Could these methods be used to show, for instance, that if $n = 2$ and $\dim A = \dim B > 5/4$, then $\mathcal{L}^2(\mathcal{S}_g(A \times B)) > 0$ for almost all $g \in O(2)$?

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