Mathematical and Computational Applications

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Article

Asymptotic Consideration of Rayleigh Waves on a Coated Orthorhombic Elastic Half-Space Reinforced Using an Elastic Winkler Foundation

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Abstract: This article derives approximate formulations for Rayleigh waves on a coated orthorhombic elastic half-space with a prescribed vertical load acting as an elastic Winkler foundation. In addition, perfect continuity conditions are imposed between the coating layer and the substrate, while suitable decaying conditions are slated along the infinite depth of the half-space. The effect of the thin layer is modeled using appropriate effective boundary conditions within the long-wave limit. By applying the Radon transform and using the perturbation method, the derived model successfully captures the physical characteristics of elastic surface waves in coated half-spaces. The model consists of a pesudo-static elliptic equation decaying over the interior of the half-space and a singularly perturbed hyperbolic equation with a pseudo-differential operator. The pseudo-differential equation gives the approximate dispersion of surface waves on the coated half-space structure and is analyzed numerically at the end.

Keywords: Rayleigh waves; coated media; orthorhombic half-space; effective boundary conditions; asymptotic formulation

Citation: Mubaraki, A.M. Asymptotic Consideration of Rayleigh Waves on a Coated Orthorhombic Elastic Half-Space Reinforced Using an Elastic Winkler Foundation. *Math. Comput. Appl.* **2023**, *28*, 109. https://doi.org/10.3390/mca28060109

Academic Editor: Nicholas Fantuzzi

Received: 6 June 2023
Revised: 12 November 2023
Accepted: 13 November 2023
Published: 15 November 2023

1. Introduction

Rayleigh surface waves are a known type of seismic wave described by Lord Rayleigh [1] that propagates along the surface of elastic media like the Earth's crust; some of the developments recorded with regards to Rayleigh surface waves can be found in references [2–4] and the references appended therein. These waves are formed due to the interaction between compressional (P-waves) and shear (S-waves) waves near the surface of the Earth. In addition, when an earthquake or any other source (like volcanic activity, explosions, and even human-made sources like traffic or construction) generates seismic waves, both P-waves and S-waves are produced. However, Rayleigh waves are slower than the P-waves and S-waves, but they have longer wavelengths and are dispersive in nonhomogenous media, upon which different frequencies travel at different speeds. Furthermore, these types of waves are comprehensively studied in seismology to understand the behavior of earthquakes, evaluate the structural integrity of buildings, and aid in the exploration of subsurface geological structures [5–7], among other applications.

Now that the Rayleigh surface wave has been found to have a vast relevance in the exploration of subsurface geological structures, we therefore further dissect coated elastic media [8–10] as a particular case of these structures. In fact, coated elastic media are structures that combine elastic properties with a protective coating or layer. Elastic media, such as elastomers or polymers, are known for their ability to deform under stress and return to their original shape when the stress is removed. More so, the imposition of the additional layer as a coating to an elastic medium can serve quite a lot of functions, including aesthetic considerations, surface modification, and protection for the underlying elastic material, to mention a few [11]. In addition, coated media are very useful in our daily activities, and are found to model numerous real-life applications. For instance, in the medical industry, coating enhances the biocompatibility of elastic material, reduces friction, and further provides a sterile barrier for medical devices like hand gloves, catheters, and bandages, to state but just a few [12]. Also, the huge relevance of coated elastic structures can equally be found in the design and modeling of coated fabrics, coated elastic bands, and coated cables/wires, among others. Please refer to references [12–22] for more information on the application of such structures amidst the influence of external forces and excitations.

In particular, as the present study aims to examine the dynamic characteristics of the propagation of Rayleigh waves on an orthorhombic-coated orthorhombic-elastic-loaded elastic half-space, it then becomes imperative to explore a little about orthorhombic material [23]. Generally, orthorhombic is a crystallographic term that is used to describe a specific type of crystal structure exhibited by certain materials [24]. In an orthorhombic crystal structure, the lattice is defined by three mutually perpendicular axes of unequal lengths with angles of 90 degrees between each axis. Furthermore, various materials can have an orthorhombic crystal structure, including, for instance, minerals and certain metals. Some examples of orthorhombic minerals include aragonite, azurite, and topaz. These minerals admit distinctive physical properties due to their crystal structure, such as optical properties and cleavage planes. In addition, we mention the notable orthorhombic crystal, titanium dioxide [25], that is formed naturally as the mineral rutile. In essence, the orthorhombic crystal structure is one of several possible arrangements in crystalline materials aside from monoclinic, cubic, hexagonal, and tetragonal materials, to mention a few [26].

In this regard, the theory of surface waves is concerned with the development of hyperbolic–elliptic asymptotic models that capture the contribution of surface waves to the overall dynamic response when surface tractions were first prescribed by Kaplunov and Kossovich [27] and Kaplunov et al. [28]. Within these formulations, the Rayleigh wave propagation is described using a hyperbolic equation along the surface (specifically, a forced wave equation), with decay into the interior governed by quasi-static elliptic equations. They are derived by perturbing the inhomogeneous dynamic equations in linear elasticity around the eigen-solution, corresponding to surface waves of arbitrary profiles that were formerly examined by Sobolev [29], Friedlander [30], and Chadwick [21], among others, for the plane strain case, and recently extended, by Kiselev and Parker [31], to the 3D setup. In addition, the approach in reference [28] was later extended to a coated isotropic elastic half-space by Dai et al. [32]. Moreover, this extension then leads to elegant explicit approximate solutions for the near-resonant regimes of a moving load on an elastic half-space (see Erbas et al. [33]; Kaplunov et al. [34]), and for examining the significance of flexural-seismic meta-surfaces (see Wootton et al. [35]). A more methodical clarification of the approach could be found in works by Ege et al. [36], Kaplunov and Prikazchikov ([37,38]), and Mubaraki and Almalki [8], among others. Later on, the approximate model of Kaplunov et al. [28] was extended to the orthorhombic elastic half-plane by Nobili and Prikazchikov [24], and to elastic half-space of arbitrary anisotropy by Fu et al. [39].

However, the current manuscript intends to make use of the asymptotic approximation method [32] to explicitly derive approximate equations of motions and the resulting dispersion relation, governing the propagation of Rayleigh waves on an orthorhombic-coated orthorhombic-elastic-loaded elastic half-space. Furthermore, the prescribed vertically loaded excitation under consideration is taken to be induced using the Winkler elastic foundation [40], as an extension case to the known work in the literature (see reference [24] and the references therewith); equally, one may read reference [41] on the refinement of the Winkler–Fuss elastic foundation. Further, suitable perfect interfacial continuity conditions are imposed between the layer and substrate of elastic half-space, while decaying boundary conditions are presumed along the depth of the half-space. Furthermore, it is our aim to derive an approximate model with the help of the long-wave limit approximation to exhaustively capture the dynamic characteristics of surface waves on the examining structure. Indeed, the propagation of Rayleigh waves on such media is presided over using a perturbed singular hyperbolic equation with a pseudo-differential operator; such an equation shall be acquired in this study, incorporating all the physical assumptions imposed. In addition, the study shall analyze the derived model with regard to some special cases of material constants in elasticity. Furthermore, the novelty of the present work is the generalization of various considerations (see references [8,24], for instance) in the case of a 3D orthorhombic-coated orthorhombic-elastic-loaded elastic half-space and is further supported by the Winkler elastic foundation. In fact, looking at the six (6) elastic constants posed by an orthorhombic material [23] is enough to figure out the generality, or rather the complexity, of the present consideration.

2. Formulation of the Problem

Let us consider a thin orthorhombic layer of thickness h coated orthorhombic elastic half-space, which occupies the domain $0 < x_j < \infty$ for $j = 1, 2$ and $x_3 \geq 0$, further subject to a prescribed surface loading, which is reinforced using the Winkler elastic foundation; see Figure 1 for a schematic vision of the coated structure.

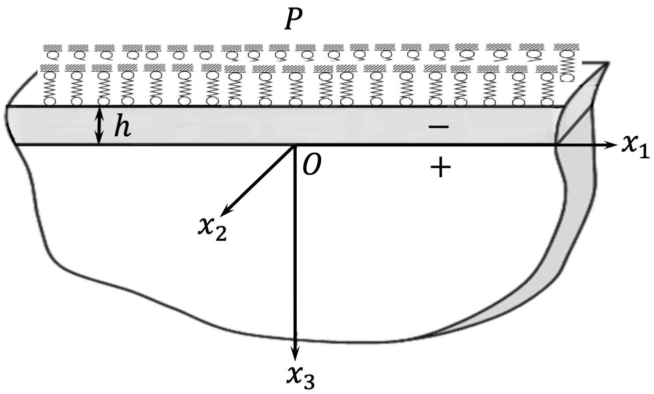

Figure 1. A coated elastic orthorhombic half-space reinforced using the Winkler elastic foundation.

The 3D equations of motion are followed by (see, e.g., Achenbach [42])

$$\begin{aligned} \sigma_{11,1}^{\mp} + \sigma_{12,2}^{\mp} + \sigma_{13,3}^{\mp} &= \rho^{\mp} u_{1,tt}^{\mp}, \\ \sigma_{21,1}^{\mp} + \sigma_{22,2}^{\mp} + \sigma_{23,3}^{\mp} &= \rho^{\mp} u_{2,tt}^{\mp}, \\ \sigma_{31,1}^{\mp} + \sigma_{32,2}^{\mp} + \sigma_{33,3}^{\mp} &= \rho^{\mp} u_{3,tt}^{\mp}, \end{aligned} \tag{1}$$

with the comma $(,)$ indicating differentiation with the corresponding variables, $u_n^{\mp} = u_n^{\mp}(x_1, x_2, x_3, t)$, and $n = 1, 2, 3$ being the plane displacements for the coating $-$ and the half-space (substrate) $+$ layers. Further, $\rho^{\mp}$ are the mass volume densities, and $\sigma_{1n}^{\mp} = \sigma_{n1}^{\mp}$, $\sigma_{2n}^{\mp} = \sigma_{n2}^{\mp}$, $\sigma_{3n}^{\mp} = \sigma_{n3}^{\mp}$ are the symmetric stress components for the orthorhombic medium, which are defined as follows:

$$\begin{aligned} \sigma_{11}^{\mp} &= c_{11}^{\mp} u_{1,1}^{\mp} + c_{12}^{\mp} u_{2,2}^{\mp} + c_{13}^{\mp} u_{3,3}^{\mp}, \qquad & \sigma_{12}^{\mp} &= c_{66}^{\mp}\left(u_{1,2}^{\mp} + u_{2,1}^{\mp}\right), \\ \sigma_{22}^{\mp} &= c_{12}^{\mp} u_{1,1}^{\mp} + c_{22}^{\mp} u_{2,2}^{\mp} + c_{23}^{\mp} u_{3,3}^{\mp}, \qquad & \sigma_{13}^{\mp} &= c_{55}^{\mp}\left(u_{1,3}^{\mp} + u_{3,1}^{\mp}\right), \\ \sigma_{33}^{\mp} &= c_{13}^{\mp} u_{1,1}^{\mp} + c_{23}^{\mp} u_{2,2}^{\mp} + c_{33}^{\mp} u_{3,3}^{\mp}, \qquad & \sigma_{23}^{\mp} &= c_{44}^{\mp}\left(u_{2,3}^{\mp} + u_{3,2}^{\mp}\right), \end{aligned} \tag{2}$$

$c_{11}^{\mp}$, $c_{12}^{\mp}$, $c_{22}^{\mp}$, $c_{13}^{\mp}$, $c_{23}^{\mp}$, $c_{33}^{\mp}$, $c_{44}^{\mp}$, $c_{55}^{\mp}$, and $c_{66}^{\mp}$ are elastic constants through Voigt notation, for the coating "$-$" and substrate "$+$" layers, respectively.

Further, upon inserting the constitutive equations for the stress-displacement relation expressed in (2) into (1), one gets the following explicit equations of motions for the governing coated half-space:

$$\begin{aligned} &c_{11}^{\mp}\,u_{1,11}^{\mp}+c_{66}^{\mp}\,u_{1,22}^{\mp}+c_{55}^{\mp}\,u_{1,33}^{\mp}+\left(c_{12}^{\mp}+c_{66}^{\mp}\right)u_{2,12}^{\mp}+\left(c_{13}^{\mp}+c_{55}^{\mp}\right)u_{3,13}^{\mp}=\rho^{\mp}\,u_{1,tt}^{\mp},\\ &c_{66}^{\mp}\,u_{2,11}^{\mp}+c_{22}^{\mp}\,u_{2,22}^{\mp}+c_{44}^{\mp}\,u_{2,33}^{\mp}+\left(c_{12}^{\mp}+c_{66}^{\mp}\right)u_{1,12}^{\mp}+\left(c_{23}^{\mp}+c_{44}^{\mp}\right)u_{3,23}^{\mp}=\rho^{\mp}\,u_{2,tt}^{\mp},\\ &c_{55}^{\mp}\,u_{3,11}^{\mp}+c_{44}^{\mp}\,u_{3,22}^{\mp}+c_{33}^{\mp}\,u_{3,33}^{\mp}+\left(c_{13}^{\mp}+c_{55}^{\mp}\right)u_{1,13}^{\mp}+\left(c_{23}^{\mp}+c_{44}^{\mp}\right)u_{2,23}^{\mp}=\rho^{\mp}\,u_{3,tt}^{\mp}. \end{aligned} \tag{3}$$

Additionally, the impulsive boundary conditions are prescribed on the surface of the coating $x_3 = -h$ as follows:

$$\sigma_{i3}^{-}=0, \qquad \text{and} \qquad \sigma_{33}^{-}=-P, \tag{4}$$

where $i = 1, 2$ and $P = P(x_1, x_2, t)$ is the prescribed vertical load, which is presumed to be induced by the Winkler elastic foundation; that is, it takes the following expression [40]:

$$P = a\,u_3^{-}, \tag{5}$$

where u_3^- is the displacement component of the coated layer, in which the load is exerted upon, while a is the stiffness of the reinforced Winkler elastic foundation.

However, the imposed perfect continuity conditions on the interface of the two layers, that is, at $x_3 = 0$, take the following expression:

$$\sigma_{n3}^{-}=\sigma_{n3}^{+}, \qquad u_n^{-}=u_n^{+}, \qquad n=1,2,3, \tag{6}$$

while the decay depth-wise boundary conditions are assumed to be $x_3 \to \infty$ as follows:

$$u_n^{+}\to 0, \qquad n=1,2,3. \tag{7}$$

Hence, the given equations of motions expressed in (3) for the propagation of Rayleigh waves on a coated orthorhombic elastic half-space will be asymptotically examined. Indeed, the prescribed impulsive boundary conditions on the surface of the coated layer, coupled with the imposed perfect continuity conditions, as expressed in (4)–(7), will be utilized for the acquisition of the resulting approximate solution, as well as the approximate equations of motions.

3. Derivation of the Effective Boundary Conditions

This section derives the required effective boundary conditions for the acquisition of the optimal approximate solution as well the approximate equations of motions for the governing formulation. Thus, we start off this approximation by suppressing the significance of the thin coated layer at the interface, that is, at $x_3 = 0$.

Here, we take into consideration the following dimensionless small parameter [34]:

$$K = kh \ll 1, \tag{8}$$

where k is the wavenumber. In fact, a very small wavenumber implies that the propagation of waves happens with a long-wave, while a very small frequency implies the propagation of waves is with a low-frequency (which is not our case). Please refer to reference [35] for related studies on the propagation of waves within a low-frequency long-wave band. Further, we assume the continuity conditions at $x_3 = 0$ to be as follows:

$$u_n^{-}=w_n^{+}, \tag{9}$$

with $w_n^+ = w_n^+(x_1, x_2, t)$ for $n = 1, 2, 3$ are the displacement components on the surface of the substrate "+".

Next, we introduce the following scaled variables:

$$\xi_i = k\,x_i, \qquad \eta = \frac{x_3}{h}, \qquad \tau^- = k\,C^-\,t, \tag{10}$$

with

$$\begin{aligned} u_n^* = k\,u_n^-, \qquad w_n^* = k\,w_n^+, \qquad a^* = \frac{1}{k\,K\,c_{44}^-}\,a, \\ \sigma_{ij}^* = \frac{1}{c_{44}^-}\,\sigma_{ij}^-, \qquad \text{and} \qquad \sigma_{n3}^* = \frac{1}{K\,c_{44}^-}\,\sigma_{n3}^-, \end{aligned} \tag{11}$$

where ξ_i for $i = 1,2$, and η are the scaled dimensionless spatial variables; τ^- is the scaled dimensionless temporal variable in the coating; a^* is the scaled dimensionless stiffness of the reinforced Winkler's foundation; u_n^* and w_n^* are scaled dimensionless displacements; σ_{ij}^* and σ_{n3}^* are scaled dimensionless stresses, all for $n = 1,2,3$, and $i \neq j = 1,2$; and C^- is the speed in the coating defined by

$$C^- = \sqrt{\frac{c_{44}^-}{\rho^-}}.$$

Indeed, the above scaling becomes imperative in order to restrain the complete dependence of the entire structure on the coating layer. Certainly, the coating layer is partially ignored, thereby utilizing its full relevance with regards to its prescribed boundary data. In this case, only the equations of motions in the substrate remain, with an infusion of the coating boundary conditions in both the substrate equations and the resulting new scaled boundary and interfacial data. Further, the equation of motions (1) and the constitutive relations expressed in (2) can then be re-expressed in terms of these new variables, as given below:

$$\begin{aligned} \sigma_{ii,\xi_i}^* + \sigma_{ji,\xi_j}^* + \sigma_{3i,\eta}^* = u_{i,\tau^-\tau^-}^*, \\ \sigma_{33,\eta}^* + K\left(\sigma_{i3,\xi_i}^* + \sigma_{j3,\xi_j}^*\right) = u_{3,\tau^-\tau^-}^*, \end{aligned} \tag{12}$$

and

$$\begin{aligned} K\,\sigma_{ii}^* &= \frac{1}{c_{44}^-}\left[c_{i3}^-\,u_{3,\eta}^* + K\left(c_{ii}^-\,u_{i,\xi_i}^* + c_{ij}^-\,u_{j,\xi_j}^*\right)\right], \\ K^2\,\sigma_{33}^* &= \frac{1}{c_{44}^-}\left[c_{33}^-\,u_{3,\eta}^* + K\left(c_{i3}^-\,u_{i,\xi_i}^* + c_{j3}^-\,u_{j,\xi_j}^*\right)\right], \\ \sigma_{ij}^* = \frac{c_{66}^-}{c_{44}^-}\left(u_{i,\xi_j}^* + u_{j,\xi_i}^*\right), &\qquad K^2\,\sigma_{31}^* = \frac{c_{55}^-}{c_{44}^-}\left(u_{1,\eta}^* + \epsilon\,u_{3,\xi_1}^*\right), \qquad K^2\,\sigma_{32}^* = u_{2,\eta}^* + \epsilon\,u_{3,\xi_2}^*. \end{aligned} \tag{13}$$

In addition, after utilizing the scaled new variables expressed above, the prescribed boundary conditions in (6) and (9) then take the following expression:

$$\begin{aligned} \sigma_{i3}^* = 0, \qquad \sigma_{33}^* = -a^*\,u_3^* \qquad \text{at} \qquad \eta = -1, \qquad \text{and} \\ u_n^* = w_n^*, \qquad \text{at} \qquad \eta = \;0. \end{aligned} \tag{14}$$

It is appropriate to express the related displacement and stress components in the following expansion form

$$\begin{pmatrix} u_n^* \\ w_n^* \\ \sigma_{mn}^* \end{pmatrix} = \begin{pmatrix} u_n^{(0)} \\ w_n^{(0)} \\ \sigma_{mn}^{(0)} \end{pmatrix} + K \begin{pmatrix} u_n^{(1)} \\ w_n^{(1)} \\ \sigma_{mn}^{(1)} \end{pmatrix} + \ldots, \qquad m,n = 1,2,3. \tag{15}$$

Therefore, upon using the above equation in (12) and (13), the following system is obtained at the leading order:

$$\begin{aligned}
&\sigma^{(0)}_{ii,\xi_i} + \sigma^{(0)}_{ji,\xi_j} + \sigma^{(0)}_{3i,\eta} = u^{(0)}_{i,\tau^-\tau^-},\\
&\sigma^{(0)}_{ij} = \frac{c^-_{66}}{c^-_{44}}\left(u^{(0)}_{i,\xi_j} + u^{(0)}_{j,\xi_i}\right),\\
&\sigma^{(0)}_{33,\eta} = u^{(0)}_{3,\tau^-\tau^-},\\
&u^*_{n,\eta} = 0,
\end{aligned} \tag{16}$$

while the corresponding boundary conditions from (14) take the following form:

$$\begin{aligned}
\sigma^{(0)}_{i3} = -a^*\, u^*_3, \quad \sigma^{(0)}_{33} = 0, \quad \text{at} \qquad \eta = -1, \quad \text{and}\\
u^{(0)}_n = w^*_n, \quad \text{at} \qquad \eta = 0.
\end{aligned} \tag{17}$$

The leading order displacement components satisfying $(16)_4$ and $(17)_2$ are then obtained in the following form:

$$u^{(0)}_n = w^*_n. \tag{18}$$

From $(16)_3$, $(17)_1$, and (18), we obtain

$$\sigma^{(0)}_{33} = (\eta + 1)\, w^*_{3,\tau^-\tau^-} - a^*\, w^*_3. \tag{19}$$

Moreover, at the next order $O(\epsilon)$, $(13)_2$ and the boundary value problem $(14)_2$ lead to the acquisition of

$$u^{(1)}_3 = \frac{1}{c^-_{33}}\left(c^-_{i3}\, u^{(0)}_{i,\xi_i} + c^-_{j3}\, u^{(0)}_{j,\xi_j}\right), \tag{20}$$

and

$$u^{(1)}_n = 0, \qquad \text{at} \qquad \eta = 0. \tag{21}$$

From (20) and the boundary conditions expressed in (21), one obtains

$$u^{(1)}_3 = -\frac{\eta}{c^-_{33}}\left(c^-_{i3}\, w^*_{i,\xi_i} + c^-_{j3}\, w^*_{j,\xi_j}\right). \tag{22}$$

By substituting (18) and (22) into $(13)_1$, we obtain

$$\sigma^{(0)}_{ii} = \frac{1}{c^-_{44}}\left[\left(c^-_{ii} - \frac{\left(c^-_{i3}\right)^2}{c^-_{33}}\right) w^*_{i,\xi_i} + \left(c^-_{ij} - \frac{c^-_{i3}\, c^-_{j3}}{c^-_{33}}\right) w^*_{j,\xi_j}\right]. \tag{23}$$

Finally, we have, from $(16)_1$, $(16)_2$, (18), and (23), at the same time satisfying $(17)_1$, the following:

$$\begin{aligned}
\sigma^{(0)}_{i3} = \frac{(1+\eta)}{c^-_{44}}\Bigg[& c^-_{44}\, w^*_{i,\tau^-\tau^-} - c^-_{66}\, w^*_{i,\xi_j\xi_j} - \left(c^-_{ii} - \frac{\left(c^-_{i3}\right)^2}{c^-_{33}}\right) w^*_{i,\xi_i\xi_i}\\
&- \left(c^-_{ii} - \frac{\left(c^-_{i3}\right)^2}{c^-_{33}}\right) w^*_{i,\xi_i\xi_i} - \left(c^-_{66} + c^-_{ij} - \frac{c^-_{i3}\, c^-_{j3}}{c^-_{33}}\right) w^*_{j,\xi_i\xi_j}\Bigg],
\end{aligned} \tag{24}$$

In the original variables, the stress components at the interface $x_3 = 0$ can then be expressed from (19) and (24) as follows:

$$\begin{aligned}
&\sigma^+_{i3} = h\left[\rho^-\, u^+_{i,tt} - c^-_{66}\, u^+_{i,jj} - \left(c^-_{ii} - \frac{\left(c^-_{i3}\right)^2}{c^-_{33}}\right) u^+_{i,ii} - \left(c^-_{66} + c^-_{ij} - \frac{c^-_{i3}\, c^-_{j3}}{c^-_{33}}\right) u^+_{j,ij}\right],\\
&\sigma^+_{33} = \rho^-\, h\, u^+_{3,tt} - a\, u^+_3.
\end{aligned} \tag{25}$$

Note that, in the absence of the effect of the Winkler elastic foundation $(a = 0)$, the conditions in (25) may obviously be affirmed to correspond to the results reported in reference [43].

4. Application of the Perturbation Technique

Now that the related effective boundary conditions are derived in (25), we then proceed to derive the resulting pseudo-differential equation for the transverse and longitudinal potentials of the governing half-space. Indeed, the model examination of the half-space $(x_3 \geq 0)$ involves the wave dynamic equations of motions expressed in (2), and subject to the derived boundary conditions in (25).

Furthermore, we consider the special case of material constants, that is, when

$$c_{11}^{\mp} = c_{22}^{\mp}, \qquad c_{13}^{\mp} = c_{23}^{\mp}, \qquad \text{and} \qquad c_{44}^{\mp} = c_{55}^{\mp}. \tag{26}$$

The above presumption, which is referred to as a pure mode, means that the displacement component is parallel everywhere to the anti-plane motion with no additional symmetry. Subsequently, it is appropriate to deploy the Radon integral transform, defined as follows [32]:

$$u_l^{(r)}(x, r, x_3, t) = \int_{-\infty}^{\infty} u_l^{+}(x\, \cos r - y \sin r,\; x\, \sin r + y \cos r, x_3, t) dy, \tag{27}$$

where

$$x = x_1\, \cos r + x_2\, \sin r, \qquad y = -x_1\, \sin r + x_2\, \cos r, \tag{28}$$

and

$$u_x^{(r)} = u_1^{(r)}\, \cos r + u_2^{(r)}\, \sin r, \qquad u_y^{(r)} = -u_1^{(r)}\, \sin r + u_2^{(r)}\, \cos r, \tag{29}$$

with $r \in \left[0, \frac{1}{2}\pi\right]$.

Now, we set $u_y^{(r)} = 0$, which means that the anti-plane dynamic motion is dissuaded by the presence of the elastic Winkler foundation. Moreover, the equations of motions expressed in (2) for the substrate $+$ are rewritten in terms of the present transformation as follows:

$$\begin{aligned} c_{11}^{+}\, u_{x,xx}^{(r)} + \beta^{+}\, u_{3,x3}^{(r)} + c_{55}^{+}\, u_{x,33}^{(r)} &= \rho^{+}\, u_{x,tt}^{(r)}, \\ c_{33}^{+}\, u_{3,33}^{(r)} + \beta^{+}\, u_{x,x3}^{(r)} + c_{55}^{+}\, u_{3,xx}^{(r)} &= \rho^{+}\, u_{3,tt}^{(r)}, \end{aligned} \tag{30}$$

and subject to

$$\begin{aligned} c_{55}^{+}\left(u_{x,3}^{(r)} + u_{3,x}^{(r)}\right) &= h\left(\rho^{-}\, u_{x,tt}^{(r)} - \delta^{-}\, u_{x,xx}^{(r)}\right), \\ c_{13}^{+}\, u_{x,x}^{(r)} + c_{33}^{+}\, u_{3,3}^{(r)} &= \rho^{-}\, h\, u_{3,tt}^{(r)} - a\, u_3^{(r)}, \end{aligned} \tag{31}$$

where

$$\beta^{+} = c_{13}^{+} + c_{55}^{+}, \qquad \text{and} \qquad \delta^{-} = c_{11}^{-} - \frac{\left(c_{13}^{-}\right)^2}{c_{33}^{-}}. \tag{32}$$

Now, let us introduce yet another scaling of the following format:

$$\xi = k(x - c_R\, t), \qquad \gamma = k\, x_3 \qquad \tau = k\, K\, c_R\, t. \tag{33}$$

where c_R is the speed of the Rayleigh wave.

Then, the transformed equations of motions expressed in (30) can now be rewritten in the latter new scaling as follows:

$$\begin{aligned}&\left(c_{11}^{+}-\rho^{+}c_R^2\right)u_{x,\xi\xi}^{(r)}+c_{55}^{+}u_{x,\gamma\gamma}^{(r)}+\beta^{+}u_{3,\xi\gamma}^{(r)}=\rho^{+}c_R^2\left(\epsilon^2u_{x,\tau\tau}^{(r)}-2\epsilon u_{x,\xi\tau}^{(r)}\right),\\&c_{33}^{+}u_{3,\gamma\gamma}^{(r)}+\beta^{+}u_{x,\xi\gamma}^{(r)}+\left(c_{55}^{+}-\rho^{+}c_R^2\right)u_{3,\xi\xi}^{(r)}=\rho^{+}c_R^2\left(\epsilon^2u_{3,\tau\tau}^{(r)}-2\epsilon c_R u_{3,\xi\tau}^{(r)}\right).\end{aligned}\tag{34}$$

Certainly, (34) can be rewritten in the form of a single partial differential equation of the fourth-order, contacted by $u_x^{(r)}$ as follows:

$$\begin{aligned}&C_1u_{x,\xi\xi\xi\xi}^{(r)}+C_2u_{x,\xi\xi\gamma\gamma}^{(r)}+C_3u_{x,\gamma\gamma\gamma\gamma}^{(r)}+K\left(D_1u_{x,\xi\xi\xi\tau}^{(r)}+D_2u_{x,\xi\gamma\gamma\tau}^{(r)}\right)\\&-K^2\left(E_1u_{x,\xi\xi\tau\tau}^{(r)}+E_2u_{x,\gamma\gamma\tau\tau}^{(r)}\right)-K^3F_1u_{x,\xi\tau\tau\tau}^{(r)}+K^4F_2u_{x,\tau\tau\tau\tau}^{(r)}=0,\end{aligned}\tag{35}$$

where the coefficients C_1, C_2, C_3, D_i, E_i, and F_i, for $i=1,2$, are specified as

$$\begin{aligned}&C_1=(c_{11}^{+}-\rho^{+}c_R^2)(c_{55}^{+}-\rho^{+}c_R^2),\quad C_2=c_{11}^{+}c_{33}^{+}+(c_{55}^{+})^2-(\beta^{+})^2-(c_{33}^{+}+c_{55}^{+})\rho^{+}c_R^2,\quad C_3=c_{33}^{+}c_{55}^{+},\\&D_1=2\rho^{+}c_R^2(c_{11}^{+}+c_{55}^{+}-2\rho^{+}c_R^2),\quad D_2=2\rho^{+}c_R^2(c_{33}^{+}+c_{55}^{+}),\quad E_1=\rho^{+}c_R^2(c_{11}^{+}+c_{55}^{+}-6\rho^{+}c_R^2),\\&E_2=\rho^{+}c_R^2(c_{33}^{+}+c_{55}^{+}),\quad F_1=4(\rho^{+})^2c_R^4,\quad\text{and}\quad F_2=(\rho^{+})^2c_R^4.\end{aligned}$$

Moreover, the boundary conditions (31) are then reformed at $\gamma=0$ as follows:

$$\begin{aligned}&u_{x,\gamma}^{(r)}+u_{3,\xi}^{(r)}=\frac{c_{55}^{-}}{c_{55}^{+}}\left[K\left(\frac{c_R^2}{c_0^2}-\frac{\delta^{-}}{c_{55}^{-}}\right)U_{x,\xi\xi}^{(r)}+\frac{c_R^2}{c_0^2}\left(K^3U_{x,\tau\tau}^{(r)}-2K^2U_{x,\xi\tau}^{(r)}\right)\right],\\&c_{13}^{+}u_{x,\xi}^{(r)}+c_{33}^{+}u_{3,\gamma}^{(r)}=\frac{c_{55}^{-}c_R^2}{c_0^2}\left[Ku_{3,\xi\xi}^{(r)}-2K^2u_{3,\xi\tau}^{(r)}+K^3u_{3,\tau\tau}^{(r)}\right]-\frac{a}{k}u_3^{(r)},\end{aligned}\tag{36}$$

where $c_0=\sqrt{c_{55}^{-}/\rho^{-}}$.

Thus, accordingly, let us now expand the displacement components $u_x^{(r)}$ and $u_3^{(r)}$ as asymptotic series as follows:

$$\begin{aligned}u_x^{(r)}&=K^{-1}U_x^{(0)}(\xi,\gamma,\tau)+U_x^{(1)}(\xi,\gamma,\tau)+\dots,\\u_3^{(r)}&=K^{-1}U_3^{(0)}(\xi,\gamma,\tau)+U_3^{(1)}(\xi,\gamma,\tau)+\dots.\end{aligned}\tag{37}$$

Then, at the leading order, $(34)_1$ becomes

$$\left(c_{11}^{+}-\rho^{+}c_R^2\right)U_{x,\xi\xi}^{(0)}+c_{55}^{+}U_{x,\gamma\gamma}^{(0)}+\beta^{+}U_{3,\xi\gamma}^{(0)}=0,\tag{38}$$

while (35) gives

$$C_1U_{x,\xi\xi\xi\xi}^{(0)}+C_2U_{x,\xi\xi\gamma\gamma}^{(0)}+C_3U_{x,\gamma\gamma\gamma\gamma}^{(0)}=0.\tag{39}$$

Undeniably, the obtained elliptic equation in (39) can alternatively be represented using an operator notation as follows:

$$\Delta_1\Delta_2U_x^{(0)}=0,\tag{40}$$

with

$$\Delta_i=\partial_{\xi\xi}^2+q_i\partial_{\gamma\gamma}^2,\qquad i=1,2,\tag{41}$$

where q_i for $i=1,2$, which is determined using

$$q_i=\sqrt{\frac{-C_2+(-1)^i\sqrt{C_2^2-4C_1C_3}}{2C_1}},\qquad i=1,2,\tag{42}$$

where $C_2^2 - 4C_1C_3 \geq 0$ for $i = 1, 2$. Indeed, this restriction allows the assumption of only real quantities, that is, $q_i \geq 0$. Therefore, the solution for (40) can be obtained with the help of a pair of plane harmonic functions as follows:

$$U_x^{(0)} = \phi^{(0)}(\xi, q_1\gamma, \tau) + \psi^{(0)}(\xi, q_2\gamma, \tau), \tag{43}$$

Then, on inserting the solution (43) into (38), amidst exploiting the application of the Cauchy–Riemann identities for the function $g(\xi, q\gamma)$, shown as

$$g_{,\gamma} = -q\,\mathcal{H}(g_{,\xi}), \qquad g_{,\xi} = \frac{1}{q}\mathcal{H}(g_{,\gamma}) \qquad \text{and} \qquad \mathcal{H}(\mathcal{H}(g)) = -g, \tag{44}$$

where $\mathcal{H}$ is Hilbert transform, then we arrive at

$$U_3^{(0)} = \alpha_1\,\mathcal{H}\left(\phi^{(0)}\right)(\xi, q_1\gamma, \tau) + \alpha_2\,\mathcal{H}\left(\psi^{(0)}\right)(\xi, q_2\gamma, \tau), \tag{45}$$

with

$$\alpha_i = \frac{\rho^+ c_R^2 - c_{11}^+ + q_i^2 c_{55}^+}{\beta^+ q_i}, \qquad i = 1, 2. \tag{46}$$

Implying (43) and (45) into leading boundary conditions (36), we deduce at the surface $\gamma = 0$ the following:

$$\begin{aligned} &(\alpha_1 - q_1)\phi_{,\xi}^{(0)} + (\alpha_2 - q_2)\psi_{,\xi}^{(0)} = 0, \\ &(c_{13}^+ + c_{33}^+\alpha_1 q_1)\phi_{,\xi}^{(0)} + (c_{13}^+ + c_{33}^+\alpha_2 q_2)\psi_{,\xi}^{(0)} = 0. \end{aligned} \tag{47}$$

Thus, the classical Rayleigh wave equation follows:

$$\text{Det}\begin{bmatrix} \alpha_1 - q_1 & \alpha_2 - q_2 \\ c_{13}^+ + c_{33}^+\alpha_1 q_1 & c_{13}^+ + c_{33}^+\alpha_2 q_2 \end{bmatrix} = 0, \tag{48}$$

having the equivalent expression

$$\lambda = \frac{\alpha_1 - q_1}{\alpha_2 - q_2} = \frac{c_{13}^+ + c_{33}^+\alpha_1 q_1}{c_{13}^+ + c_{33}^+\alpha_2 q_2}. \tag{49}$$

Then, the elastic potentials $\psi^{(0)}$ and $\phi^{(0)}$ can easily be related to each other as follows:

$$\psi^{(0)} = -\lambda\,\phi^{(0)} \qquad \text{at} \qquad \gamma = 0. \tag{50}$$

Therefore, the solution obtained in (45) may be expressed in terms of only one potential function $\psi^{(0)}$ or $\phi^{(0)}$ as follows:

$$U_3^{(0)} = (\alpha_1 - \alpha_2\lambda)\,\mathcal{H}\left(\phi^{(0)}\right)(\xi, 0, \tau) = \frac{1}{\lambda}(\lambda\alpha_2 - \alpha_1)\,\mathcal{H}\left(\psi^{(0)}\right)(\xi, 0, \tau). \tag{51}$$

Furthermore, upon going further to the next order, $(34)_1$ and (35) then take the following expressions:

$$\left(c_{11}^+ - \rho^+ c_R^2\right)U_{x,\xi\xi}^{(1)} + c_{55}^+ U_{x,\gamma\gamma}^{(1)} + \beta^+ U_{3,\xi\gamma}^{(1)} = -2\rho^+ c_R^2\, U_{x,\xi\tau}^{(0)}, \tag{52}$$

and

$$C_3\,\Delta_1\,\Delta_2\,U_x^{(1)} = -2\rho^+ c_R^2\left[\left(c_{11}^+ + c_{55}^+ - 2\rho^+ c_R^2\right)U_{x,\xi\xi\xi\tau}^{(0)} + (c_{33}^+ + c_{55}^+)U_{x,\xi\gamma\gamma\tau}^{(0)}\right]. \tag{53}$$

The general solutions for $U_x^{(1)}$ and $U_3^{(1)}$ are obtained in a similar manner to reference [24] as follows:

$$U_x^{(1)}(\xi,\gamma,\tau) = \phi^{(1)}(\xi,q_1\gamma,\tau) + \psi^{(1)}(\xi,q_2\gamma,\tau) + \frac{\gamma}{2C_3(q_2^2-q_1^2)}\left[\frac{\vartheta_1}{q_1}\bar{\phi}^{(0)}_{,\tau} - \frac{\vartheta_2}{q_2}\mathcal{H}\left(\psi^{(0)}_{,\tau}\right)\right], \tag{54}$$

$$\begin{aligned} U^{(1)}_{3,\gamma}(\xi,\gamma,\tau) &= \alpha_1 q_1 \phi^{(1)}_{,\xi}(\xi,q_1\gamma,\tau) + \alpha_2 q_2 \psi^{(1)}_{,\xi}(\xi,q_2\gamma,\tau) \\ &- \frac{1}{\beta^+}\left[2\rho^+ c_R^2 + \frac{\vartheta_1}{c^+_{3333}(q_2^2-q_1^2)}\right]\phi^{(0)}_{,\tau} - \frac{1}{\beta^+}\left[2\rho^+ c_R^2 + \frac{\vartheta_2}{c^+_{3333}(q_1^2-q_2^2)}\right]\psi^{(0)}_{,\tau} \\ &+ \frac{\gamma}{2C_3(q_2^2-q_1^2)}\left[\vartheta_1\alpha_1\mathcal{H}\left(\phi^{(0)}_{,\xi\tau}\right) - \vartheta_2\alpha_2\mathcal{H}\left(\psi^{(0)}_{,\xi\tau}\right)\right], \end{aligned} \tag{55}$$

and

$$\begin{aligned} U^{(1)}_{3,\xi}(\xi,\gamma,\tau) &= \alpha_1\mathcal{H}\left(\phi^{(1)}_{,\xi}\right) + \alpha_2\mathcal{H}\left(\psi^{(1)}_{,\xi}\right) - \frac{1}{q_1\beta^+}\left[2\rho^+ c_R^2 + \frac{\vartheta_1(2c^+_{55}q_1 - \alpha_1\beta^+)}{2C_2 q_1(q_2^2-q_1^2)}\right]\bar{\phi}^{(0)}_{,\tau} \\ &- \frac{1}{q_2\beta^+}\left[2\rho^+ c_R^2 + \frac{\vartheta_2(2c^+_{55}q_2 - \alpha_2\beta^+)}{2C_2 q_2(q_1^2-q_2^2)}\right]\mathcal{H}\left(\psi^{(0)}_{,\tau}\right) + \frac{\gamma}{2C_2(q_2^2-q_1^2)}\left[\frac{\vartheta_2\alpha_2}{q_2}\psi^{(0)}_{,\xi\tau} - \frac{\vartheta_1\alpha_1}{q_1}\phi^{(0)}_{,\xi\tau}\right], \end{aligned} \tag{56}$$

where

$$\vartheta_i = -2\rho^+ c_R^2\left[c^+_{11} + c^+_{55} - (c^+_{33}+c^+_{55})q_i^2 - 2\rho^+ c_R^2\right]. \tag{57}$$

At order $O(1)$, the boundary conditions (36) lead to

$$\begin{aligned} &U^{(1)}_{x,\gamma} + U^{(1)}_{3,\xi} = \frac{c^-_{55}}{c^+_{55}}\left(\frac{c_R^2}{c_0^2} - \frac{\delta^-}{c^-_{55}}\right)U^{(1)}_{x,\xi\xi}, \\ &c^+_{13}U^{(1)}_{x,\xi} + c^+_{33}U^{(1)}_{3,\gamma} = \frac{c^-_{55}c_R^2}{c_0^2}U^{(0)}_{3,\xi\xi} - \frac{a}{k}U^{(0)}_3, \qquad \text{at} \qquad \gamma = 0. \end{aligned} \tag{58}$$

Now, upon putting the solutions acquired in (43), (45), and (54)–(56) into (58), and via the application of the Cauchy–Riemann relations earlier expressed in (44), one obtains

$$\begin{aligned} &(\alpha_1-q_1)\mathcal{H}\left(\phi^{(1)}_{,\xi}\right) + (\alpha_2-q_2)\mathcal{H}\left(\psi^{(1)}_{,\xi}\right) - (\alpha_1-q_1)\left(\delta_{11}\mathcal{H}\left(\phi^{(0)}_{,\tau}\right) + \delta_{12}\mathcal{H}\left(\psi^{(0)}_{,\tau}\right)\right) \\ &\qquad - \frac{c^-_{55}}{c^+_{55}}\left(\frac{c_R^2}{c_0^2} - \frac{\delta^-}{c^-_{1313}}\right)\left(\phi^{(0)}_{,\xi\xi} + \psi^{(0)}_{,\xi\xi}\right) = 0, \\ &(c^+_{13} + c^+_{33}\alpha_1 q_1)\phi^{(1)}_{,\xi} + (c^+_{13} + c^+_{33}\alpha_2 q_2)\psi^{(1)}_{,\xi} - (c^+_{13} + c^+_{33}\alpha_1 q_1)\left(\delta_{21}\phi^{(0)}_{,\tau} + \delta_{22}\psi^{(0)}_{,\tau}\right) \\ &- \frac{c^-_{55}c_R^2}{c_0^2}\left(\alpha_1\mathcal{H}\left(\phi^{(0)}_{,\xi\xi}\right) + \alpha_2\mathcal{H}\left(\psi^{(0)}_{,\xi\xi}\right)\right) = -\frac{a}{k}(\alpha_1 - \alpha_2\lambda)\mathcal{H}\left(\phi^{(0)}\right), \qquad \text{at} \qquad \gamma = 0, \end{aligned} \tag{59}$$

where

$$\begin{aligned} (\alpha_1-q_1)\delta_{1i} &= \frac{1}{q_i\beta^+}\left[2\rho^+ c_R^2 + \vartheta_j\frac{(\alpha_j\beta^+ + (\beta^+ - 2c^+_{55})q_i)}{2C_2 q_i\left(q_i^2-q_j^2\right)}\right], \\ \text{and} \quad (c^+_{13} + c^+_{33}\alpha_1 q_1)\delta_{2i} &= \frac{1}{\beta^+}\left[2\rho^+ c^+_{33}c_R^2 - \frac{\vartheta_i}{\left(q_i^2-q_j^2\right)}\right]. \end{aligned} \tag{60}$$

Using (50) and implicit differentiation with respect to ξ, we arrive at

$$\left[\frac{(c_{13}^{+}+c_{33}^{+}\,\alpha_2\,q_2)}{(c_{13}^{+}+c_{33}^{+}\,\alpha_1\,q_1)}-\frac{(\alpha_2-q_2)}{(\alpha_1-q_1)}\right]\psi^{(1)}_{,\xi\xi}-[(\delta_{21}-\delta_{11})-\lambda(\delta_{22}-\delta_{12})]\phi^{(0)}_{,\xi\tau}$$
$$-\left[\frac{(\rho^{-}\,c_R^2-\delta^{-})(1-\lambda)}{c_{55}^{+}(\alpha_1-q_1)}+\frac{\rho^{-}\,c_R^2(\alpha_1-\lambda\,\alpha_2)}{(c_{13}^{+}+c_{33}^{+}\,\alpha_1\,q_1)}\right]\mathcal{H}\left(\phi^{(0)}_{,\xi\xi\xi}\right) \tag{61}$$
$$=-\frac{a(\alpha_1-\alpha_2\,\lambda)\,\mathcal{H}\left(\phi^{(0)}_{,\xi}\right)}{k(c_{13}^{+}+c_{33}^{+}\,\alpha_1\,q_1)},\quad \text{at}\quad \gamma=0.$$

By simplifying this formula, we obtain

$$2\,\phi^{(0)}_{,\xi\tau}+\frac{b}{q_1}\,\phi^{(0)}_{,\xi\xi\gamma}=\frac{a(\alpha_1-\alpha_2\,\lambda)\,\phi^{(0)}_{,\gamma}}{k\,q_1(c_{13}^{+}+c_{33}^{+}\,\alpha_1\,q_1)},\quad \text{at}\quad \gamma=0, \tag{62}$$

where the constant b inherits the properties of both coating and substrate, given explicitly by

$$b=\frac{1}{B}\left[\frac{(\rho^{-}\,c_R^2-\delta^{-})(1-\lambda)}{c_{55}^{+}(\alpha_1-q_1)}+\frac{\rho^{-}\,c_R^2(\alpha_1-\lambda\,\alpha_2)}{(c_{13}^{+}+c_{33}^{+}\alpha_1\,q_1)}\right], \tag{63}$$

which takes both positive and negative values corresponding to the local minimum and maximum of the phase velocity equal to the Rayleigh wave speed, while the constant B contains properties of the substrate only takes the following form:

$$B=-\frac{1}{2}[\delta_{21}-\delta_{11}-\lambda(\delta_{22}-\delta_{12})]. \tag{64}$$

Then, on re-expressing (62) in the form of the original variables (x, x_3, t), one obtains

$$\phi^{(r)}_{,xx}-\frac{1}{c_R^2}\,\phi^{(r)}_{,tt}+\frac{bh}{q_1}\phi^{(r)}_{,xx3}=\frac{a(\alpha_1-\alpha_2\,\lambda)\,\phi^{(0)}_{,3}}{B\,q_1(c_{13}^{+}+c_{33}^{+}\alpha_1\,q_1)},\quad \text{at}\quad x_3=0. \tag{65}$$

In addition, the elliptic equation for the potentials $\phi^{(r)}$ and $\psi^{(r)}$ are then found to be

$$\phi^{(r)}_{,xx}+q_1\,\phi^{(r)}_{,33}=0,\qquad \text{and}\qquad \psi^{(r)}_{,xx}+q_2\,\psi^{(r)}_{,33}=0. \tag{66}$$

Moreover, let us now introduce their respective inverse transforms as follows:

$$\psi^{(r)}_1=\psi^{(r)}\,\cos r,\qquad \text{and}\qquad \psi^{(r)}_2=\psi^{(r)}\,\sin r.$$

Hence, (65) and (66) may be reinterpreted by inverting the respective transforms as follows:

$$\Delta\,\phi+q_1\,\phi_{,33}=0,\qquad \text{and}\qquad \Delta\,\psi_i+q_2\,\psi_{i,33}=0, \tag{67}$$

which govern the decay along the depth of the half-space $(x_3 \geq 0)$, the interior, and in addition to the boundary conditions at $x_3 = 0$ given below:

$$\Delta\,\phi-\frac{1}{c_R^2}\,\phi_{,tt}+\frac{bh}{q_1}\,\Delta\,\phi_{,3}=\frac{a(\alpha_1-\alpha_2\,\lambda)\,\phi_{,3}}{B\,q_1(c_{13}^{+}+c_{33}^{+}\alpha_1\,q_1)}, \tag{68}$$

and

$$\Delta\,\psi_i-\frac{1}{c_R^2}\,\psi_{i,tt}+\frac{bh}{q_2}\,\Delta\,\psi_{i,3}=\frac{a(\lambda\,\alpha_2-\alpha_1)\,\psi_{i,3}}{\lambda\,B\,q_2(c_{13}^{+}+c_{33}^{+}\alpha_1\,q_1)}. \tag{69}$$

Once again, let us say that Equation (68) can be presented in terms of a pseudo-differential equation on the surface $x_3 = 0$ of the coated structure as

$$\Delta\,\phi - \frac{1}{c_R^2}\,\phi_{,tt} - bh\sqrt{-\Delta}\,\Delta\,\phi = -\frac{a(\alpha_1 - \alpha_2\,\lambda)\sqrt{-\Delta}\,\phi}{B\,\left(c_{13}^{+} + c_{33}^{+}\alpha_1\,q_1\right)}, \tag{70}$$

where $\sqrt{-\Delta}$ is the pseudo-differential operator (for more details, see reference [32]).

Therefore, (70) reduces to the plane strain problem (x_1, x_3, t); that is, it takes the following form:

$$\phi_{,11} - \frac{1}{c_R^2}\,\phi_{,tt} - bh\sqrt{-\partial_{11}^2}\,\phi_{,11} = -\frac{a(\alpha_1 - \alpha_2\,\lambda)\,\sqrt{-\partial_{11}^2}\,\phi}{B\left(c_{13}^{+} + c_{33}^{+}\alpha_1\,q_1\right)} \qquad \text{at} \qquad x_3 = 0. \tag{71}$$

Certainly, Equation (71) leads to the acquisition of the resulting approximate dispersion by using the solution $\phi(x_1, 0, t) = f(0)\,e^{ik(x_1 - ct)}$, where c is the phase velocity, and $f(0)$ is an arbitrary function, which further results in obtaining

$$\frac{c}{c_R} = \sqrt{1 - b\,K + \frac{a\,h(\alpha_2\,\lambda - \alpha_1)}{KB\left(c_{13}^{+} + c_{33}^{+}\alpha_1\,q_1\right)}}, \tag{72}$$

with c_R in the latter equation equally denoting the speed of the Rayleigh wave.

5. Model Verification

In the case of the absence of the coating layer and the Winkler elastic foundation, that is, when $h = 0$ and $a = 0$, then (71) may be identical to the hyperbolic Equation (38) in reference [24]. Moreover, for an isotropic material case, the specified elastic constants $c_{mn}^{\mp}$ $(m = n = 1, 2, 3)$ for the orthorhombic elastic now take the following reduced form:

$$c_{11}^{\mp} = c_{33}^{\mp} = \lambda^{\mp} + 2\mu^{\mp}, \qquad c_{13}^{\mp} = \lambda^{\mp}, \qquad c_{55}^{\mp} = 2\mu^{\mp}, \tag{73}$$

where $\lambda^{\mp}$ and $\mu^{\mp}$ are the respective Lame's elastic constants [42], then the equation expressed in (70) may be compared with Equation (5.3) of Dai et al. [32] while disregarding the effect of external loading as follows:

$$\Delta\,\phi - \frac{1}{c_R^2}\,\phi_{,tt} - b_0 h\sqrt{-\Delta}\,\Delta\,\phi = 0, \tag{74}$$

where

$$b_0 = \frac{\mu(1 - \beta_R^2)}{2B_0}\left[\frac{\rho^-\,c_R^2}{\mu^-}(\alpha_R + \beta_R) - 4\beta_R\left(1 - \kappa_0^{-2}\right)\right], \tag{75}$$

with

$$B_0 = \frac{\beta_R}{\alpha_R}\left(1 - \alpha_R^2\right) + \frac{\alpha_R}{\beta_R}\left(1 - \beta_R^2\right) - \left(1 - \beta_R^4\right), \quad \alpha_R = \sqrt{1 - \frac{\rho^+ c_R^2}{\lambda^+ + 2\mu^+}}, \quad \beta_R = \sqrt{1 - \frac{\rho^+ c_R^2}{\mu^+}}, \tag{76}$$

and

$$\mu = \frac{\mu^-}{\mu^+} \qquad \kappa_0 = \sqrt{\frac{\lambda^- + 2\mu^-}{\mu^-}}. \tag{77}$$

Furthermore, the approximate dispersion relation expressed in (72) may then be reduced, leading to a similar approximate dispersion relation, as reported in reference [8], given as

$$\frac{c}{c_R} = \sqrt{1 - b_0\,K - \frac{b_1\,\zeta}{K}}, \tag{78}$$

where

$$b_1 = \frac{\mu\, \alpha_R \left(1 - \beta_R^2\right)}{2B_0}, \qquad \text{and} \qquad \zeta = \frac{h\, a}{\mu^-}. \tag{79}$$

In short, it is part of the novelty of the present work to categorically state that various considerations have been generalized by our examination. As an example, in the absence of the reinforcement induced by the elastic Winkler foundation, the results of the present study are reduced to those of Nobili and Prikazchikov, as reported in reference [24]. Further, when the isotropic material case is considered, the present study matches the results obtained by Dai et al. [32]. In the same vein, our finding coincides with the recent results of Mubaraki and Almalki [8] in the absence of the action of magnetic field force.

In this regard, we endeavour here to numerically analyze the significance of the imposed loading on the coated substrate, which was presided over by the elastic Winkler foundation. In light of this, the obtained approximate dispersion relation in (72) for orthorhombic-coated orthorhombic half-space, is simulated numerically, considering the combination of soft-stiff and stiff-soft materials, respectively. In fact, the cilicon (Si) material [44] is sought after as a soft material, which admits the following material data:

$$c_{11}^- = 11.6 \text{ GPa}, \;\; c_{13}^- = 5.4 \text{ GPa}, \;\; c_{33}^- = 16.6 \text{ GPa}, \;\; c_{55}^- = 9.5 \text{ GPa}, \;\; \rho^- = 2329 \text{ kgm}^3,$$

while for the stiff material, aluminum nitride (AIN) material [45] is considered, which has the following physical data:

$$c_{11}^+ = 34.5 \text{ GPa}, \;\; c_{13}^+ = 12.0 \text{ GPa}, \;\; c_{33}^+ = 39.5 \text{ GPa}, \;\; c_{55}^+ = 11.8 \text{ GPa}, \;\; \rho^+ = 3260 \text{ kgm}^3.$$

Thus, we have portrayed in Figures 2 and 3 the influence of the scaled dimensionless Winkler foundation parameter $\zeta^+ = \frac{ha}{c_{55}^+}$ on scaled dimensionless phase speed $\frac{c}{c_R}$ against the dimensionless wavenumber K. More precisely, Figures 2 and 3 shows the variational significance of the Winkler foundation parameter on the dispersion of waves in a soft-coated stiff-substrate, and in stiff-coated soft-substrate structures, respectively. Notably, from Figure 2, it is noted that an increase in the Winkler foundation parameter lessens the acquired approximate dispersion relation through the phase speed versus the wavenumber curve. Moreover, one can also observe that the dispersion relation attains its maximum in the absence of the elastic Winkler foundation. Additionally, when a swap of materials is made between the material constants of the coating and that of the half-space substrate layers, as portrayed in Figure 3, that is, a media with stiff-coated soft-substrate, an opposite tread is realized. Thus, the choice of material or combination of materials is very important with regard to the vibration analysis and design of dissimilar single, coated, and multilayered structures.

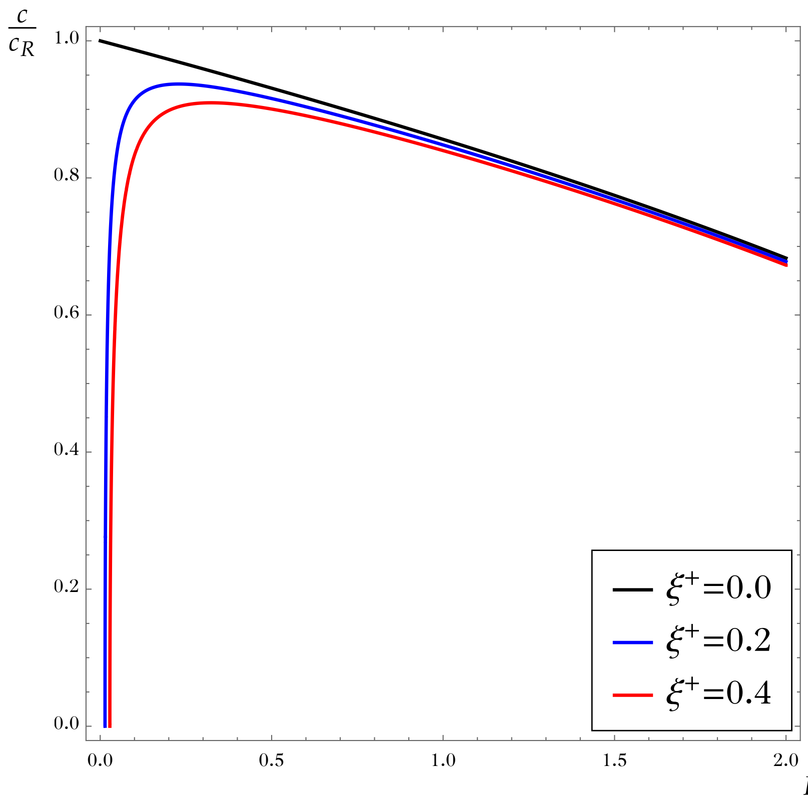

Figure 2. Influence of the scaled dimensionless Winkler foundation parameter ζ^+ on scaled dimensionless phase speed $\frac{c}{c_R}$ versus the dimensionless wave number K on a Si-coated AIN-substrate.

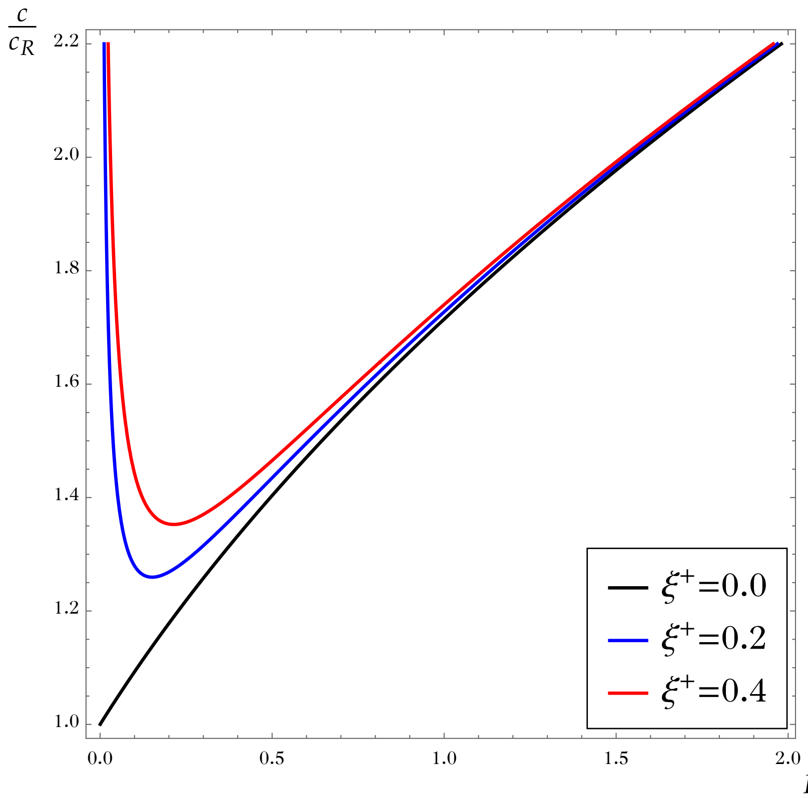

Figure 3. Influence of the scaled dimensionless Winkler foundation parameter ζ^+ on scaled dimensionless phase speed $\frac{c}{c_R}$ versus the dimensionless wavenumber K on an AIN-coated Si-substrate.

6. Conclusions

The present study asymptotically derived the approximate equations of motions and dispersion relation governing the propagation of Rayleigh waves on a loaded orthorhombic-coated orthorhombic elastic half-space. More precisely, the prescribed vertically loaded excitation was presumed to be in favor of an elastic Winkler foundation. Indeed, perfect continuity conditions were imposed between the coated layer and elastic half-space. Certainly, the derived model was found to comprehensively capture the physical characteristics of elastic surface waves, where the propagation of Rayleigh waves on the governing media was described using a singularly perturbed hyperbolic equation, admitting a pseudo-differential operator. Furthermore, upon utilizing the long-wave limit approximation for elastic surface waves, the decay over the interior of the half-space was described using a pseudo-static elliptic equation, through the acquisition of appropriate effective boundary conditions; further, the significance of the thin coating layer on the dispersion of surface waves on the coated structure was equally examined. Finally, as the orthorhombic material happened to generalize several other materials of real-life relevance, the present study then serves as an interesting monograph for the examination of the dispersion of surface waves on coated media in the fields of linear elasticity and material science.

Funding: This research received no external funding.

Acknowledgments: The researcher would like to acknowledge Deanship of Scientific Research, Taif University for funding this work.

Conflicts of Interest: The author declares that he has no conflict of interest.

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