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Article

Fractional Hermite–Hadamard-Type Inequalities for Differentiable Preinvex Mappings and Applications to Modified Bessel and q-Digamma Functions

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Abstract: The theory of convexity pertaining to fractional calculus is a well-established concept that has attracted significant attention in mathematics and various scientific disciplines for over a century. In the realm of applied mathematics, convexity, particularly in relation to fractional analysis, finds extensive and remarkable applications. In this manuscript, we establish new fractional identities. Employing these identities, some extensions of the fractional H-H type inequality via generalized preinvexities are explored. Finally, we discuss some applications to the q-digamma and Bessel functions via the established results. We believe that the methodologies and approaches presented in this work will intrigue and spark the researcher's interest even more.

Keywords: convex function; invex sets; preinvex functions; Hölder's inequality; power mean inequality

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1. Introduction

Convex inequalities are mathematical inequalities involving convex functions. A convex inequality is similar to the definition of a convex function, but it applies to the inequalities formed by these functions. In order to design constraints that limit the viable region to convex sets, convex inequalities are crucial in optimization issues. Convexity is well known as playing a significant and critical role in a range of domains such as economics, finance, optimization, game theory, statistical theory, quality management, and numerous sciences. For the literature regarding convexity, see the references [1–14].

Inequalities are an amazing mathematical tool due to their importance in fractional calculus, traditional calculus, quantum calculus, stochastic, time-scale calculus, fractal sets, and other fields. The crucial mathematical tool that connects integrals and inequalities, integral inequalities provide insights into the behavior of functions over particular intervals. For the literature regarding inequalities, see the references [15–19].

Fractional calculus, which focuses on fractional integration across complex domains, has recently acquired popularity due to its practical applications and has piqued the curiosity of mathematicians. The research of well-known inequalities, such as Ostrowski, Simpson, and Hadamard, inspired the study of fractional integral inequality. Transform theory, engineering, modeling, finance, mathematical biology, fluid flow, natural phenomenon prediction, healthcare, and image processing are all domains where fractional calculus is used.

The goal of this article is to prove some integral inequalities for derivable mapping whose absolute values are preinvex. Next, we will review some concepts in invexity analysis that will be utilized throughout the paper (see [20–24] and references therein). The idea of convexity is a strong and magnificent tool for dealing with a huge range of applied and pure science problems. Many researchers have recently devoted themselves to researching the properties and inequalities associated with the topic of convexity in different areas, (see [25,26] and the references therein).

We constructed this manuscript in the following way: first, we explore some fundamental ideas and definitions in Section 2. In Section 3, we investigate and prove new integral identities. In Section 4, we investigate some applications involving modified Bessel functions and q-digamma functions. Lastly, in Section 5, future directions and conclusions of the newly discussed concept are elaborated.

2. Preliminaries

The main objective of this section is to remember and discuss specific related ideas and concepts that are pertinent to our analysis in later sections of this paper.

Jensen introduced the term convexity for the first time in the following manner:

Definition 1 ([27]). *A mapping $\Pi : \mathfrak{K} \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if*

$$\Pi(\iota\tau_1 + (1-\iota)\tau_2) \leq \iota\Pi(\tau_1) + (1-\iota)\Pi(\tau_2),$$

for all $\tau_1, \tau_2 \in \mathfrak{K}$ and $\iota \in [0,1]$.

Definition 2 ([28]). *The term invexity (ξ-connected set) is defined on a set $\mathfrak{K} \subset \mathbb{R}^n$ with respect to $\xi(*,*)$, if $\tau_1, \tau_2 \in \mathfrak{K}$ and $\iota \in [0,1]$*

$$\tau_1 + \iota\xi(\tau_2, \tau_1) \in \mathfrak{K}.$$

It is self-evident that every convex set is invex in terms of $\xi(\tau_2, \tau_1) = \tau_2 - \tau_1$. However, there are invex sets that are not convex [20].

Definition 3 ([29]). *Let $\mathbb{I} \subseteq \mathbb{R}^n$, then $\mathbb{I}$ is m–invex w.r.t $\xi : \mathbb{I} \times \mathbb{I} \times (0,1] \to \mathbb{R}^n$, if*

$$m\tau_1 + \iota\xi(\tau_2, \tau_1, m) \in \mathbb{I},$$

for every $\tau_1, \tau_2 \in \mathbb{I}$, $m \in (0,1]$ and $\iota \in [0,1]$.

Example 1 ([29]). *Suppose $m = \frac{1}{4}$, $\mathbb{I} = [\frac{-\pi}{2}, 0) \bigcup (0, \frac{1}{2}]$ and*

$$\xi(\tau_2, \tau_1, m) = \begin{cases} m\cos(\tau_2 - \tau_1) & if \ \tau_1 \in (0, \frac{\pi}{2}], \tau_2 \in (0, \frac{\pi}{2}]; \\ -m\cos(\tau_2 - \tau_1) & if \ \tau_1 \in [\frac{-\pi}{2}, 0), \tau_2 \in [\frac{-\pi}{2}, 0); \\ m\cos(\tau_1) & if \ \tau_1 \in (0, \frac{\pi}{2}], \tau_2 \in [\frac{-\pi}{2}, 0); \\ -m\cos(\tau_1) & if \ \tau_1 \in [-\frac{\pi}{2}, 0), \tau_2 \in (0, \frac{\pi}{2}]. \end{cases}$$

Then, $\mathbb{I}$ is an m-invex set with respect to ξ for $\iota \in [0,1]$ and $m = \frac{1}{4}$. It is obvious that $\mathbb{I}$ is not a convex set.

In the year 1988, Mond and Weir [30] explored the idea of invex set to introduce the idea of preinvexity.

Definition 4 ([30]). *A function $\Pi : \mathfrak{K} \to \mathbb{R}^n$ is said to be preinvex with respect to ξ, if*

$$\Pi(\tau_1 + \iota\xi(\tau_2, \tau_1)) \leq (1-\iota)\,\Pi(\tau_1) + \iota\,\Pi(\tau_2), \quad \forall \tau_1, \tau_2 \in \mathfrak{K}, \ \iota \in [0,1].$$

It is very important to mark that every convex is a preinvex function, but the converse is not true [21]. For example, $\Pi(\iota) = -|\iota|, \ \forall \iota \in \mathbb{R}$, is preinvex but not convex with respect to

$$\xi(\tau_2,\tau_1) = \begin{cases} \tau_2 - \tau_1 & if \ \tau_1\tau_2 \geq 0 \\ \tau_1 - \tau_2 & if \ \tau_1\tau_2 < 0. \end{cases}$$

Recently, Deng [31] introduced m–preinvex function, which is defined as:

Definition 5. *A function $\Pi : \mathbb{I} \to \mathbb{R}$ is said to be generalized m–preinvex with respect to $\xi : \mathbb{I} \times \mathbb{I} \times (0,1] \to \mathbb{R}^n$ for $m \in (0,1]$, if*

$$\Pi(m\tau_1 + \iota\xi(\tau_2,\tau_1,m)) \leq m(1-\iota)\Pi(\tau_1) + \iota\Pi(\tau_2), \tag{1}$$

for every $\tau_1, \tau_2 \in \mathbb{I}$, $\iota \in [0,1]$.

The following condition C was explored and discussed for the first time by Mohan and Neogy [32].

Condition-C: Assume that $\mathfrak{K} \subset \mathbb{R}^n$ is an open invex subset with respect to $\xi : \mathfrak{K} \times \mathfrak{K} \to \mathbb{R}$. We say the ξ satisfies the condition C if for any $\tau_1, \tau_2 \in \mathfrak{K}$ and $\iota \in [0,1]$,

$$\begin{aligned} \xi(\tau_2, \tau_2 + \iota\,\xi(\tau_1,\tau_2)) &= \ -\iota\,\xi(\tau_1,\tau_2) \\ \xi(\tau_1, \tau_2 + \iota\,\xi(\tau_1,\tau_2)) &= \ (1-\iota)\,\xi(\tau_1,\tau_2). \end{aligned} \tag{2}$$

For any $\tau_1, \tau_2 \in \mathfrak{K}$ and $\iota_1, \iota_2 \in [0,1]$ from condition C, we have

$$\xi(\tau_2 + \iota_2\,\xi(\tau_1,\tau_2)\,,\ \tau_2 + \iota_1\,\xi(\tau_1,\tau_2)) = \ (\iota_2 - \iota_1)\xi(\tau_1,\tau_2).$$

If Π is a preinvex on $[\tau_1, \tau_1 + \xi(\tau_2,\tau_1)]$ and ξ satisfies condition C, then for each $\iota \in [0,1]$, from above Equation (2), it yields

$$\begin{aligned} |\Pi(\tau_1 + \iota\xi(\tau_2,\tau_1))| &= |\Pi(\tau_1 + \xi(\tau_2,\tau_1)) + (1-\iota)\xi(\tau_1, \tau_1 + \xi(\tau_2,\tau_1))| \\ &\leq \iota\,|\Pi(\tau_1 + \xi(\tau_2,\tau_1))| + (1-\iota)|\Pi(\tau_1)| \end{aligned}$$

and

$$\begin{aligned} |\Pi(\tau_1 + (1-\iota)\xi(\tau_2,\tau_1))| &= |\Pi(\tau_1 + \xi(\tau_2,\tau_1)) + \iota\xi(\tau_1, \tau_1 + \xi(\tau_2,\tau_1))| \\ &\leq (1-\iota)\,|\Pi(\tau_1 + \xi(\tau_2,\tau_1))| + \iota|\Pi(\tau_1)|. \end{aligned}$$

The following generalized Condition C first time introduced by Du [33] in the aspect of m–preinvex.

Extended Condition-C: Assume that $\mathfrak{K} \subset \mathbb{R}^n$ be an open invex subset with respect to $\xi : \mathfrak{K} \times \mathfrak{K} \times (0,1] \to \mathbb{R}$. We say the ξ satisfies the extended condition C, for any $\tau_1, \tau_2 \in \mathfrak{K}$, $\iota \in [0,1]$ and $m \in (0,1]$, if

$$\begin{aligned} \xi(\tau_1, m\tau_1 + \iota\xi(\tau_2,\tau_1,m), m) &= -\iota\,\xi(\tau_2,\tau_1,m), \\ \xi(\tau_2, m\tau_1 + \iota\,\xi(\tau_2,\tau_1,m), m) &= (1-\iota)\,\xi(\tau_2,\tau_1,m), \\ \xi(\tau_2,\tau_1,m) &= -\xi(\tau_1,\tau_2,m). \end{aligned}$$

If Π is a m-preinvex on $[m\tau_1, m\tau_1 + \xi(\tau_2,\tau_1,m)]$ and ξ satisfies extended condition C, then for each $\iota \in [0,1]$, from above equation, it yields

$$\begin{aligned} |\Pi(m\tau_1 + \iota\xi(\tau_2,\tau_1,m))| &= |\Pi(m\tau_1 + \xi(\tau_2,\tau_1,m)) + (1-\iota)\xi(m\tau_1, m\tau_1 + \xi(\tau_2,\tau_1,m))| \\ &\leq \iota\,|\Pi(m\tau_1 + \xi(\tau_2,\tau_1,m))| + (1-\iota)|\Pi(m\tau_1)| \end{aligned}$$

and

$$
\begin{aligned}
|\Pi(m\tau_1+(1-\iota)\xi(\tau_2,\tau_1,m))| &= |\Pi(m\tau_1+\xi(\tau_2,\tau_1,m))+\iota\xi(m\tau_1,m\tau_1+\xi(\tau_2,\tau_1,m))| \\
&\leq (1-\iota)\,|\Pi(m\tau_1+\xi(\tau_2,\tau_1,m))|+m\iota|\Pi(\tau_1)|.
\end{aligned}
$$

There are numerous vector functions that meet the condition C in [28], with trivial case $\xi(\tau_1,\tau_2)=\tau_1-\tau_2$.

For example, suppose $\mathfrak{K}=\mathbb{R}\backslash\{0\}$ and

$$
\xi(\tau_2,\tau_1)=\begin{cases} \tau_2-\tau_1 & if\;\; \tau_1>0,\tau_2>0 \\ \tau_2-\tau_1 & if\;\; \tau_1<0,\tau_2<0 \\ -\tau_2, & otherwise \end{cases}
$$

The set $\mathfrak{K}$ is invex set and the condition C is satisfied by ξ.

In Noor [34], the following H-H type inequalities were demonstrated.

Theorem 1. *Assume that function $\Pi:\mathfrak{K}=[\tau_1,\tau_1+\xi(\tau_2,\tau_1)]\to(0,\infty)$ is preinvex on $\mathfrak{K}^0$ with $\xi(\tau_2,\tau_1)>0$. Then:*

$$
\Pi\left(\frac{2\tau_1+\xi(\tau_2,\tau_1)}{2}\right)\leq\frac{1}{\xi(\tau_2,\tau_1)}\int_{\tau_1}^{\tau_1+\,\xi(\tau_2,\tau_1)}\Pi(x)\,dx\leq\frac{\Pi(\tau_1)\;+\;\Pi(\tau_2)}{2}.
$$

Definition 6 ([35])**.** *Suppose $\Pi\in\mathcal{L}[\tau_1,\tau_2]$. The left-sided and right-sided Riemann–Liouville fractional integrals of order $\varrho>0$ defined by*

$$
J^{\varrho}_{\tau_1}\Pi(\tau)=\frac{1}{\Gamma(\varrho)}\int_{\tau_1}^{\tau}(\tau-\mu)^{\varrho-1}\Pi(\mu)d\mu,\qquad \tau_1<\tau
$$

and

$$
J^{\varrho}_{\tau_2}\Pi(\tau)=\frac{1}{\Gamma(\varrho)}\int_{\tau}^{\tau_2}(\mu-\tau)^{\varrho-1}\Pi(\mu)d\mu,\qquad \tau<\tau_2.
$$

The gamma function is defined as $\Gamma(\varrho)=\int_0^\infty e^{-u}u^{\varrho-1}du$.

Note that $J^0_{\tau_1}f(\tau)=J^0_{\tau_2}\Pi(\tau)=\Pi(\tau)$.

Throughout the paper, we will consider that $\Gamma(.)$ is the gamma function and $\varrho>0$.

3. Main Results

Lemma 1. *Let an open invex subset $\mathfrak{K}\subseteq\mathbb{R}$ with respect to $\xi:\mathfrak{K}\times\mathfrak{K}\longrightarrow\mathbb{R}$ and $\tau_1,\tau_2\in\mathfrak{K}$ with $m\tau_1<m\tau_1+\xi(\tau_2,\tau_1,m)$. Assume that $\Pi:\mathfrak{K}\to\mathbb{R}$ is differentiable function on $\mathfrak{K}$ such that $\Pi'\in\mathcal{L}([m\tau_1,m\tau_1+\xi(\tau_2,\tau_1,m)])$. Then:*

$$
\begin{aligned}
&\frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1,m)]^{\varrho}}\,J^{\varrho}_{m\tau_1^+}\Pi(m\tau_1+\xi(\tau_2,\tau_1,m))\;-\Pi\left(m\tau_1+\frac{\varrho}{\varrho+1}\xi(\tau_2,\tau_1,m)\right) \\
&=\;\xi(\tau_2,\tau_1,m)\left[-\int_0^{\frac{\varrho}{\varrho+1}}\iota^{\varrho}\Pi'(m\tau_1+\iota\xi(\tau_2,\tau_1,m))d\iota+\int_{\frac{\varrho}{\varrho+1}}^1(1-\iota^{\varrho})\Pi'(m\tau_1+\iota\xi(\tau_2,\tau_1,m))d\iota\right].
\end{aligned}
\tag{3}
$$

Proof. By applying the integration by parts to the right hand side of (3), we obtain

$$
\xi(\tau_2,\tau_1,m)\left[-\int_0^{\frac{\varrho}{\varrho+1}}\iota^{\varrho}\Pi'(m\tau_1+\iota\xi(\tau_2,\tau_1,m))d\iota+\int_{\frac{\varrho}{\varrho+1}}^1(1-\iota^{\varrho})\Pi'(m\tau_1+\iota\xi(\tau_2,\tau_1,m))d\iota\right]
$$

$$
=\;\xi(\tau_2,\tau_1,m)\left[-\int_0^1\iota^{\varrho}\Pi'(m\tau_1+\iota\xi(\tau_2,\tau_1,m))d\iota+\int_{\frac{\varrho}{\varrho+1}}^1\Pi'(m\tau_1+\iota\xi(\tau_2,\tau_1,m))d\iota\right]
$$

$$= -\Pi(m\tau_1 + \xi(\tau_2,\tau_1,m)) + \varrho\int_0^1 \iota^{\varrho-1}\Pi(m\tau_1+\iota\xi(\tau_2,\tau_1,m))d\iota + \Pi(m\tau_1+\xi(\tau_2,\tau_1,m))$$
$$-\Pi\left(m\tau_1 + \frac{\varrho}{\varrho+1}\xi(\tau_2,\tau_1,m)\right)$$
$$= \frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1,m)]^{\varrho}} J^{\varrho}_{m\tau_1^+}\Pi(m\tau_1+\xi(\tau_2,\tau_1,m)) - \Pi\left(m\tau_1+\frac{\varrho}{\varrho+1}\xi(\tau_2,\tau_1,m)\right).$$

This ends the proof. □

Lemma 2. *Let an open invex subset $\mathfrak{K} \subseteq \mathbb{R}$ with respect to $\xi : \mathfrak{K}\times\mathfrak{K} \longrightarrow \mathbb{R}$ and $\tau_1, \tau_2 \in \mathfrak{K}$ with $m\tau_1 < m\tau_1 + \xi(\tau_2,\tau_1,m)$. Assume that $\Pi : \mathfrak{K} \to \mathbb{R}$ is twice differentiable function on $\mathfrak{K}$ such that $\Pi'' \in \mathcal{L}([m\tau_1, m\tau_1+\xi(\tau_2,\tau_1,m)])$. Then:*

$$\frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1,m)]^{\varrho+1}}\left\{J^{\varrho-1}_{\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)^-}\Pi(m\tau_1) + J^{\varrho-1}_{\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)^+}\Pi(m\tau_1+\xi(\tau_2,\tau_1,m))\right\} \tag{4}$$
$$-\frac{\varrho\,\Pi\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)}{2^{\varrho-2}[\xi(\tau_2,\tau_1,m)]^2}$$
$$= \int_0^{\frac{1}{2}} \iota^{\varrho}\,\Pi''(m\tau_1+\iota\,\xi(\tau_2,\tau_1,m))\,d\iota + \int_{\frac{1}{2}}^1 (1-\iota)^{\varrho}\,\Pi''(m\tau_1+\iota\,\xi(\tau_2,\tau_1,m))\,d\iota.$$

Proof. It suffices to write that

$$\begin{aligned} I &= \int_0^{\frac{1}{2}} \iota^{\varrho}\,\Pi''(m\tau_1+\iota\,\xi(\tau_2,\tau_1,m))\,d\iota + \int_{\frac{1}{2}}^1 (1-\iota)^{\varrho}\,\Pi''(m\tau_1+\iota\,\xi(\tau_2,\tau_1,m))\,d\iota \\ &= I_1 + I_2, \end{aligned} \tag{5}$$

where

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{2}} \iota^{\varrho}\,\Pi''(m\tau_1+\iota\,\xi(\tau_2,\tau_1,m))\,d\iota \\ &= \frac{\iota^{\varrho}\,\Pi'(m\tau_1+\iota\,\xi(\tau_2,\tau_1,m))}{\xi(\tau_2,\tau_1,m)}\Bigg|_0^{\frac{1}{2}} - \frac{\varrho}{\xi(\tau_2,\tau_1,m)}\int_0^{\frac{1}{2}} \iota^{\varrho-1}\,\Pi'(m\tau_1+\iota\,\xi(\tau_2,\tau_1,m))\,d\iota \\ &= \frac{\Pi'\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)}{2^{\varrho}\,\xi(\tau_2,\tau_1,m)} - \frac{\varrho\,\Pi\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)}{2^{\varrho-1}[\xi(\tau_2,\tau_1,m)]^2} \\ &\quad + \frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1,m)]^{\varrho+1}} J^{\varrho-1}_{\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)^-}\Pi(m\tau_1) \end{aligned} \tag{6}$$

and

$$\begin{aligned} I_2 &= \int_{\frac{1}{2}}^1 (1-\iota)^{\varrho}\,\Pi''(m\tau_1+\iota\,\xi(\tau_2,\tau_1,m))\,d\iota \\ &= \frac{(1-\iota)^{\varrho}\,\Pi'(m\tau_1+\iota\,\xi(\tau_2,\tau_1,m))}{\xi(\tau_2,\tau_1,m)}\Big|_{\frac{1}{2}}^{1} + \frac{\varrho}{\xi(\tau_2,\tau_1,m)}\int_{\frac{1}{2}}^1 (1-\iota)^{\varrho-1}\,\Pi'(m\tau_1+\iota\,\xi(\tau_2,\tau_1,m))\,d\iota \\ &= -\frac{\Pi'\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)}{2^{\varrho}\xi(\tau_2,\tau_1,m)} + \frac{\varrho}{\xi(\tau_2,\tau_1,m)}\int_{\frac{1}{2}}^1 (1-\iota)^{\varrho-1}\,\Pi'(m\tau_1+\iota\,\xi(\tau_2,\tau_1,m))\,d\iota \end{aligned}$$

$$= -\frac{\Pi'\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)}{2^{\varrho}\,\xi(\tau_2,\tau_1,m)} - \frac{\varrho\,\Pi\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)}{2^{\varrho-1}[\xi(\tau_2,\tau_1,m)]^2}$$
$$+\frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1,m)]^{\varrho+1}}\; J^{\varrho-1}_{\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)^+}\;\Pi(m\tau_1+\xi(\tau_2,\tau_1,m)). \tag{7}$$

Combine Equations (6) and (7) with (5), and obtain Equation (4). □

Theorem 2. *Let all the conditions in Lemma 1 are satisfied. If $|\Pi'|$ is m-preinvex on $[m\tau_1, m\tau_1 +\xi(\tau_2,\tau_1,m)]$, then, for fractional integrals, the following inequality with $\varrho>0$ holds:*

$$\left|\frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1,m)]^{\varrho}}\, J^{\varrho}_{m\tau_1^+}\Pi(m\tau_1+\xi(\tau_2,\tau_1,m)) \;-\Pi\left(m\tau_1+\frac{\varrho}{\varrho+1}\xi(\tau_2,\tau_1,m)\right)\right| \tag{8}$$
$$\leq\; \xi(\tau_2,\tau_1,m)\left[\frac{m\varrho}{2(\varrho+1)^2(\varrho+2)}|\Pi'(\tau_1)|+\frac{\varrho(-\varrho^{\varrho+1}+2\varrho^{\varrho}+(\varrho+1)^{\varrho})}{2(\varrho+1)^{\varrho+2}(\varrho+2)}\,|\Pi'(\tau_2)|\right].$$

Proof. From inequality (3) and the m-preinvexity of $|\Pi'|$, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1,m)]^{\varrho}}\, J^{\varrho}_{m\tau_1^+}\Pi(m\tau_1+\xi(\tau_2,\tau_1,m)) \;-\Pi\left(m\tau_1+\frac{\varrho}{\varrho+1}\xi(\tau_2,\tau_1,m)\right)\right|\\
\leq\;& \xi(\tau_2,\tau_1,m)\left[\int_0^{\frac{\varrho}{\varrho+1}}\iota^{\varrho}|\Pi'(m\tau_1+\iota\xi(\tau_2,\tau_1,m))|\,d\iota\right.\\
&\left.+\int_{\frac{\varrho}{\varrho+1}}^1(1-\iota^{\varrho})|\Pi'(m\tau_1+\iota\xi(\tau_2,\tau_1,m))|d\iota\right]\\
\leq\;& \xi(\tau_2,\tau_1,m)\left[\int_0^{\frac{\varrho}{\varrho+1}}\iota^{\varrho}\{m(1-\iota)|\Pi'(\tau_1)|+\iota|\Pi'(\tau_2)|\}d\iota\right.\\
&\left.+\int_{\frac{\varrho}{\varrho+1}}^1(1-\iota^{\varrho})\{m(1-\iota)|\Pi'(\tau_1)|+\iota|\Pi'(\tau_2)|\}d\iota\right]\\
\leq\;& \xi(\tau_2,\tau_1,m)\left[m|\Pi'(\tau_1)|\int_0^{\frac{\varrho}{\varrho+1}}\iota^{\varrho}(1-\iota)d\iota+|\Pi'(\tau_2)|\int_0^{\frac{\varrho}{\varrho+1}}\iota^{\varrho+1}d\iota\right.\\
&\left.+m|\Pi'(\tau_1)|\int_{\frac{\varrho}{\varrho+1}}^1(1-\iota^{\varrho})(1-\iota)d\iota+|\Pi'(\tau_2)|\int_{\frac{\varrho}{\varrho+1}}^1(1-\iota^{\varrho})\iota d\iota\right]\\
=\;& \xi(\tau_2,\tau_1,m)\left[\frac{m\varrho}{2(\varrho+1)^2(\varrho+2)}|\Pi'(\tau_1)|+\frac{\varrho(-\varrho^{\varrho+1}+2\varrho^{\varrho}+(\varrho+1)^{\varrho})}{2(\varrho+1)^{\varrho+2}(\varrho+2)}\,|\Pi'(\tau_2)|\right],
\end{aligned}$$

where

$$\begin{aligned}
\int_0^{\frac{\varrho}{\varrho+1}}\iota^{\varrho+1}d\iota &= \frac{\varrho^{\varrho+2}}{(\varrho+1)^{\varrho+2}(\varrho+2)},\\
\int_0^{\frac{\varrho}{\varrho+1}}\left(\iota^{\varrho}-\iota^{\varrho+1}\right)d\iota &= \frac{2\varrho^{\varrho+1}}{(\varrho+1)^{\varrho+2}(\varrho+2)},\\
\int_{\frac{\varrho}{\varrho+1}}^1\left(\iota-\iota^{\varrho+1}\right)d\iota &= \frac{2\varrho^{\varrho+1}+\varrho(\varrho+1)^{\varrho}}{2(\varrho+1)^{\varrho+2}(\varrho+2)},\\
\int_{\frac{\varrho}{\varrho+1}}^1(1-\iota^{\varrho})(1-\iota)d\iota &= \frac{4\varrho^{\varrho+1}-\varrho(\varrho+1)^{\varrho}}{2(\varrho+1)^{\varrho+2}(\varrho+2)}.
\end{aligned}$$

This ends the proof. □

Remark 1. *In inequality (8), if we take* $\xi(\tau_2, \tau_1, m) = \tau_2 - m\tau_1$ *and* $\varrho = m = 1$*, then we get the inequality proven in [36], Theorem 2.2.*

Corollary 1. *In inequality (8), if we take* $\xi(\tau_2, \tau_1, m) = \tau_2 - m\tau_1$*, then*

$$\begin{aligned}
&\left|\frac{\Gamma(\varrho+1)}{[(\tau_2 - m\tau_1)]^{\varrho}} J^{\varrho}_{m\tau_1^+}\Pi(\tau_2) - \Pi\left(\frac{m\tau_1 + \varrho\tau_2}{\varrho+1}\right)\right| \\
&\leq (\tau_2 - m\tau_1)\left[\frac{m\varrho}{2(\varrho+1)^2(\varrho+2)}\left|\Pi'(\tau_1)\right| + \frac{\varrho\left(-\varrho^{\varrho+1} + 2\varrho^{\varrho} + (\varrho+1)^{\varrho}\right)}{2(\varrho+1)^{\varrho+2}(\varrho+2)}\left|\Pi'(\tau_2)\right|\right].
\end{aligned} \tag{9}$$

Corollary 2. *In inequality (8), if we take* $\xi(\tau_2, \tau_1, m) = \tau_2 - m\tau_1$ *and* $m = 1$*, then*

$$\begin{aligned}
&\left|\frac{\Gamma(\varrho+1)}{[(\tau_2 - \tau_1)]^{\varrho}} J^{\varrho}_{\tau_1^+}\Pi(\tau_2) - \Pi\left(\frac{\tau_1 + \varrho\tau_2}{\varrho+1}\right)\right| \\
&\leq (\tau_2 - \tau_1)\left[\frac{\varrho}{2(\varrho+1)^2(\varrho+2)}\left|\Pi'(\tau_1)\right| + \frac{\varrho\left(-\varrho^{\varrho+1} + 2\varrho^{\varrho} + (\varrho+1)^{\varrho}\right)}{2(\varrho+1)^{\varrho+2}(\varrho+2)}\left|\Pi'(\tau_2)\right|\right].
\end{aligned} \tag{10}$$

Corollary 3. *If* ξ *satisfies the extended condition C, then by definition of the m-preinvexity of* $|\Pi'|$*, we obtain*

$$\begin{aligned}
\left|\Pi'(m\tau_1 + \iota\xi(\tau_2, \tau_1, m))\right| &= \left|\Pi'(m\tau_1 + \xi(\tau_2, \tau_1, m)) + (1-\iota)\xi(m\tau_1, m\tau_1 + \xi(\tau_2, \tau_1, m))\right| \\
&\leq \iota\left|\Pi'(m\tau_1 + \xi(\tau_2, \tau_1, m))\right| + m(1-\iota)\left|\Pi'(\tau_1)\right|.
\end{aligned} \tag{11}$$

Using inequality (11) in the proof of Theorem 2, the inequality (8) becomes

$$\begin{aligned}
&\left|\frac{\Gamma(\varrho+1)}{[\xi(\tau_2, \tau_1, m)]^{\varrho}} J^{\varrho}_{m\tau_1^+}\Pi(m\tau_1 + \xi(\tau_2, \tau_1, m)) - \Pi\left(m\tau_1 + \frac{\varrho}{\varrho+1}\xi(\tau_2, \tau_1, m)\right)\right| \\
&\leq \xi(\tau_2, \tau_1, m) \\
&\times \left[\frac{m\varrho}{2(\varrho+1)^2(\varrho+2)}\left|\Pi'(\tau_1)\right| + \frac{\varrho\left(-\varrho^{\varrho+1} + 2\varrho^{\varrho} + (\varrho+1)^{\varrho}\right)}{2(\varrho+1)^{\varrho+2}(\varrho+2)}\left|\Pi'(m\tau_1 + \xi(\tau_2, \tau_1, m))\right|\right].
\end{aligned} \tag{12}$$

We observe that, by employing the m-preinvexity of $|\Pi'|$, we have

$$\left|\Pi'(m\tau_1 + \xi(\tau_2, \tau_1, m))\right| \leq \left|\Pi'(\tau_2)\right|.$$

Therefore, inequality (12) is better than inequality (8).

Corollary 4. *If* ξ *satisfies the condition C and* $m = 1$*, then by definition of the preinvexity of* $|\Pi'|$*, we obtain*

$$\begin{aligned}
\left|\Pi'(\tau_1 + \iota\xi(\tau_2, \tau_1))\right| &= \left|\Pi'(\tau_1 + \xi(\tau_2, \tau_1)) + (1-\iota)\xi(m\tau_1 + \xi(\tau_2, \tau_1))\right| \\
&\leq \iota\left|\Pi'(\tau_1 + \xi(\tau_2, \tau_1))\right| + (1-\iota)\left|\Pi'(\tau_1)\right|.
\end{aligned} \tag{13}$$

Using inequality (13) in proof of Theorem 2, inequality (8) becomes the following:

$$\begin{aligned}
&\left|\frac{\Gamma(\varrho+1)}{[\xi(\tau_2, \tau_1)]^{\varrho}} J^{\varrho}_{\tau_1^+}\Pi(\tau_1 + \xi(\tau_2, \tau_1)) - \Pi\left(\tau_1 + \frac{\varrho}{\varrho+1}\xi(\tau_2, \tau_1)\right)\right| \\
&\leq \xi(\tau_2, \tau_1) \\
&\times \left[\frac{\varrho}{2(\varrho+1)^2(\varrho+2)}\left|\Pi'(\tau_1)\right| + \frac{\varrho\left(-\varrho^{\varrho+1} + 2\varrho^{\varrho} + (\varrho+1)^{\varrho}\right)}{2(\varrho+1)^{\varrho+2}(\varrho+2)}\left|\Pi'(\tau_1 + \xi(\tau_2, \tau_1))\right|\right].
\end{aligned} \tag{14}$$

We observe that, by employing the preinvexity of $|\Pi'|$, we have

$$\left|\Pi'(\tau_1+\xi(\tau_2,\tau_1))\right| \leq \left|\Pi'(\tau_2)\right|.$$

Therefore, inequality (14) is better than inequality (8).

Theorem 3. *Let all conditions in Lemma 1 be satisfied. If $|\Pi'|^q$ is m-preinvex on $[m\tau_1, m\tau_1+\xi(\tau_2,\tau_1,m)]$ for $y \geq 1$, then, for fractional integrals, the following inequality holds:*

$$\begin{aligned}
&\left|\frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1,m)]^{\varrho}}\, J^{\varrho}_{m\tau_1^+}\Pi(m\tau_1+\xi(\tau_2,\tau_1,m)) \; - \Pi\left(m\tau_1+\frac{\varrho}{\varrho+1}\xi(\tau_2,\tau_1,m)\right)\right| \qquad (15)\\
&\leq \xi(\tau_2,\tau_1,m)\left(\frac{\varrho^{\varrho+1}}{(\varrho+1)^{\varrho+2}}\right)^{1-\frac{1}{y}}\\
&\times\left\{\left(\frac{2m\varrho^{\varrho+1}}{(\varrho+1)^{\varrho+2}(\varrho+2)}\left|\Pi'(\tau_1)\right|^y+\frac{\varrho^{\varrho+2}}{(\varrho+1)^{\varrho+2}(\varrho+2)}\left|\Pi'(\tau_2)\right|^y\right)^{\frac{1}{y}}\right.\\
&\left.+\left(\frac{4\varrho^{\varrho+1}-\varrho(\varrho+1)^{\varrho}}{2(\varrho+1)^{\varrho+2}(\varrho+2)}m\left|\Pi'(\tau_1)\right|^y+\frac{2\varrho^{\varrho+1}+\varrho(\varrho+1)^{\varrho}}{2(\varrho+1)^{\varrho+2}(\varrho+2)}\left|\Pi'(\tau_2)\right|^y\right)^{\frac{1}{y}}\right\},
\end{aligned}$$

where $x^{-1}=1-y^{-1}$.

Proof. From inequality (3), by utilizing power-mean inequality and definition of m-preinvexity of $|\Pi'|^q$, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1,m)]^{\varrho}}\, J^{\varrho}_{m\tau_1^+}\Pi(m\tau_1+\xi(\tau_2,\tau_1,m)) \; - \Pi\left(m\tau_1+\frac{\varrho}{\varrho+1}\xi(\tau_2,\tau_1,m)\right)\right|\\
\leq\;& \xi(\tau_2,\tau_1,m)\left[\int_0^{\frac{\varrho}{\varrho+1}}\iota^{\varrho}\Pi'(m\tau_1+\iota\xi(\tau_2,\tau_1,m))d\iota+\int_{\frac{\varrho}{\varrho+1}}^1(1-\iota^{\varrho})\Pi'(m\tau_1+\iota\xi(\tau_2,\tau_1,m))d\iota\right]\\
\leq\;& \xi(\tau_2,\tau_1,m)\left\{\left(\int_0^{\frac{\varrho}{\varrho+1}}\iota^{\varrho}d\iota\right)^{1-\frac{1}{y}}\left(\int_0^{\frac{\varrho}{\varrho+1}}\iota^{\varrho}\left|\Pi'(m\tau_1+\iota\xi(\tau_2,\tau_1,m))\right|d\iota\right)^{\frac{1}{y}}\right.\\
&\left.+\left(\int_{\frac{\varrho}{\varrho+1}}^1(1-\iota^{\varrho})d\iota\right)^{1-\frac{1}{y}}\left(\int_{\frac{\varrho}{\varrho+1}}^1(1-\iota^{\varrho})\left|\Pi'(m\tau_1+\iota\xi(\tau_2,\tau_1,m))\right|d\iota\right)^{\frac{1}{y}}\right\}\\
\leq\;& \xi(\tau_2,\tau_1,m)\left(\frac{\varrho^{\varrho+1}}{(\varrho+1)^{\varrho+2}}\right)^{1-\frac{1}{y}}\left\{\left(\int_0^{\frac{\varrho}{\varrho+1}}\iota^{\varrho}\left(m(1-\iota)\left|\Pi'(\tau_1)\right|^y+\iota\left|\Pi'(\tau_2)\right|^y\right)d\iota\right)^{\frac{1}{y}}\right.\\
&\left.+\int_{\frac{\varrho}{\varrho+1}}^1(1-\iota^{\varrho})\left(m(1-\iota)\left|\Pi'(\tau_1)\right|^y+\iota\left|\Pi'(\tau_2)\right|^y\right)d\iota\right\}\\
=\;& \xi(\tau_2,\tau_1,m)\left(\frac{\varrho^{\varrho+1}}{(\varrho+1)^{\varrho+2}}\right)^{1-\frac{1}{y}}\\
&\times\left\{\left(\frac{2m\varrho^{\varrho+1}}{(\varrho+1)^{\varrho+2}(\varrho+2)}\left|\Pi'(\tau_1)\right|^y+\frac{\varrho^{\varrho+2}}{(\varrho+1)^{\varrho+2}(\varrho+2)}\left|\Pi'(\tau_2)\right|^y\right)^{\frac{1}{y}}\right.\\
&\left.+\left(\frac{4\varrho^{\varrho+1}-\varrho(\varrho+1)^{\varrho}}{2(\varrho+1)^{\varrho+2}(\varrho+2)}m\left|\Pi'(\tau_1)\right|^y+\frac{2\varrho^{\varrho+1}+\varrho(\varrho+1)^{\varrho}}{2(\varrho+1)^{\varrho+2}(\varrho+2)}\left|\Pi'(\tau_2)\right|^y\right)^{\frac{1}{y}}\right\}.
\end{aligned}$$

This ends the proof. □

Corollary 5. *In inequality (15), if we take* $\xi(\tau_2, \tau_1, m) = \tau_2 - m\tau_1$, *then*

$$\begin{aligned}
&\left|\frac{\Gamma(\varrho+1)}{[(\tau_2 - m\tau_1)]^{\varrho}} J^{\varrho}_{m\tau_1^+}\Pi(\tau_2) - \Pi\left(\frac{m\tau_1+\varrho\tau_2}{\varrho+1}\right)\right| \\
&\leq \xi(\tau_2 - m\tau_1)\left(\frac{\varrho^{\varrho+1}}{(\varrho+1)^{\varrho+2}}\right)^{1-\frac{1}{y}} \\
&\times\left\{\left(\frac{2m\varrho^{\varrho+1}}{(\varrho+1)^{\varrho+2}(\varrho+2)}|\Pi'(\tau_1)|^y + \frac{\varrho^{\varrho+2}}{(\varrho+1)^{\varrho+2}(\varrho+2)}|\Pi'(\tau_2)|^y\right)^{\frac{1}{y}}\right. \\
&\left.+\left(\frac{4\varrho^{\varrho+1}-\varrho(\varrho+1)^{\varrho}}{2(\varrho+1)^{\varrho+2}(\varrho+2)}m|\Pi'(\tau_1)|^y + \frac{2\varrho^{\varrho+1}+\varrho(\varrho+1)^{\varrho}}{2(\varrho+1)^{\varrho+2}(\varrho+2)}|\Pi'(\tau_2)|^y\right)^{\frac{1}{y}}\right\}.
\end{aligned} \tag{16}$$

Corollary 6. *In inequality (15), if we take* $\xi(\tau_2, \tau_1, m) = \tau_2 - m\tau_1$ *and* $m = 1$, *then*

$$\begin{aligned}
&\left|\frac{\Gamma(\varrho+1)}{[(\tau_2 - \tau_1)]^{\varrho}} J^{\varrho}_{\tau_1^+}\Pi(\tau_2) - \Pi\left(\frac{\tau_1+\varrho\tau_2}{\varrho+1}\right)\right| \\
&\leq \xi(\tau_2 - \tau_1)\left(\frac{\varrho^{\varrho+1}}{(\varrho+1)^{\varrho+2}}\right)^{1-\frac{1}{y}} \\
&\times\left\{\left(\frac{2\varrho^{\varrho+1}}{(\varrho+1)^{\varrho+2}(\varrho+2)}|\Pi'(\tau_1)|^y + \frac{\varrho^{\varrho+2}}{(\varrho+1)^{\varrho+2}(\varrho+2)}|\Pi'(\tau_2)|^y\right)^{\frac{1}{y}}\right. \\
&\left.+\left(\frac{4\varrho^{\varrho+1}-\varrho(\varrho+1)^{\varrho}}{2(\varrho+1)^{\varrho+2}(\varrho+2)}|\Pi'(\tau_1)|^y + \frac{2\varrho^{\varrho+1}+\varrho(\varrho+1)^{\varrho}}{2(\varrho+1)^{\varrho+2}(\varrho+2)}|\Pi'(\tau_2)|^y\right)^{\frac{1}{y}}\right\}.
\end{aligned} \tag{17}$$

Corollary 7. *In inequality (15), if we set* $\xi(\tau_2, \tau_1, m) = \tau_2 - m\tau_1$ *and* $\varrho = 1$, *then we obtain the following midpoint-type inequality:*

$$\begin{aligned}
&\left|\frac{1}{\tau_2 - m\tau_1}\int_{m\tau_1}^{\tau_2}\Pi(x)dx - \Pi\left(\frac{\tau_1+\tau_2}{2}\right)\right| \\
&\leq \frac{\tau_2 - m\tau_1}{8}\left\{\left(\frac{m|\Pi'(\tau_1)|^y + 2\left|\Pi'(\tau_2)\right|^y}{3}\right)^{\frac{1}{y}} + \left(\frac{2m|\Pi'(\tau_1)|^y + |\Pi'(\tau_2)|^y}{3}\right)^{\frac{1}{y}}\right\}.
\end{aligned} \tag{18}$$

Corollary 8. *In inequality (15), if we set* $\xi(\tau_2, \tau_1) = \tau_2 - \tau_1$, $m = 1$ *and* $\varrho = 1$, *then we obtain the following midpoint-type inequality*

$$\begin{aligned}
&\left|\frac{1}{\tau_2 - \tau_1}\int_{\tau_1}^{\tau_2}\Pi(x)dx - \Pi\left(\frac{\tau_1+\tau_2}{2}\right)\right| \\
&\leq \frac{\tau_2 - \tau_1}{8}\left\{\left(\frac{|\Pi'(\tau_1)|^y + 2\left|\Pi'(\tau_2)\right|^y}{3}\right)^{\frac{1}{y}} + \left(\frac{2|\Pi'(\tau_1)|^y + |\Pi'(\tau_2)|^y}{3}\right)^{\frac{1}{y}}\right\}.
\end{aligned} \tag{19}$$

Corollary 9. *In inequality (15), considering that* ξ *meets the extended condition C and using inequality (3), we obtain*

$$\left|\frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1,m)]^{\varrho}} J^{\varrho}_{m\tau_1^+}\Pi(m\tau_1+\xi(\tau_2,\tau_1,m)) - \Pi\left(m\tau_1 + \frac{\varrho}{\varrho+1}\xi(\tau_2,\tau_1,m)\right)\right|$$

$$\leq \xi(\tau_2,\tau_1,m)\left(\frac{\varrho^{\varrho+1}}{(\varrho+1)^{\varrho+2}}\right)^{1-\frac{1}{y}}$$
$$\times\left\{\left(\frac{2m\varrho^{\varrho+1}}{(\varrho+1)^{\varrho+2}(\varrho+2)}|\Pi'(\tau_1)|^y+\frac{\varrho^{\varrho+2}}{(\varrho+1)^{\varrho+2}(\varrho+2)}|\Pi'(m\tau_1+\xi(\tau_2,\tau_1,m))|^y\right)^{\frac{1}{y}}\right.$$
$$\left.+\left(\frac{4\varrho^{\varrho+1}-\varrho(\varrho+1)^{\varrho}}{2(\varrho+1)^{\varrho+2}(\varrho+2)}m|\Pi'(\tau_1)|^y+\frac{2\varrho^{\varrho+1}+\varrho(\varrho+1)^{\varrho}}{2(\varrho+1)^{\varrho+2}(\varrho+2)}|\Pi'(m\tau_1+\xi(\tau_2,\tau_1,m))|^y\right)^{\frac{1}{y}}\right\}.$$

Corollary 10. *In inequality (15), considering that ξ meets the extended condition C, $m=1$ and using inequality (3), we obtain*

$$\left|\frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1,m)]^{\varrho}}\,J^{\varrho}_{\tau_1^+}\Pi(\tau_1+\xi(\tau_2,\tau_1))\ -\Pi\left(\tau_1+\frac{\varrho}{\varrho+1}\xi(\tau_2,\tau_1)\right)\right|$$
$$\leq \xi(\tau_2,\tau_1)\left(\frac{\varrho^{\varrho+1}}{(\varrho+1)^{\varrho+2}}\right)^{1-\frac{1}{y}}$$
$$\times\left\{\left(\frac{2\varrho^{\varrho+1}}{(\varrho+1)^{\varrho+2}(\varrho+2)}|\Pi'(\tau_1)|^y+\frac{\varrho^{\varrho+2}}{(\varrho+1)^{\varrho+2}(\varrho+2)}|\Pi'(\tau_1+\xi(\tau_2,\tau_1))|^y\right)^{\frac{1}{y}}\right.$$
$$\left.+\left(\frac{4\varrho^{\varrho+1}-\varrho(\varrho+1)^{\varrho}}{2(\varrho+1)^{\varrho+2}(\varrho+2)}|\Pi'(\tau_1)|^y+\frac{2\varrho^{\varrho+1}+\varrho(\varrho+1)^{\varrho}}{2(\varrho+1)^{\varrho+2}(\varrho+2)}|\Pi'(\tau_1+\xi(\tau_2,\tau_1))|^y\right)^{\frac{1}{y}}\right\}.$$

Theorem 4. *Let all conditions in Lemma 1 be satisfied. If $|\Pi'|^q$ is m-preinvex on $[m\tau_1, m\tau_1 +\xi(\tau_2,\tau_1,m)]$ for $y>1$, then, for fractional integrals, the following inequality holds:*

$$\left|\frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1,m)]^{\varrho}}\,J^{\varrho}_{m\tau_1^+}\Pi(m\tau_1+\xi(\tau_2,\tau_1,m))\ -\Pi\left(m\tau_1+\frac{\varrho}{\varrho+1}\xi(\tau_2,\tau_1,m)\right)\right|$$
$$\leq \xi(\tau_2,\tau_1,m)\left\{\Big(M(\varrho,x)\Big)^{\frac{1}{x}}\left(\frac{\varrho^2+2\varrho}{2(\varrho+1)^2}m|\Pi'(\tau_1)|^y+\frac{\varrho^2}{2(\varrho+1)^2}|\Pi'(\tau_2)|^y\right)^{\frac{1}{y}}\right.$$
$$\left.+\Big(N(\varrho,x)\Big)^{\frac{1}{x}}\left(\frac{m}{2(\varrho+1)^2}|\Pi'(\tau_1)|^y+\frac{2\varrho+1}{2(\varrho+1)^2}|\Pi'(\tau_2)|^y\right)^{\frac{1}{y}}\right\}, \tag{20}$$

where

$$M(\varrho,x)=\int_0^{\frac{\varrho}{\varrho+1}}\iota^{\varrho x}d\iota,$$
$$N(\varrho,x)=\int_{\frac{\varrho}{\varrho+1}}^1(1-\iota^{\varrho})^x d\iota,$$

where $x^{-1}+y^{-1}=1$.

Proof. From inequality (3), from the Hölder integral inequality and the m-preinvexity of $|\Pi'|^q$, we have

$$\left|\frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1,m)]^{\varrho}}\,J^{\varrho}_{\tau_1^+}\Pi(m\tau_1+\xi(\tau_2,\tau_1,m))\ -\Pi\left(m\tau_1+\frac{\varrho}{\varrho+1}\xi(\tau_2,\tau_1,m)\right)\right|$$
$$\leq \xi(\tau_2,\tau_1,m)\left[\int_0^{\frac{\varrho}{\varrho+1}}\iota^{\varrho}|\Pi'(\tau_1+\iota\xi(\tau_2,\tau_1,m))|d\iota\right.$$
$$\left.+\int_{\frac{\varrho}{\varrho+1}}^1(1-\iota^{\varrho})|\Pi'(m\tau_1+\iota\xi(\tau_2,\tau_1,m))|d\iota\right]$$

$$\leq \xi(\tau_2,\tau_1,m)\Bigg\{\left(\int_0^{\frac{\varrho}{\varrho+1}}\iota^{\varrho x}d\iota\right)^{\frac{1}{x}}\left(\int_0^{\frac{\varrho}{\varrho+1}}|\Pi'(m\tau_1+\iota\xi(\tau_2,\tau_1,m))|^y d\iota\right)^{\frac{1}{y}}$$

$$+\left(\int_{\frac{\varrho}{\varrho+1}}^1(1-\iota^{\varrho})^x d\wp\right)^{\frac{1}{x}}\left(\int_{\frac{\varrho}{\varrho+1}}^1|\Pi'(m\tau_1+\iota\xi(\tau_2,\tau_1,m))|^y d\iota\right)^{\frac{1}{y}}\Bigg\}$$

$$\leq \xi(\tau_2,\tau_1,m)\Bigg\{\left(\int_0^{\frac{\varrho}{\varrho+1}}\iota^{\varrho x}d\iota\right)^{\frac{1}{x}}\left(\int_0^{\frac{\varrho}{\varrho+1}}\{m(1-\iota)|\Pi'(\tau_1)|^y+\iota|\Pi'(\tau_2)|^y\}d\iota\right)^{\frac{1}{y}}$$

$$+\left(\int_{\frac{\varrho}{\varrho+1}}^1(1-\iota^{\varrho})^x d\iota\right)^{\frac{1}{x}}\left(\int_{\frac{\varrho}{\varrho+1}}^1\{m(1-\iota)|\Pi'(\tau_1)|^y+\iota|\Pi'(\tau_2)|^y\}d\iota\right)^{\frac{1}{y}}\Bigg\}$$

$$= \xi(\tau_2,\tau_1,m)\Bigg\{\Big(M(\varrho,x)\Big)^{\frac{1}{x}}\left(\frac{\varrho^2+2\varrho}{2(\varrho+1)^2}m|\Pi'(\tau_1)|^y-\frac{\varrho^2}{2(\varrho+1)^2}|\Pi'(\tau_2)|^y\right)^{\frac{1}{y}}$$

$$+\Big(N(\varrho,x)\Big)^{\frac{1}{x}}\left(\frac{m}{2(\varrho+1)^2}|\Pi'(\tau_1)|^y+\frac{2\varrho+1}{2(\varrho+1)^2}|\Pi'(\tau_2)|^y\right)^{\frac{1}{y}}\Bigg\}.$$

This ends the proof. □

Remark 2. *In inequality (20), if we take* $\xi(\tau_2,\tau_1,m)=\tau_2-m\tau_1$ *and* $\varrho=1$*, then we obtain the inequality proved in [36], Theorem 2.3.*

Corollary 11. *In inequality (20), if we take* $\xi(\tau_2,\tau_1,m)=\tau_2-m\tau_1$*, then*

$$\left|\frac{\Gamma(\varrho+1)}{[(\tau_2-m\tau_1)]^{\varrho}}J^{\varrho}_{m\tau_1^+}\Pi(\tau_2)-\Pi\left(\frac{m\tau_1+\varrho\tau_2}{\varrho+1}\right)\right|$$

$$\leq(\tau_2-m\tau_1)\Bigg\{\Big(M(\varrho,x)\Big)^{\frac{1}{x}}\left(\frac{\varrho^2+2\varrho}{2(\varrho+1)^2}m|\Pi'(\tau_1)|^y+\frac{\varrho^2}{2(\varrho+1)^2}|\Pi'(\tau_2)|^y\right)^{\frac{1}{y}}$$

$$+\Big(N(\varrho,x)\Big)^{\frac{1}{x}}\left(\frac{m}{2(\varrho+1)^2}|\Pi'(\tau_1)|^y+\frac{2\varrho+1}{2(\varrho+1)^2}|\Pi'(\tau_2)|^y\right)^{\frac{1}{y}}\Bigg\}.$$

Corollary 12. *In inequality (20), if we take* $\xi(\tau_2,\tau_1)=\tau_2-\tau_1$ *and* $m=1$*, then*

$$\left|\frac{\Gamma(\varrho+1)}{[(\tau_2-\tau_1)]^{\varrho}}J^{\varrho}_{\tau_1^+}\Pi(\tau_2)-\Pi\left(\frac{\tau_1+\varrho\tau_2}{\varrho+1}\right)\right|$$

$$\leq(\tau_2-\tau_1)\Bigg\{\Big(M(\varrho,x)\Big)^{\frac{1}{x}}\left(\frac{\varrho^2+2\varrho}{2(\varrho+1)^2}|\Pi'(\tau_1)|^y+\frac{\varrho^2}{2(\varrho+1)^2}|\Pi'(\tau_2)|^y\right)^{\frac{1}{y}}$$

$$+\Big(N(\varrho,x)\Big)^{\frac{1}{x}}\left(\frac{1}{2(\varrho+1)^2}|\Pi'(\tau_1)|^y+\frac{2\varrho+1}{2(\varrho+1)^2}|\Pi'(\tau_2)|^y\right)^{\frac{1}{y}}\Bigg\}.$$

Corollary 13. *In inequality (20), considering that* ξ *meets the extended condition C and using inequality (3), we obtain*

$$\left|\frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1,m)]^{\varrho}}J^{\varrho}_{m\tau_1^+}\Pi(m\tau_1+\xi(\tau_2,\tau_1,m))-\Pi\left(m\tau_1+\frac{\varrho}{\varrho+1}\xi(\tau_2,\tau_1,m)\right)\right|$$

$$\leq \xi(\tau_2,\tau_1,m)$$

$$\times\Bigg\{\Big(M(\varrho,x)\Big)^{\frac{1}{x}}\left(\frac{\varrho^2+2\varrho}{2(\varrho+1)^2}m|\Pi'(\tau_1)|^y+\frac{\varrho^2}{2(\varrho+1)^2}|\Pi'(m\tau_1+\xi(\tau_2,\tau_1,m))|^y\right)^{\frac{1}{y}}$$

$$+\left(N(\varrho,x)\right)^{\frac{1}{x}}\left(\frac{m}{2(\varrho+1)^2}|\Pi'(\tau_1)|^y+\frac{2\varrho+1}{2(\varrho+1)^2}|\Pi'(m\tau_1+\xi(\tau_2,\tau_1,m))|^y\right)^{\frac{1}{y}}\Bigg\}.$$

Corollary 14. *In inequality (20), considering that ξ meets the extended condition C, $m=1$ and using inequality (3), we obtain*

$$\begin{aligned}&\left|\frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1)]^{\varrho}}J^{\varrho}_{\tau_1^+}\Pi(\tau_1+\xi(\tau_2,\tau_1))-\Pi\left(\tau_1+\frac{\varrho}{\varrho+1}\xi(\tau_2,\tau_1)\right)\right|\\ \leq\ &\xi(\tau_2,\tau_1)\Bigg\{\left(M(\varrho,x)\right)^{\frac{1}{x}}\left(\frac{\varrho^2+2\varrho}{2(\varrho+1)^2}|\Pi'(\tau_1)|^y+\frac{\varrho^2}{2(\varrho+1)^2}|\Pi'(\tau_1+\xi(\tau_2,\tau_1))|^y\right)^{\frac{1}{y}}\\ &+\left(N(\varrho,x)\right)^{\frac{1}{x}}\left(\frac{1}{2(\varrho+1)^2}|\Pi'(\tau_1)|^y+\frac{2\varrho+1}{2(\varrho+1)^2}|\Pi'(\tau_1+\xi(\tau_2,\tau_1))|^y\right)^{\frac{1}{y}}\Bigg\}.\end{aligned}$$

Theorem 5. *Let all conditions in Lemma 2 be satisfied. If $|\Pi''|^q$ is m-preinvex on $[m\tau_1, m\tau_1+\xi(\tau_2,\tau_1,m)]$ for $y>1$, then, for fractional integrals, the following inequality holds:*

$$\begin{aligned}&\Bigg|\frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1,m)]^{\varrho+1}}\left\{J^{\varrho-1}_{(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m))^-}\Pi(m\tau_1)+J^{\varrho-1}_{(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m))^+}\Pi(m\tau_1+\xi(\tau_2,\tau_1,m))\right\}\\ &-\frac{\varrho\Pi\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)}{2^{\varrho-2}[\xi(\tau_2,\tau_1,m)]^2}\Bigg|\\ \leq\ &\left(\frac{2^{-\varrho-1}}{\varrho+1}\right)^{\frac{1}{x}}\times\left\{\left[m|\,\Pi''(\tau_1)|^y\left(\frac{(\varrho+3)2^{-\varrho-2}}{(\varrho+1)(\varrho+2)}\right)+|\,\Pi''(\tau_2)|^y(\frac{2^{-\varrho-2}}{\varrho+2})\right]^{\frac{1}{q}}\right.\\ &\left.+\left[m|\,\Pi''(\tau_1)|^y(\frac{2^{-\varrho-2}}{\varrho+2})+|\,\Pi''(\tau_2)|^y\left(\frac{4-(\varrho+3)2^{-\varrho}}{4(\varrho+1)(\varrho+2)}\right)\right]^{\frac{1}{y}}\right\},\end{aligned}\tag{21}$$

where $x^{-1}+y^{-1}=1$.

Proof. From inequality (4) and Hölder's integral inequality, we have

$$\begin{aligned}&\Bigg|\frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1,m)]^{\varrho+1}}\left\{J^{\varrho-1}_{(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m))^-}\Pi(m\tau_1)+J^{\varrho-1}_{(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m))^+}\Pi(m\tau_1+\xi(\tau_2,\tau_1,m))\right\}\\ &-\frac{\varrho\Pi\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)}{2^{\varrho-2}[\xi(\tau_2,\tau_1,m)]^2}\Bigg|\\ \leq\ &\int_0^{\frac{1}{2}}\iota^{\varrho}|\Pi''(m\tau_1+\iota\,\xi(\tau_2,\tau_1,m))|\,d\iota+\int_{\frac{1}{2}}^1(1-\iota)^{\varrho}\,|\Pi''(m\tau_1+\iota\,\xi(\tau_2,\tau_1,m))|\,d\iota\\ \leq\ &\left(\int_0^{\frac{1}{2}}\iota^{\varrho}\,d\iota\right)^{\frac{1}{x}}\left(\int_0^{\frac{1}{2}}\iota^{\varrho}|\Pi''(m\tau_1+\iota\,\xi(\tau_2,\tau_1,m))|^y\,d\iota\right)^{\frac{1}{y}}\\ &+\left(\int_0^{\frac{1}{2}}(1-\iota)^{\varrho}\,d\iota\right)^{\frac{1}{x}}\left(\int_0^{\frac{1}{2}}(1-\iota)^{\varrho}\,|\Pi''(m\tau_1+\iota\,\xi(\tau_2,\tau_1,m))|^y\,d\iota\right)^{\frac{1}{y}}.\end{aligned}\tag{22}$$

Since $|\Pi''|^q$ is m-preinvex function on $[m\tau_1, m\tau_1+\xi(\tau_2,\tau_1,m)]$, we have

$$\int_0^{\frac{1}{2}}\iota^{\varrho}|\Pi''(m\tau_1+\iota\,\xi(\tau_2,\tau_1,m))|^y\,d\iota\leq\int_0^{\frac{1}{2}}\iota^{\varrho}\left\{m(1-\iota)|\,\Pi''(\tau_1)|^y+\iota\,|\,\Pi''(\tau_2)|^y\right\}d\iota\tag{23}$$

$$\leq\ m|\,\Pi''(\tau_1)|^y\left(\frac{(\varrho+3)2^{-\varrho-2}}{(\varrho+1)(\varrho+2)}\right)+|\,\Pi''(\tau_2)|^y\left(\frac{2^{-\varrho-2}}{\varrho+2}\right)$$

and

$$\int_{\frac{1}{2}}^{1}(1-\iota)^{\varrho}\big|\Pi''(m\tau_1+\iota\,\xi(\tau_2,\tau_1,m))\big|^y\ d\iota\leq\int_{\frac{1}{2}}^{1}(1-\iota)^{\varrho}\left\{m(1-\iota)|\,\Pi''(\tau_1)|^y+\iota\,|\,\Pi''(\tau_2)|^y\right\}d\iota$$
$$\leq\ m|\,\Pi''(\tau_1)|^y\left(\frac{2^{-\varrho-2}}{\varrho+2}\right)+|\,\Pi''(\tau_2)|^y\left(\frac{4-(\varrho+3)2^{-\varrho}}{4(\varrho+1)(\varrho+2)}\right).\tag{24}$$

Using Equations (23) and (24) in (22) and obtaining the result of (21) completes the proof. □

Corollary 15. *In inequality (21), if we take $\xi(\tau_2,\tau_1)=\tau_2-m\tau_1$, then*

$$\Bigg|\frac{\Gamma(\varrho+1)}{(\tau_2-m\tau_1)^{\varrho+1}}\left\{J^{\varrho-1}_{\left(\frac{m\tau_1+\tau_2}{2}\right)^-}\Pi(m\tau_1)+J^{\varrho-1}_{\left(\frac{m\tau_1+\tau_2}{2}\right)^+}\Pi(\tau_2)\right\}$$
$$-\frac{\varrho\Pi\left(\frac{m\tau_1+\tau_2}{2}\right)}{2^{\varrho-2}(\tau_2-m\tau_1)^2}\Bigg|$$
$$\leq\left(\frac{2^{-\varrho-1}}{\varrho+1}\right)^{\frac{1}{x}}\times\left\{\left[m|\,\Pi''(\tau_1)|^y\left(\frac{(\varrho+3)2^{-\varrho-2}}{(\varrho+1)(\varrho+2)}\right)+|\,\Pi''(\tau_2)|^y(\frac{2^{-\varrho-2}}{\varrho+2})\right]^{\frac{1}{y}}\right.$$
$$\left.+\left[m|\,\Pi''(\tau_1)|^y(\frac{2^{-\varrho-2}}{\varrho+2})+|\,\Pi''(\tau_2)|^y\left(\frac{4-(\varrho+3)2^{-\varrho}}{4(\varrho+1)(\varrho+2)}\right)\right]^{\frac{1}{y}}\right\}.$$

Corollary 16. *In inequality (21), if we take $\xi(\tau_2,\tau_1)=\tau_2-m\tau_1$ and $m=1$, then*

$$\Bigg|\frac{\Gamma(\varrho+1)}{(\tau_2-\tau_1)^{\varrho+1}}\left\{J^{\varrho-1}_{\left(\frac{\tau_1+\tau_2}{2}\right)^-}\Pi(\tau_1)+J^{\varrho-1}_{\left(\frac{\tau_1+\tau_2}{2}\right)^+}\Pi(\tau_2)\right\}-\frac{\varrho\Pi\left(\frac{\tau_1+\tau_2}{2}\right)}{2^{\varrho-2}(\tau_2-\tau_1)^2}\Bigg|$$
$$\leq\left(\frac{2^{-\varrho-1}}{\varrho+1}\right)^{\frac{1}{x}}\times\left\{\left[|\,\Pi''(\tau_1)|^y\left(\frac{(\varrho+3)2^{-\varrho-2}}{(\varrho+1)(\varrho+2)}\right)+|\,\Pi''(\tau_2)|^y(\frac{2^{-\varrho-2}}{\varrho+2})\right]^{\frac{1}{y}}\right.$$
$$\left.+\left[|\,\Pi''(\tau_1)|^y(\frac{2^{-\varrho-2}}{\varrho+2})+|\,\Pi''(\tau_2)|^y\left(\frac{4-(\varrho+3)2^{-\varrho}}{4(\varrho+1)(\varrho+2)}\right)\right]^{\frac{1}{y}}\right\}.$$

Corollary 17. *In inequality (21), considering that ξ meets the extended condition C and using inequality (4), we obtain*

$$\Bigg|\frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1,m)]^{\varrho+1}}\left\{J^{\varrho-1}_{(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m))^-}\Pi(\tau_1)+J^{\varrho-1}_{(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m))^+}\Pi(m\tau_1+\xi(\tau_2,\tau_1,m))\right\}$$
$$-\frac{\varrho\Pi\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)}{2^{\varrho-2}[\xi(\tau_2,\tau_1,m)]^2}\Bigg|$$
$$\leq\left(\frac{2^{-\varrho-1}}{\varrho+1}\right)^{\frac{1}{x}}\times\left\{\left[m|\,\Pi''(\tau_1)|^y\left(\frac{(\varrho+3)2^{-\varrho-2}}{(\varrho+1)(\varrho+2)}\right)+|\Pi''(m\tau_1+\xi(\tau_2,\tau_1,m))|^y(\frac{2^{-\varrho-2}}{\varrho+2})\right]^{\frac{1}{y}}\right.$$
$$\left.+\left[m|\,\Pi''(\tau_1)|^y(\frac{2^{-\varrho-2}}{\varrho+2})+|\Pi''(m\tau_1+\xi(\tau_2,\tau_1,m))|^y\left(\frac{4-(\varrho+3)2^{-\varrho}}{4(\varrho+1)(\varrho+2)}\right)\right]^{\frac{1}{y}}\right\}.$$

Corollary 18. *In inequality (21), considering that ξ meets the extended condition C, $m = 1$ and using inequality (4), we obtain*

$$\left|\frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1)]^{\varrho+1}}\left\{J^{\varrho-1}_{\left(\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1)\right)^-}\Pi(\tau_1)+J^{\varrho-1}_{\left(\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1)\right)^+}\Pi(\tau_1+\xi(\tau_2,\tau_1))\right\}\right.$$
$$\left.-\frac{\varrho\Pi\left(\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1)\right)}{2^{\varrho-2}[\xi(\tau_2,\tau_1)]^2}\right|$$
$$\leq \left(\frac{2^{-\varrho-1}}{\varrho+1}\right)^{\frac{1}{x}}\times\left\{\left[|\,\Pi''(\tau_1)|^y\left(\frac{(\varrho+3)2^{-\varrho-2}}{(\varrho+1)(\varrho+2)}\right)+|\Pi''(\tau_1+\xi(\tau_2,\tau_1))|^y\left(\frac{2^{-\varrho-2}}{\varrho+2}\right)\right]^{\frac{1}{y}}\right.$$
$$\left.+\left[|\,\Pi''(\tau_1)|^y\left(\frac{2^{-\varrho-2}}{\varrho+2}\right)+|\Pi''(\tau_1+\xi(\tau_2,\tau_1))|^y\left(\frac{4-(\varrho+3)2^{-\varrho}}{4(\varrho+1)(\varrho+2)}\right)\right]^{\frac{1}{y}}\right\}.$$

Theorem 6. *Let all the conditions in Lemma 2 be satisfied. If $|\Pi''|^q$ is m-preinvex function on $[m\tau_1, m\tau_1+\xi(\tau_2,\tau_1,m)]$ for $y > 1$, $y \geq r$, $s \geq 0$. Then, for fractional integrals, the following inequality is satisfied:*

$$\left|\frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1,m)]^{\varrho+1}}\left\{J^{\varrho-1}_{\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)^-}\Pi(m\tau_1)+J^{\varrho-1}_{\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)^+}\Pi(m\tau_1+\xi(\tau_2,\tau_1,m))\right\}\right.$$
$$\left.-\frac{\varrho\Pi\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)}{2^{\varrho-2}[\xi(\tau_2,\tau_1,m)]^2}\right|$$
$$\leq \left(\frac{1}{2^{\varrho\left(\frac{y-r}{y-1}\right)+1}\cdot\varrho\left(\frac{y-r}{y-1}\right)+1}\right)^{1-\frac{1}{y}}\left\{\left[m|\,\Pi''(\tau_1)|^y\left(\frac{(\varrho r+3)2^{-\varrho r-2}}{(\varrho r+1)(\varrho r+2)}\right)+|\,\Pi''(\tau_2)|^y\left(\frac{2^{-\varrho r-2}}{\varrho r+2}\right)\right]^{\frac{1}{y}}\right\}$$
$$+\left(\frac{1}{2^{\varrho\left(\frac{y-s}{y-1}\right)+1}\cdot\varrho\left(\frac{y-s}{y-1}\right)+1}\right)^{1-\frac{1}{y}}\left\{\left[m|\,\Pi''(\tau_1)|^y\left(\frac{2^{-\varrho s-2}}{\varrho s+2}\right)+|\,\Pi''(\tau_2)|^y\left(\frac{(\varrho s+3)e^{-\ln(2)\varrho s}}{4(\varrho s+1)(\varrho s+2)}\right)\right]^{\frac{1}{y}}\right\}. \tag{25}$$

where $x^{-1}+y^{-1}=1$.

Proof. From inequality (4) and Hölder's integral inequality, we have

$$\left|\frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1,m)]^{\varrho+1}}\left\{J^{\varrho-1}_{\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)^-}\Pi(m\tau_1)+J^{\varrho-1}_{\left(\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)^+}\Pi(m\tau_1+\xi(\tau_2,\tau_1,m))\right\}\right.$$
$$\left.-\frac{\varrho\Pi\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)}{2^{\varrho-2}[\xi(\tau_2,\tau_1,m)]^2}\right|$$
$$\leq\left\{\left(\int_0^{\frac{1}{2}}\iota^{\varrho\left(\frac{y-r}{y-1}\right)}\,d\iota\right)^{1-\frac{1}{y}}\left(\int_0^{\frac{1}{2}}\varrho^{\varrho r}\,|\Pi''(m\tau_1+\varrho\,\xi(\tau_2,\tau_1,m))|^y\,d\iota\right)^{\frac{1}{y}}\right.$$
$$\left.+\left(\int_{\frac{1}{2}}^1(1-\iota)^{\varrho\left(\frac{y-s}{y-1}\right)}\,d\iota\right)^{1-\frac{1}{y}}\left(\int_{\frac{1}{2}}^1(1-\varrho)^{\varrho s}|\Pi''(m\tau_1+\varrho\,\xi(\tau_2,\tau_1,m))|^y\,d\iota\right)^{\frac{1}{y}}\right\}. \tag{26}$$

Since $|\Pi''|^y$ is m-preinvex function on $[m\tau_1, m\tau_1+\xi(\tau_2,\tau_1,m)]$ we have

$$\int_0^{\frac{1}{2}}\iota^{\varrho r}\,|\Pi''(m\tau_1+\iota\,\xi(\tau_2,\tau_1,m))|^y\,d\iota\leq\int_0^{\frac{1}{2}}\iota^{\varrho r}\left\{m(1-\varrho)|\,\Pi''(\tau_1)|^y+\iota\,|\,\Pi''(\tau_2)|^y\right\}d\iota$$
$$\leq\; m|\,\Pi''(\tau_1)|^y\left(\frac{(\varrho r+3)2^{-\varrho r-2}}{(\varrho r+1)(\varrho r+2)}\right)+|\,\Pi''(\tau_2)|^y\left(\frac{2^{-\varrho r-2}}{\varrho r+2}\right) \tag{27}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}(1-\iota)^{\varrho s}\left|\Pi''(m\tau_1+\iota\,\xi(\tau_2,\tau_1,m))\right|^{y}d\iota \\
\leq\ &\int_{\frac{1}{2}}^{1}(1-\iota)^{\varrho s}\left\{m(1-\iota)|\,\Pi''(\tau_1)|^{y}+\iota\,|\,\Pi''(\tau_2)|^{y}\right\}d\iota \\
\leq\ &m|\,\Pi''(\tau_1)|^{y}\left(\frac{2^{-\varrho s-2}}{\varrho s+2}\right)+|\,\Pi''(\tau_2)|^{y}\left(\frac{(\varrho s+3)e^{-\ln(2)\varrho s}}{4(\varrho s+1)(\varrho s+2)}\right). \qquad (28)
\end{aligned}$$

Use Equations (27) and (28) in (26) and obtain (25). This ends the proof. □

Corollary 19. *In inequality (25), if we take $\xi(\tau_2,\tau_1)=\tau_2-m\tau_1$, then*

$$\begin{aligned}
&\left|\frac{\Gamma(\varrho+1)}{(\tau_2-m\tau_1)^{\varrho+1}}\left\{J^{\varrho-1}_{\left(\frac{m\tau_1+\tau_2}{2}\right)^-}\Pi(m\tau_1)+J^{\varrho-1}_{\left(\frac{m\tau_1+\tau_2}{2}\right)^+}\Pi(\tau_2)\right\}-\frac{\varrho\Pi\left(\frac{m\tau_1+\tau_2}{2}\right)}{2^{\varrho-2}(\tau_2-m\tau_1)^2}\right| \\
\leq\ &\left(\frac{1}{2^{\varrho\left(\frac{y-r}{y-1}\right)+1}.\,\varrho\left(\frac{y-r}{y-1}\right)+1}\right)^{1-\frac{1}{y}}\left\{\left[m|\,\Pi''(\tau_1)|^{y}\left(\frac{(\varrho r+3)2^{-\varrho r-2}}{(\varrho r+1)(\varrho r+2)}\right)+|\,\Pi''(\tau_2)|^{y}\left(\frac{2^{-\varrho r-2}}{\varrho r+2}\right)\right]^{\frac{1}{y}}\right\} \\
&+\left(\frac{1}{2^{\varrho\left(\frac{y-s}{y-1}\right)+1}.\,\varrho\left(\frac{y-s}{y-1}\right)+1}\right)^{1-\frac{1}{y}}\left\{\left[m|\,\Pi''(\tau_1)|^{y}\left(\frac{2^{-\varrho s-2}}{\varrho s+2}\right)+|\,\Pi''(\tau_2)|^{y}\left(\frac{(\varrho s+3)e^{-\ln(2)\varrho s}}{4(\varrho s+1)(\varrho s+2)}\right)\right]^{\frac{1}{y}}\right\}.
\end{aligned}$$

Corollary 20. *In inequality (25), if we take $\xi(\tau_2,\tau_1)=\tau_2-m\tau_1$ and $m=1$, then*

$$\begin{aligned}
&\left|\frac{\Gamma(\varrho+1)}{(\tau_2-\tau_1)^{\varrho+1}}\left\{J^{\varrho-1}_{\left(\frac{\tau_1+\tau_2}{2}\right)^-}\Pi(\tau_1)+J^{\varrho-1}_{\left(\frac{\tau_1+\tau_2}{2}\right)^+}\Pi(\tau_2)\right\}-\frac{\varrho\Pi\left(\frac{\tau_1+\tau_2}{2}\right)}{2^{\varrho-2}(\tau_2-\tau_1)^2}\right| \\
\leq\ &\left(\frac{1}{2^{\varrho\left(\frac{y-r}{y-1}\right)+1}.\,\varrho\left(\frac{y-r}{y-1}\right)+1}\right)^{1-\frac{1}{y}}\left\{\left[|\,\Pi''(\tau_1)|^{y}\left(\frac{(\varrho r+3)2^{-\varrho r-2}}{(\varrho r+1)(\varrho r+2)}\right)+|\,\Pi''(\tau_2)|^{y}\left(\frac{2^{-\varrho r-2}}{\varrho r+2}\right)\right]^{\frac{1}{y}}\right\} \\
&+\left(\frac{1}{2^{\varrho\left(\frac{y-s}{y-1}\right)+1}.\,\varrho\left(\frac{y-s}{y-1}\right)+1}\right)^{1-\frac{1}{y}}\left\{\left[|\,\Pi''(\tau_1)|^{y}\left(\frac{2^{-\varrho s-2}}{\varrho s+2}\right)+|\,\Pi''(\tau_2)|^{y}\left(\frac{(\varrho s+3)e^{-\ln(2)\varrho s}}{4(\varrho s+1)(\varrho s+2)}\right)\right]^{\frac{1}{y}}\right\}.
\end{aligned}$$

Corollary 21. *In inequality (25), considering that ξ meets the extended condition C and using inequality (4), we obtain*

$$\begin{aligned}
&\left|\frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1,m)]^{\varrho+1}}\left\{J^{\varrho-1}_{\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)^-}\Pi(m\tau_1)+J^{\varrho-1}_{\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)^+}\Pi(m\tau_1+\xi(\tau_2,\tau_1,m))\right\}\right. \\
&\left.-\frac{\varrho\Pi\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)}{2^{\varrho-2}[\xi(\tau_2,\tau_1,m)]^2}\right| \\
\leq\ &\left(\frac{1}{2^{\varrho\left(\frac{y-r}{y-1}\right)+1}.\,\varrho\left(\frac{y-r}{y-1}\right)+1}\right)^{1-\frac{1}{y}} \\
&\times\left\{\left[m|\,\Pi''(\tau_1)|^{y}\left(\frac{(\varrho r+3)2^{-\varrho r-2}}{(\varrho r+1)(\varrho r+2)}\right)+|\Pi''(m\tau_1+\xi(\tau_2,\tau_1,m))|^{y}\left(\frac{2^{-\varrho r-2}}{\varrho r+2}\right)\right]^{\frac{1}{y}}\right\} \\
&+\left(\frac{1}{2^{\varrho\left(\frac{y-s}{y-1}\right)+1}.\,\varrho\left(\frac{y-s}{y-1}\right)+1}\right)^{1-\frac{1}{y}}
\end{aligned}$$

$$\times\left\{\left[m\left|\Pi''(\tau_1)\right|^y\left(\frac{2^{-\varrho s-2}}{\varrho s+2}\right)+\left|\Pi''(m\tau_1+\xi(\tau_2,\tau_1,m))\right|^y\left(\frac{(\varrho s+3)e^{-\ln(2)\varrho s}}{4(\varrho s+1)(\varrho s+2)}\right)\right]^{\frac{1}{y}}\right\}.$$

Corollary 22. *In inequality (25), considering that ξ meets the extended condition C, $m=1$ and using inequality (4), we obtain*

$$\begin{aligned}&\Bigg|\frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1)]^{\varrho+1}}\left\{J^{\varrho-1}_{(\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1))^-}\Pi(\tau_1)+J^{\varrho-1}_{(\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1))^+}\Pi(\tau_1+\xi(\tau_2,\tau_1))\right\}\\&-\frac{\varrho\Pi\left(\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1)\right)}{2^{\varrho-2}[\xi(\tau_2,\tau_1)]^2}\Bigg|\\ \leq\ &\left(\frac{1}{2^{\varrho\left(\frac{y-r}{y-1}\right)+1}\cdot\varrho\left(\frac{y-r}{y-1}\right)+1}\right)^{1-\frac{1}{y}}\\&\times\left\{\left[\left|\Pi''(\tau_1)\right|^y\left(\frac{(\varrho r+3)2^{-\varrho r-2}}{(\varrho r+1)(\varrho r+2)}\right)+\left|\Pi''(\tau_1+\xi(\tau_2,\tau_1))\right|^y\left(\frac{2^{-\varrho r-2}}{\varrho r+2}\right)\right]^{\frac{1}{y}}\right\}\\&+\left(\frac{1}{2^{\varrho\left(\frac{y-s}{y-1}\right)+1}\cdot\varrho\left(\frac{y-s}{y-1}\right)+1}\right)^{1-\frac{1}{y}}\\&\times\left\{\left[\left|\Pi''(\tau_1)\right|^y\left(\frac{2^{-\varrho s-2}}{\varrho s+2}\right)+\left|\Pi''(\tau_1+\xi(\tau_2,\tau_1))\right|^y\left(\frac{(\varrho s+3)e^{-\ln(2)\varrho s}}{4(\varrho s+1)(\varrho s+2)}\right)\right]^{\frac{1}{y}}\right\}.\end{aligned}$$

Corollary 23. *In inequality (25), when $r=s$, we have*

$$\begin{aligned}&\Bigg|\frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1,m)]^{\varrho+1}}\left\{J^{\varrho-1}_{(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m))^-}\Pi(m\tau_1)+J^{\varrho-1}_{(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m))^+}\Pi(m\tau_1+\xi(\tau_2,\tau_1,m))\right\}\\&-\frac{\varrho\Pi\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)}{2^{\varrho-2}[\xi(\tau_2,\tau_1,m)]^2}\Bigg|\\ \leq\ &\left(\frac{1}{2^{\varrho\left(\frac{y-r}{y-1}\right)+1}\varrho\left(\frac{y-r}{y-1}\right)+1}\right)^{1-\frac{1}{y}}\times\left\{\left\{\left[m\left|\Pi''(\tau_1)\right|^y\left(\frac{(\varrho r+3)2^{-\varrho r-2}}{(\varrho r+1)(\varrho r+2)}\right)\ +\ \left|\Pi''(\tau_2)\right|^y\left(\frac{2^{-\varrho r-2}}{\varrho r+2}\right)\right]^{\frac{1}{y}}\right\}\right.\\&\left.+\left\{\left[m\left|\Pi''(\tau_1)\right|^y\left(\frac{2^{-\varrho r-2}}{\varrho r+2}\right)\ +\ \left|\Pi''(\tau_2)\right|^y\left(\frac{(\varrho r+3)e^{-\ln(2)\varrho r}}{4(\varrho r+1)(\varrho r+2)}\right)\right]^{\frac{1}{y}}\right\}\right\}.\end{aligned}$$

Corollary 24. *In inequality (25), when $r=s$ and $m=1$, we have*

$$\begin{aligned}&\Bigg|\frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1)]^{\varrho+1}}\left\{J^{\varrho-1}_{(\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1))^-}\Pi(\tau_1)+J^{\varrho-1}_{(\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1))^+}\Pi(\tau_1+\xi(\tau_2,\tau_1))\right\}\\&-\frac{\varrho\Pi\left(\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1)\right)}{2^{\varrho-2}[\xi(\tau_2,\tau_1)]^2}\Bigg|\\ \leq\ &\left(\frac{1}{2^{\varrho\left(\frac{y-r}{y-1}\right)+1}\cdot\varrho\left(\frac{y-r}{y-1}\right)+1}\right)^{1-\frac{1}{y}}\times\left\{\left\{\left[\left|\Pi''(\tau_1)\right|^y\left(\frac{(\varrho r+3)2^{-\varrho r-2}}{(\varrho r+1)(\varrho r+2)}\right)\ +\ \left|\Pi''(\tau_2)\right|^y\left(\frac{2^{-\varrho r-2}}{\varrho r+2}\right)\right]^{\frac{1}{y}}\right\}\right.\\&\left.+\left\{\left[\left|\Pi''(\tau_1)\right|^y\left(\frac{2^{-\varrho r-2}}{\varrho r+2}\right)\ +\ \left|\Pi''(\tau_2)\right|^y\left(\frac{(\varrho r+3)e^{-\ln(2)\varrho r}}{4(\varrho r+1)(\varrho r+2)}\right)\right]^{\frac{1}{y}}\right\}\right\}.\end{aligned}$$

Corollary 25. *In inequality (25), when $r = 0 = s$, we have*

$$\left|\frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1,m)]^{\varrho+1}}\left\{J^{\varrho-1}_{\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)^-}\Pi(m\tau_1)+J^{\varrho-1}_{\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)^+}\Pi(m\tau_1+\xi(\tau_2,\tau_1,m))\right\}\right.$$
$$\left.-\frac{\varrho\Pi\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)}{2^{\varrho-2}[\xi(\tau_2,\tau_1,m)]^2}\right|$$
$$\leq \left(\frac{1}{2^{\varrho\left(\frac{y}{y-1}\right)+1}.\,\varrho\left(\frac{y}{y-1}\right)+1}\right)^{1-\frac{1}{y}}$$
$$\times\left\{\left(m|\Pi''(\tau_1)|^y\frac{3}{8}+|\Pi''(\tau_2)|^y\frac{1}{8}\right)^{\frac{1}{y}}+\left(m|\Pi''(\tau_1)|^y\frac{1}{8}+|\Pi''(\tau_2)|^y\frac{3}{8}\right)^{\frac{1}{y}}\right\}.$$

Corollary 26. *In inequality (25), when $r = 0 = s$ and $m = 1$, we have*

$$\left|\frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1)]^{\varrho+1}}\left\{J^{\varrho-1}_{\left(\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1)\right)^-}\Pi(\tau_1)+J^{\varrho-1}_{\left(\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1)\right)^+}\Pi(\tau_1+\xi(\tau_2,\tau_1))\right\}\right.$$
$$\left.-\frac{\varrho\Pi\left(\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1)\right)}{2^{\varrho-2}[\xi(\tau_2,\tau_1)]^2}\right|$$
$$\leq \left(\frac{1}{2^{\varrho\left(\frac{y}{y-1}\right)+1}.\,\varrho\left(\frac{y}{y-1}\right)+1}\right)^{1-\frac{1}{y}}$$
$$\times\left\{\left(|\Pi''(\tau_1)|^y\frac{3}{8}+|\Pi''(\tau_2)|^y\frac{1}{8}\right)^{\frac{1}{y}}+\left(|\Pi''(\tau_1)|^y\frac{1}{8}+|\Pi''(\tau_2)|^y\frac{3}{8}\right)^{\frac{1}{y}}\right\}.$$

Corollary 27. *In inequality (25), when $r = s = y$, we have*

$$\left|\frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1)]^{\varrho+1}}\left\{J^{\varrho-1}_{\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)^-}\Pi(m\tau_1)+J^{\varrho-1}_{\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)^+}\Pi(m\tau_1+\xi(\tau_2,\tau_1,m))\right\}\right.$$
$$\left.-\frac{\varrho\Pi\left(m\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1,m)\right)}{2^{\varrho-2}[\xi(\tau_2,\tau_1,m)]^2}\right|$$
$$\leq \left(\frac{1}{2}\right)^{1-\frac{1}{y}}\times\left\{\left(m|\Pi''(\tau_1)|^y\left(\frac{(\varrho y+3)2^{-\varrho y-2}}{(\varrho y+1)(\varrho y+2)}\right)+|\Pi''(\tau_2)|^y\left(\frac{2^{-\varrho y-2}}{(\varrho y+2)}\right)\right)^{\frac{1}{y}}\right.$$
$$\left.+\left(m|\Pi''(\tau_1)|^y\left(\frac{2^{-\varrho y-2}}{(\varrho y+2)}\right)+|\Pi''(\tau_2)|^y\left(\frac{(\varrho y+3)e^{-\ln(2)\varrho y}}{4(\varrho y+1)(\varrho y+2)}\right)\right)^{\frac{1}{y}}\right\}.$$

Corollary 28. *In inequality (25), when $r = s = y$ and $m = 1$, we have*

$$\left|\frac{\Gamma(\varrho+1)}{[\xi(\tau_2,\tau_1)]^{\varrho+1}}\left\{J^{\varrho-1}_{\left(\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1)\right)^-}\Pi(\tau_1)+J^{\varrho-1}_{\left(\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1)\right)^+}\Pi(\tau_1+\xi(\tau_2,\tau_1))\right\}\right.$$
$$\left.-\frac{\varrho\Pi\left(\tau_1+\frac{1}{2}\xi(\tau_2,\tau_1)\right)}{2^{\varrho-2}[\xi(\tau_2,\tau_1)]^2}\right|$$
$$\leq \left(\frac{1}{2}\right)^{1-\frac{1}{y}}\times\left\{\left(|\Pi''(\tau_1)|^y\left(\frac{(\varrho y+3)2^{-\varrho y-2}}{(\varrho y+1)(\varrho y+2)}\right)+|\Pi''(\tau_2)|^y\left(\frac{2^{-\varrho y-2}}{(\varrho y+2)}\right)\right)^{\frac{1}{y}}\right.$$

$$+\left(\left|\Pi''(\tau_1)\right|^y\left(\frac{2^{-\varrho y-2}}{(\varrho y+2)}\right)+\left|\Pi''(\tau_2)\right|^y\left(\frac{(\varrho y+3)e^{-\ln(2)\varrho y}}{4(\varrho y+1)(\varrho y+2)}\right)\right)^{\frac{1}{y}}\Bigg\}.$$

4. Applications to Some Special Functions

4.1. q-Digamma Function

Let $0<\mathbf{q}<1$, the mathematically $\mathbf{q}$-digamma function $\varphi_{\mathbf{q}}$ (see [37,38]), which is given as:

$$\begin{aligned}\varphi_{\mathbf{q}} &= -\ln(1-\mathbf{q})+\ln\mathbf{q}\sum_{k=0}^{\infty}\frac{\mathbf{q}^{k+\zeta}}{1-\mathbf{q}^{k+\zeta}}\\ &= -\ln(1-\mathbf{q})+\ln\mathbf{q}\sum_{k=0}^{\infty}\frac{\mathbf{q}^{k\zeta}}{1-\mathbf{q}^{k\zeta}}.\end{aligned}$$

For $\mathbf{q}>1$ and $\zeta>0$, $\mathbf{q}$-digamma function $\varphi_{\mathbf{q}}$ can be given as:

$$\begin{aligned}\varphi_{\mathbf{q}} &= -\ln(\mathbf{q}-1)+\ln\mathbf{q}\left[\zeta-\frac{1}{2}-\sum_{k=0}^{\infty}\frac{\mathbf{q}^{-(k+\zeta)}}{1-\mathbf{q}^{-(k+\zeta)}}\right]\\ &= -\ln(\mathbf{q}-1)+\ln\mathbf{q}\left[\zeta-\frac{1}{2}-\sum_{k=0}^{\infty}\frac{\mathbf{q}^{-k\zeta}}{1-\mathbf{q}^{-k\zeta}}\right].\end{aligned}$$

Proposition 1. *Assume that $\tau_1,\tau_2\in R$ such that $0<\tau_1<\tau_2$ and $0<q<1$. Then:*

$$\left|\frac{1}{\tau_2-\tau_1}\int_{\tau_1}^{\tau_2}\varphi_q(\varepsilon)\,d\varepsilon-\varphi_q\left(\frac{\tau_1+\tau_2}{2}\right)\right|\leq\left(\frac{\tau_2-\tau_1}{8}\right)\left\{\left(\frac{\left|\varphi_q^{(1)}(\tau_1)\right|^y+2\left|\varphi_q^{(1)}(\tau_2)\right|^y}{3}\right)^{\frac{1}{y}}+\left(\frac{2\left|\varphi_q^{(1)}(\tau_1)\right|^y+\left|\varphi_q^{(1)}(\tau_2)\right|^y}{3}\right)^{\frac{1}{y}}\right\}. \quad (29)$$

Proof. The assertion can be obtained immediately by inequality (18), when $m=1$, $\Pi(\varepsilon)=\varphi_{\mathbf{q}}(\varepsilon)$ and $\varepsilon>0$, since $\Pi'(\varepsilon)=\varphi'_{\mathbf{q}}(\varepsilon)$ is convex on $(0,+\infty)$. □

4.2. Modified Bessel Function

This section contains multiple uses related to the prediction of a few special functions, specifically modified Bessel functions. Such functions can be observed in statistical mechanics, non-uniform beams, transmission line studies, and statistical treatment of relativistic gas. First, we add the mathematical form of modified Bessel function $\Im_\rho$, in the first sense, which is given by (see [37], p. 77)

$$\Im_\rho(\zeta)=\Sigma_{n\geq 0}\frac{\left(\frac{\zeta}{2}\right)^{\rho+2n}}{n!\Gamma(\rho+n+1)}.$$

where $\zeta\in\mathbb{R}$ and $\rho>-1$, while the mathematical form of modified Bessel function $\mathfrak{K}_\rho$ in the second sense (see [37], p. 78) is usually explored as

$$\mathfrak{K}_\rho(\zeta)=\frac{\pi}{2}\,\frac{\Im_{-\rho}(\zeta)-\Im_\rho(\zeta)}{\sin\rho\pi}.$$

Consider the function $\Omega_\rho(\zeta):\mathbb{R}\to[1,\infty)$ defined by

$$\Omega_\rho(\zeta)=2^\rho\Gamma(\rho+1)\zeta^{-\rho}\Im_\rho(\zeta).$$

The first order derivative formula of $\Omega_\rho(\zeta)$ is given by [37]:

$$\Omega'_\rho(\zeta) = \frac{\zeta}{2(\rho+1)}\Omega_{\rho+1}(\zeta), \tag{30}$$

and the second derivative can be attained easily from (30) to be

$$\Omega''_\rho(\zeta) = \frac{\zeta^2\Omega_{\rho+2}(\zeta)}{4(\rho+1)(\rho+2)} + \frac{\Omega_{\rho+1}(\zeta)}{2(\rho+1)}. \tag{31}$$

Proposition 2. *Suppose that $\rho > -1$ and $0 < \tau_1 < \tau_2$. Then, we have*

$$\begin{aligned}
&\left|\frac{\tau_1+\tau_2}{4(\rho+1)}\Omega_{\rho+1}\left(\frac{\tau_1+\tau_2}{8}\right) - \frac{\Omega_\rho(\tau_2)-\Omega_\rho(\tau_1)}{\tau_2-\tau_1}\right| \\
&\leq \frac{\tau_2-\tau_1}{8}\left[\frac{1}{3}\left\{\left(\frac{\tau_1^2\Omega_{\rho+2}(\tau_1)}{4(\rho+1)(\rho+2)} + \frac{\Omega_{\rho+1}(\tau_1)}{2(\rho+1)}\right)^q + 2\left(\frac{\tau_2^2\Omega_{\rho+2}(\tau_2)}{4(\rho+1)(\rho+2)} + \frac{\Omega_{\rho+1}(\tau_2)}{2(\rho+1)}\right)^q\right\}^{\frac{1}{q}}\right. \\
&\left.+\frac{1}{3}\left\{2\left(\frac{\tau_1^2\Omega_{\rho+2}(\tau_1)}{4(\rho+1)(\rho+2)} + \frac{\Omega_{\rho+1}(\tau_1)}{2(\rho+1)}\right)^q + \left(\frac{\tau_2^2\Omega_{\rho+2}(\tau_2)}{4(\rho+1)(\rho+2)} + \frac{\Omega_{\rho+1}(\tau_2)}{2(\rho+1)}\right)^q\right\}^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. Applying the inequality (18) to the mapping $\Pi(\zeta) = \Omega'_\rho(\zeta)$, $\zeta > 0$, $m = 1$ and the identities (30) and (31) we have the result. (Note that all assumptions are satisfied). □

5. Conclusions

The work on integral inequalities associated with fractional operators has proven to be an abundant source of inspiration for numerous researchers in a variety of fields. Improvements and generalizations achieved with the concept of preinvexity result in better and sharper bounds when compared to convex functions. First, in this work, we established a few fractional identities. Employing these new notations and identities, we derived some Hermite–Hadamard-type inequalities applicable to the R-L fractional integrals. Furthermore, various examples are provided to demonstrate the accuracy of the results. With the help of power mean and Hölder inequality, we derived the generalizations of H-H inequality that brought the work more aesthetic appeal. Our findings provide improvements and modifications to prior investigations, encouraging additional investigation.

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