Mathematical and Computational Applications

MDPI

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Noether Symmetries of the Triple Degenerate DNLS Equations

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Abstract: In this paper, Lie symmetries and Noether symmetries along with the corresponding conservation laws are derived for weakly nonlinear dispersive magnetohydrodynamic wave equations, also known as the triple degenerate derivative nonlinear Schrödinger equations. The main goal of this study is to obtain Noether symmetries of the second-order Lagrangian density for these equations using the Noether symmetry approach with a gauge term. For this Lagrangian density, we compute the conserved densities and fluxes corresponding to the Noether symmetries with a gauge term, which differ from the conserved densities obtained using Lie symmetries in Webb et al. (*J. Plasma Phys.* **1995**, *54*, 201–244; *J. Phys. A Math. Gen.* **1996**, *29*, 5209–5240). Furthermore, we find some new Lie symmetries of the dispersive triple degenerate derivative nonlinear Schrödinger equations for non-vanishing integration functions $K_i(t)$ $(i = 1, 2, 3)$.

Keywords: nonlinear Schrödinger equation; Alfvén waves; Noether symmetry; Lie symmetry

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Citation: Camci, U. Noether Symmetries of the Triple Degenerate DNLS Equations. *Math. Comput. Appl.* **2024**, *29*, 60. https://doi.org/10.3390/mca29040060

Academic Editor: Mehmet Pakdemirli

Received: 7 July 2024
Revised: 26 July 2024
Accepted: 28 July 2024
Published: 30 July 2024

1. Introduction

Nonlinear finite amplitude Alfvén waves and magnetosonic waves are types of magnetohydrodynamic (MHD) waves that propagate in a plasma, which is a state of matter consisting of charged particles (ions and electrons) and magnetic fields. These waves are described by the MHD equations, a set of fluid-like equations that govern the behavior of magnetized plasmas [1]. Both types of waves play crucial roles in the dynamics of magnetized plasmas, which are commonly found in astrophysical environments such as the solar wind and magnetospheres of planets, as well as in laboratory plasma experiments. Understanding the nonlinear aspects of these waves is essential to reaching a more accurate description of the complex interactions and behaviors that occur in such environments.

Alfvén waves are characterized by the propagation of perturbations in the magnetic field, with the plasma particles moving in spiral or helical paths around the magnetic field lines. If the wave's amplitude is not infinitesimally small and the wave behavior cannot be described by linear theory, then we refer to the "nonlinear finite amplitude" of the wave. In the nonlinear regime, the amplitude of the wave becomes significant and interactions between different wave components become important. Alfvén waves obey the derivative nonlinear Schrödinger (DNLS) equation [2,3]:

$$\frac{\partial \psi}{\partial t} + \frac{V_A^3}{4(V_A^2 - a_g^2)} \frac{\partial}{\partial x}\left(|\psi|^2 \psi\right) + \frac{i}{2}\chi V_A \frac{\partial^2 \psi}{\partial x^2} = 0 \tag{1}$$

where V_A and a_g are respectively the Alfvén speed and the gas sound speed, $\chi = V_A/\Omega_p$ is the ion inertial length, Ω_p is the proton gyro-frequency, and $\psi = v + iw$ is the normalized complex transverse-wave magnetic field perturbation.

Magnetosonic waves, also known as MHD fast waves, are another type of MHD wave which involve both magnetic and acoustic perturbations in a plasma [4]. They consist of a combination of the fast magnetosonic mode and the entropy mode. The fast magnetosonic mode is characterized by compressional waves that involve variations in the magnetic field and plasma density. Magnetosonic waves can propagate in various directions with respect to the magnetic field, and their behavior is influenced by plasma conditions such as density, temperature, and magnetic field strength. Similar to nonlinear Alfvén waves, nonlinear finite-amplitude magnetosonic waves involve situations where the amplitude of the wave is non-negligible and nonlinear effects become important.

If a physical system exhibits continuous symmetry in its action, i.e., the action remains unchanged under certain continuous transformations, then there exists a conserved quantity associated with that symmetry. Symmetries of an action correspond to transformations of the system that leave the action unchanged. Noether's theorem establishes that these symmetries are associated with conserved quantities, often referred to as 'Noether charges' or 'Noether currents' (see the recent review in [5] on the physical significance of Noether symmetries). Furthermore, in the quantum field theory context, Noether's theorem plays a crucial role in connecting symmetries to the conservation of various quantities, including energy, momentum, angular momentum, and charge.

The rest of this paper is arranged as follows: in Section 2, we introduce the triple degenerate DNLS system; in Section 3, we derive both Noether symmetries and Lie point symmetries of the latter system; and in final section, Section 4, the conclusions are presented.

2. The TDNLS System

In order to derive the DNLS Equation (1), it is assumed that the Alfvén and sound speeds are well separated and distinct. Previous papers [2,3] considered the appropriate form of the wave evolution equations for quasiparallel propagation of the Alfvén and magneto-acoustic modes near the triple umbilic point, in which the sound speed and Alfvén speed are almost equal, that is, $a_g^2/V_A^2 - 1 = \epsilon\Delta$, where ϵ is the perturbation parameter representing the wave amplitude and Δ is a constant of order unity. The coefficient of the nonlinear term for this limit in the DNLS Equation (1) diverges, and a modified version of the method of multiple scales must be used considering the fact that $\epsilon\Delta$ is a small quantity. If the Alfvén, fast magneto-acoustic, and slow magneto-acoustic waves have the same phase speed, then the resulting equations at the lowest order in ϵ are appropriately described as the triple degenerate DNLS system (the TDNLS system).

Considering waves which propagate in the x direction and assuming that all quantities are dependent on x and t only, the dimensional and dimensionless forms of the TDNLS equations are as presented in [2,3]. In this paper, we consider the dimensionless forms of the TDNLS equations provided by

$$u_t + \frac{\partial}{\partial x}\left[\frac{1}{2}\left(\Gamma u^2 + v^2 + w^2\right)\right] = 0\,, \tag{2}$$

$$v_t + \frac{\partial}{\partial x}[(u-\Delta)v - \chi w_x] = 0\,, \tag{3}$$

$$w_t + \frac{\partial}{\partial x}[(u-\Delta)w + \chi v_x] = 0\,, \tag{4}$$

where u represents the density or x component of the fluid velocity perturbation and $\Gamma = \gamma_g + 1$, with γ_g being the gas adiabatic index. It has been shown [2] that the TDNLS equations admit both Lagrangian and Hamiltonian variational formulations. By taking $u = U_x, v = V_x, w = W_x$ and $\psi = v + iw = \Psi_x$, the TDNLS Equations (2)–(4) become

$$\frac{\partial}{\partial x}\left[U_t + \frac{1}{2}\left(\Gamma\, U_x^2 + \Psi_x \Psi_x^*\right)\right] = 0\,, \tag{5}$$

$$\frac{\partial}{\partial x}[\Psi_t + (U_x - \Delta)\Psi_x + i\chi\Psi_{xx}] = 0\,, \tag{6}$$

$$\frac{\partial}{\partial x}[\Psi_t^* + (U_x - \Delta)\Psi_x^* - i\chi\Psi_{xx}^*] = 0\,, \tag{7}$$

where $*$ denotes complex conjugation and U, V, W, and $\Psi = V + iW$ are potentials for u, v, w, and ψ. An alternative form of the above equations in terms of U, V, and W is

$$E_1 \equiv \quad U_t + \frac{1}{2}\left(\Gamma\, U_x^2 + V_x^2 + W_x^2\right) - K_1(t) = 0\,, \tag{8}$$

$$E_2 \equiv \quad V_t + (U_x - \Delta)V_x - \chi W_{xx} - K_2(t) = 0\,, \tag{9}$$

$$E_3 \equiv \quad W_t + (U_x - \Delta)W_x + \chi V_{xx} - K_3(t) = 0\,, \tag{10}$$

where $K_1(t)$, $K_2(t)$ and $K_3(t)$ are integration functions. The latter equations are called the potential form of the TDNLS equations in [2].

We can obtain Equations (5)–(7) by extremizing the variational functional

$$\mathcal{A}[U, \Psi, \Psi^*] = \iint \mathcal{L}\, dxdt\,, \tag{11}$$

where the Lagrangian density L is provided by

$$\mathcal{L} = \frac{1}{3}\Gamma\, U_x^3 + (U_x - \Delta)\Psi_x\Psi_x^* + U_x U_t + \frac{1}{2}(\Psi_x\Psi_t^* + \Psi_x^*\Psi_t) + \frac{i}{2}\chi(\Psi_x^*\Psi_{xx} - \Psi_x\Psi_{xx}^*)\,, \tag{12}$$

which can be written as

$$\mathcal{L} = \frac{1}{3}\Gamma\, U_x^3 + (U_x - \Delta)\left(V_x^2 + W_x^2\right) + U_x U_t + V_x V_t + W_x W_t + \chi(W_x V_{xx} - V_x W_{xx})\,. \tag{13}$$

By varying the functional (11) with respect to U, Ψ^*, and Ψ, we find Equations (5), (6) and (7), respectively. These equations can actually be written as

$$\frac{\delta\mathcal{A}}{\delta U} = E_1\,, \quad \frac{\delta\mathcal{A}}{\delta\Psi^*} = \frac{1}{2}E_2\,, \quad \frac{\delta\mathcal{A}}{\delta\Psi} = \frac{1}{2}E_3\,, \tag{14}$$

where $\delta\mathcal{A}/\delta U^i$ denotes the variational derivatives of $\mathcal{A}$ with respect to $U^i = (U, \Psi, \Psi^*)$.

3. Symmetries and Conservation Laws

The purpose of this article is to study the Lie symmetries and Noether symmetries with the gauge function and the corresponding conservation laws of the TDNLS equations, which we deal with in the following.

3.1. Noether Symmetries

The usual Noether symmetry approach with a boundary (or gauge) term is valuable in addressing a variety of problems in physics and applied mathematics. In order to study the Lagrangian symmetries for the TDNLS equations, we can consider a Lagrange function $\mathcal{L}(x^\alpha, q^j, q^j_{,\alpha}, q^j_{,\alpha\beta})$ depending on second-order derivatives, which gives rise to the third-order Euler–Lagrange equations

$$\frac{\delta\mathcal{L}}{\delta q^i} \equiv \frac{\partial\mathcal{L}}{\partial q^i} - D_\mu\left(\frac{\partial\mathcal{L}}{\partial q^i_{,\mu}}\right) + D_\mu D_\nu\left(\frac{\partial\mathcal{L}}{\partial q^i_{,\mu\nu}}\right) = 0\,, \tag{15}$$

where $D_\mu = \partial_{x^\mu} + q^i_{,\mu}\partial_{q^i} + q^i_{,\mu\nu}\partial_{q^i_{,\nu}} + q^i_{,\mu\alpha\beta}\partial_{q^i_{,\alpha\beta}}$ is the total derivative operator on the 2-th jet space, $q^j_{,\mu} = \partial q^j/\partial x^\mu$ and $q^j_{,\mu\nu} = \partial^2 q^j/\partial x^\mu \partial x^\nu$, and so on. Here, the indexes $i, j, \ldots$ and $\mu, \nu, \alpha, \ldots$ represent the dependent and independent variables, respectively.

Let us consider, in particular, an action integral $S = \int \mathcal{L}(x^\alpha, q^j, q^j_{,\alpha}, q^j_{,\alpha\beta})dx^\mu$ which remains invariant under the one-parameter infinitesimal transformation $\bar{x}^\mu = x^\mu + \epsilon\,\xi^\mu(x^\nu, q^i)$,

$\bar{q}^i = q^i + \epsilon\,\eta^i(x^\nu, q^j)$, in which ϵ is an infinitesimal parameter such that $\epsilon^2 \to 0$ if and only if $\bar{S} = S$, that is,

$$\xi^\mu \frac{\partial \mathcal{L}}{\partial x^\mu} + \eta^i \frac{\partial \mathcal{L}}{\partial q^i} + \eta^i_\alpha \frac{\partial \mathcal{L}}{\partial q^i_{,\alpha}} + \eta^i_{\alpha\beta} \frac{\partial \mathcal{L}}{\partial q^i_{,\alpha\beta}} + \mathcal{L}(D_\mu \xi^\mu) = D_\mu f^\mu\,, \tag{16}$$

which is equivalent to $\mathbf{X}^{[2]}\mathcal{L} + \mathcal{L}(D_\mu\xi^\mu) = D_\mu f^\mu$, where the coefficients η^i_α and $\eta^i_{\alpha\beta}$ are defined as $\eta^i_\alpha = D_\alpha \eta^i - q^i_{,\beta} D_\alpha \xi^\beta$ and $\eta^i_{\alpha\beta} = D_\beta \eta^i_\alpha - q^i_{,\alpha\mu} D_\beta \xi^\mu$, respectively, and f^μ is a boundary (or gauge) term which should be determined. The expression in (16) is Noether's first theorem, and the generator $\mathbf{X} = \xi^\mu \partial_{x^\mu} + \eta^i\,\partial_{q^i}$ is called Noether symmetry for the Lagrangian $\mathcal{L}$. Noether's second theorem indicates that for every Noether point symmetry $\mathbf{X}$ there exists a conserved vector $\mathbf{J} = J^\mu \partial_{x^\mu}$, or a current density, where J^μ is defined as [6]

$$J^\mu = f^\mu - \left[\xi^\mu \mathcal{L} + \left(\eta^i - \xi^\alpha q^i_{,\alpha}\right)\left(\frac{\partial \mathcal{L}}{\partial q^i_{,\mu}} - D_\nu \frac{\partial \mathcal{L}}{\partial q^i_{,\mu\nu}}\right)\right] - D_\nu(\eta^i - q^i_{,\alpha}\xi^\alpha)\frac{\partial \mathcal{L}}{\partial q^i_{,\mu\nu}}\,, \tag{17}$$

which satisfies the local conservation law $D_\mu J^\mu = 0$. For the TDNLS equations, the independent variables and dependent variables (or the generalized coordinates) are respectively denoted as $x^\mu = \{t, x\}$ with $\mu = 0, 1$ and $q^i = \{U, V, W\}$ with $i = 1, 2, 3$. When we apply the conservation law, the resulting conserved flow vector components related to the Lagrangian (13) are as follows:

$$\begin{aligned} J^0 = &-\xi^0\left[\frac{1}{3}\Gamma\, U_x^3 + (U_x - \Delta)\left(V_x^2 + W_x^2\right) + \chi(W_x V_{xx} - V_x W_{xx})\right] \\ &+\xi^1\left(U_x^2 + V_x^2 + W_x^2\right) - \eta^1 U_x - \eta^2 V_x - \eta^3 W_x + f^0\,, \end{aligned} \tag{18}$$

$$\begin{aligned} J^1 = &\,\xi^0\Big\{U_t^2 + V_t^2 + W_t^2 + U_t\left(\Gamma\, U_x^2 + V_x^2 + W_x^2\right) + 2(U_x - \Delta)(V_t V_x + W_t W_x) + 2\,\chi(W_t V_{xx} - V_t W_{xx}) \\ &+\chi(W_x V_{tx} - V_x W_{tx})\Big\} + \xi^1\left\{\frac{2}{3}\Gamma\, U_x^3 + (2U_x - \Delta)\left(V_x^2 + W_x^2\right) + 2\,\chi(W_x V_{xx} - V_x W_{xx})\right\} \\ &+\chi(W_x V_t - V_x W_t)\left(\xi^0_{,x} + U_x \xi^0_{,U} + V_x \xi^0_{,V} + W_x \xi^0_{,W}\right) - \eta^1\left(U_t + \Gamma\, U_x^2 + V_x^2 + W_x^2\right) \\ &-\eta^2[V_t + 2(U_x - \Delta)V_x - 2\,\chi W_{xx}] - \eta^3[W_t + 2(U_x - \Delta)W_x + 2\,\chi V_{xx}] \\ &-\chi\, W_x\left(\eta^2_{,x} + U_x \eta^2_{,U} + V_x \eta^2_{,V} + W_x \eta^2_{,W}\right) + \chi\, V_x\left(\eta^3_{,x} + U_x \eta^3_{,U} + V_x \eta^3_{,V} + W_x \eta^3_{,W}\right) + f^1\,, \end{aligned} \tag{19}$$

where J^0 and J^1 are the conserved density and flux, respectively. For these quantities, the conservation law of the TDNLS system is of the form

$$\mathbf{D} \cdot \mathbf{J} = 0 \quad \Leftrightarrow \quad D_t J^0 + D_x J^1 = 0\,, \tag{20}$$

where $\mathbf{D} = (D_t, D_x)$, $\mathbf{J} = (J^0, J^2)$, and the differential operators D_t and D_x are

$$\begin{aligned} D_t = &\frac{\partial}{\partial t} + U_t \frac{\partial}{\partial U} + V_t \frac{\partial}{\partial V} + W_t \frac{\partial}{\partial W} + U_{tx}\frac{\partial}{\partial U_x} + U_{tt}\frac{\partial}{\partial U_t} + V_{tx}\frac{\partial}{\partial V_x} + V_{tt}\frac{\partial}{\partial V_t} \\ &+W_{tx}\frac{\partial}{\partial W_x} + W_{tt}\frac{\partial}{\partial W_t} + V_{txx}\frac{\partial}{\partial V_{xx}} + W_{txx}\frac{\partial}{\partial W_{xx}}\,, \end{aligned} \tag{21}$$

$$\begin{aligned} D_x = &\frac{\partial}{\partial x} + U_x \frac{\partial}{\partial U} + V_x \frac{\partial}{\partial V} + W_x \frac{\partial}{\partial W} + U_{xx}\frac{\partial}{\partial U_x} + U_{tx}\frac{\partial}{\partial U_t} + V_{xx}\frac{\partial}{\partial V_x} + V_{tx}\frac{\partial}{\partial V_t} \\ &+W_{xx}\frac{\partial}{\partial W_x} + W_{tx}\frac{\partial}{\partial W_t} + V_{xxx}\frac{\partial}{\partial V_{xx}} + W_{xxx}\frac{\partial}{\partial W_{xx}} + V_{txx}\frac{\partial}{\partial V_{tx}} + W_{txx}\frac{\partial}{\partial W_{tx}}\,. \end{aligned} \tag{22}$$

The energy functional associated with the Lagrangian $\mathcal{L}$ is defined by

$$E_{\mathcal{L}} = q^j_{,t}\frac{\partial \mathcal{L}}{\partial q^j_{,t}} - \mathcal{L}\,. \tag{23}$$

This is actually the Hamiltonian of the system. Hence, the $E_{\mathcal{L}}$ associated with (13) has the following form:

$$E_{\mathcal{L}} = -\left[\frac{1}{3}\Gamma\,U_x^3 + (U_x - \Delta)\left(V_x^2 + W_x^2\right) + \chi(W_x V_{xx} - V_x W_{xx})\right]. \tag{24}$$

For the TDNLS equations, the dependencies of the Lagrangian (13) yield

$$\mathcal{L} = \mathcal{L}(t, x, U, V, W, U_t, V_t, W_t, U_x, V_x, W_x, V_{xx}, W_{xx}).$$

It is explicitly observed that the Lagrangian (13) leads to third-order equations of motion, i.e., the TDNLS Equations (5)–(7). Therefore, we consider the Noether symmetry condition (16) to search for the Noether symmetry generator such that

$$\mathbf{X} = \xi^0\frac{\partial}{\partial t} + \xi^1\frac{\partial}{\partial x} + \eta^1\frac{\partial}{\partial U} + \eta^2\frac{\partial}{\partial V} + \eta^3\frac{\partial}{\partial W}\,, \tag{25}$$

where the components $\xi^0, \xi^2, \eta^1, \eta^2$, and η^3 are dependent on x, t, U, V, and W. Substituting the Lagrangian (13) into the Noether symmetry condition (16) yields the following set of partial differential equations:

$$\begin{aligned}
&\xi^0_{,x} = 0\,,\quad \xi^0_{,U} = 0\,,\quad \xi^0_{,V} = 0\,,\quad \xi^0_{,W} = 0\,,\quad \xi^1_{,U} = 0\,,\quad \xi^1_{,V} = 0\,,\quad \chi\xi^1_{,W} = 0\,,\\
&\eta^1_{,U} = 0\,,\quad \eta^1_{,V} = 0\,,\quad \eta^2_{,x} = 0\,,\quad \eta^2_{,U} = 0\,,\quad \eta^2_{,V} = 0\,,\quad \chi\eta^3_{,U} = 0\,,\quad \eta^3_{,W} = 0\,,\\
&\chi\eta^2_{,WW} = 0\,,\quad \chi\eta^3_{,VV} = 0\,,\quad \eta^1_{,W} - \Delta\xi^1_{,W} = 0\,,\quad \eta^1_{,W} + \eta^3_{,U} = 0\,,\quad \eta^2_{,W} + \eta^3_{,V} = 0\,,\\
&\Gamma\eta^1_{,V} + 2\eta^2_{,U} = 0\,,\qquad \Gamma\eta^1_{,x} - \xi^1_{,t} = 0\,,\qquad \Gamma(3\,\eta^1_{,U} + \xi^0_{,t} - 2\xi^1_{,x}) = 0\,,\\
&f^0_{,t} + f^1_{,x} = 0\,,\quad \eta^1_{,t} - f^1_{,U} = 0\,,\quad \eta^1_{,x} - f^0_{,U} = 0\,,\quad \eta^2_{,x} - f^0_{,V} = 0\,,\quad \eta^3_{,x} - f^0_{,W} = 0\,,\\
&\Gamma\eta^1_{,W} + 2\eta^3_{,U} = 0\,,\quad 2\Delta\eta^3_{,x} - \eta^3_{,t} + f^1_{,W} = 0\,,\quad 2\Delta\eta^2_{,x} - \eta^2_{,t} + \chi\,\eta^3_{,xx} + f^1_{,V} = 0\,,\\
&\xi^0_{,t} - 2\xi^1_{,x} = 0\,,\quad \eta^1_{,x} - \xi^1_{,t} + \Delta(\xi^1_{,x} - \xi^0_{,t}) = 0\,,\quad 2\chi\,\eta^3_{,xV} - \eta^1_{,x} + \xi^1_{,t} - \Delta(\xi^1_{,x} - \xi^0_{,t}) = 0\,,\\
&\eta^3_{,x} - \Delta\,\eta^3_{,U} = 0\,,\quad \eta^1_{,W} - \Delta\,\xi^1_{,W} = 0\,,\quad \chi(\xi^1_{,xx} - \xi^1_{,tt}) - 2\,\Delta(\eta^2_{,W} + \eta^3_{,V}) = 0\,.
\end{aligned} \tag{26}$$

Now, Equation (26) can be solved to obtain the components $\xi^0, \xi^2, \eta^1, \eta^2$, and η^3 of the Noether symmetry generator in (25) and the gauge vector components f^0 and f^1 in cases $\Gamma \neq 1$ and $\Gamma = 1$. The case where $\Gamma \neq 1$ is only of physical interest due to the relation $\Gamma = \gamma_g + 1$. We deal with these two cases separately below.

3.1.1. Noether Symmetries for $\Gamma \neq 1$

For $\Gamma \neq 1$, the solution of the Noether symmetry Equation (26) yields

$$\xi^0 = c_1 + c_2 t\,,\quad \xi^1 = c_2\left[\frac{\Gamma(x - \Delta\,t) - x}{2(\Gamma - 1)}\right] + c_3\,, \tag{27}$$

$$\eta^1 = -c_2\frac{\Delta\,x}{2(\Gamma - 1)} + F_1(t)\,,\quad \eta^2 = c_4 W + F_2(t)\,,\quad \eta^3 = -c_4 V + F_3(t)\,, \tag{28}$$

$$f^0 = -c_2\frac{\Delta\,x}{2(\Gamma - 1)} + F_4(t, x)\,,\ f^1 = F_{1,t}U + F_{2,t}V + F_{3,t}W - \int F_{4,t}dx + F_5(t)\,, \tag{29}$$

where $c_1, ..., c_4$ are constant parameters and $F_1, ..., F_5$ are integration functions. Therefore, linearly independent Noether symmetries provided by the generator in (25) are of the form

$$\mathbf{X}_1 = \frac{\partial}{\partial t}, \quad \mathbf{X}_2 = \frac{\partial}{\partial x}, \quad \mathbf{X}_3 = W\frac{\partial}{\partial V} - V\frac{\partial}{\partial W}, \tag{30}$$

$$\mathbf{X}_4 = 2\,t\frac{\partial}{\partial t} + \left(x - \frac{\Delta\,\Gamma\,t}{\Gamma-1}\right)\frac{\partial}{\partial x} - \frac{\Delta\,x}{(\Gamma-1)}\frac{\partial}{\partial U} \quad \text{with} \quad f^0 = -\frac{\Delta\,U}{\Gamma-1},\ f^1 = 0, \tag{31}$$

$$\mathbf{X}_5 = F_1(t)\frac{\partial}{\partial U} \quad \text{with} \quad f^0 = 0, \quad f^1 = F_{1,t}U, \tag{32}$$

$$\mathbf{X}_6 = F_2(t)\frac{\partial}{\partial V} \quad \text{with} \quad f^0 = 0, \quad f^1 = F_{2,t}V, \tag{33}$$

$$\mathbf{X}_7 = F_3(t)\frac{\partial}{\partial W} \quad \text{with} \quad f^0 = 0, \quad f^1 = F_{3,t}W. \tag{34}$$

The components of the conserved vector (17), corresponding to the integration functions F_4 and F_5 of the gauge functions, which possess the property $J^\mu = f^\mu$, are as follows: $f^0 = F_4(t,x)$ and $f^1 = F_5(t) - \int F_{4,t}dx$. These satisfy the conservation law $D_t f^0 + D_x f^1 = 0$. It is noteworthy that these conserved quantities will arise in all possible cases; thus, further mention is unnecessary. The vector fields $\mathbf{X}_1, \mathbf{X}_2$, and $\mathbf{X}_3$ in Equation (30) correspond to time translation invariance, translation invariance in the x direction, and rotational invariance of the (V,W) variables, respectively. The vector fields $\mathbf{X}_5, \mathbf{X}_6$, and $\mathbf{X}_7$ in Equations (32)–(34) represent boost symmetries along the U, W, and W directions if $F_1(t), F_2(t)$, and $F_3(t)$ are linear polynomial functions of time. They indicate translations in the U, W, and W directions if $F_1(t), F_2(t)$, and $F_3(t)$ are nonzero constants. When $\Delta = 0$ in (31), then $\mathbf{X}_4$ represents a scaling symmetry, a property of a system that remains unchanged under rescaling. Although characterizing the symmetry $\mathbf{X}_4$ might be challenging, it can be considered a combination of scaling symmetry in the x and t directions, boost symmetry in the x direction, and dilatation symmetry along the U direction with respect to the variable x.

After considering Equations (8) and (9), the conserved densities J_a^0 and fluxes J_a^1 $(a = 1, \ldots, 7)$ corresponding to the Noether symmetries $\mathbf{X}_1, \ldots, \mathbf{X}_7$ in (30)–(34) are listed as follows:

$$J_1^0 = -\frac{1}{3}\Gamma U_x^3 - (U_x - \Delta)(V_x^2 + W_x^2) - \chi(W_x V_{xx} - V_x W_{xx}), \tag{35}$$

$$\begin{aligned} J_1^1 = {} & 2\,(U_x - \Delta)(K_2 V_x + K_3 W_x) - \frac{1}{4}\left(\Gamma U_x^2 + V_x^2 + W_x^2\right)^2 - [(U_x - \Delta)V_x - \chi W_{xx}]^2 \\ & -[(U_x - \Delta)W_x + \chi V_{xx}]^2 + \chi(W_x V_{tx} - V_x W_{tx}), \end{aligned} \tag{36}$$

$$J_2^0 = U_x^2 + V_x^2 + W_x^2, \qquad J_2^1 = \frac{2}{3}\Gamma U_x^3 + (2\,U_x - \Delta)(V_x^2 + W_x^2) + 2\,\chi(W_x V_{xx} - V_x W_{xx}), \tag{37}$$

$$J_3^0 = V W_x - W V_x, \tag{38}$$

$$J_3^1 = V[(U_x - \Delta)W_x + \chi V_{xx}] - W[(U_x - \Delta)V_x - \chi W_{xx}] - \chi\left(V_x^2 + W_x^2\right) - K_2 W + K_3 V, \tag{39}$$

$$J_4^0 = t\,J_1^0 + \frac{1}{2}\left(x - \frac{\Delta\,\Gamma}{\Gamma-1}t\right)J_2^0 + \frac{\Delta\,x\,U_x}{2(\Gamma-1)} - \frac{\Delta\,U}{2(\Gamma-1)}, \tag{40}$$

$$J_4^1 = t\,J_1^1 + \frac{1}{2}\left(x - \frac{\Delta\,\Gamma}{\Gamma-1}t\right)J_2^1 + \frac{\Delta\,x}{2(\Gamma-1)}\left(K_1 + \frac{1}{2}(\Gamma U_x^2 + V_x^2 + W_x^2)\right), \tag{41}$$

$$J_5^0 = -F_1(t)U_x\,, \quad J_5^1 = F_{1,t}U - F_1\left[K_1 + \frac{1}{2}(\Gamma U_x^2 + V_x^2 + W_x^2)\right], \tag{42}$$

$$J_6^0 = -F_2(t)V_x\,, \quad J_6^1 = F_{2,t}V - F_2[K_2 + (U_x - \Delta)V_x - \chi W_{xx}]\,, \tag{43}$$

$$J_7^0 = -F_3(t)W_x\,, \quad J_7^1 = F_{3,t}W - F_3[K_3 + (U_x - \Delta)W_x + \chi V_{xx}]\,. \tag{44}$$

In [2], it was shown that the Hamiltonian density of the system is $\mathcal{H} = -J_1^0$, i.e., $\mathcal{H} = -E_{\mathcal{L}}$, due to the relation in (24). The conserved vectors $\mathbf{J}_1 = (J_1^0, J_1^1)$, $\mathbf{J}_2 = (J_2^0, J_2^1)$, and $\mathbf{J}_3 = (J_3^0, J_3^1)$ respectively correspond to energy conservation, momentum conservation, and helicity conservation, as noted by Hada [7]. The conserved vector $\mathbf{J}_4 = (J_4^0, J_4^1)$ for $\mathbf{X}_4$ can be written in the following form:

$$\mathbf{J}_4 = t\left(\mathbf{J}_1 - \frac{\Delta}{2(\Gamma-1)}\mathbf{J}_2\right) + \frac{x}{2}\left(\mathbf{J}_2 + \frac{\Delta}{\Gamma-1}\mathbf{J}_0\right) \tag{45}$$

where $\mathbf{J}_0$ is defined as

$$\mathbf{J}_0 = \left(U_x - \frac{1}{x}U\,,\, K_1 + \frac{1}{2}(\Gamma U_x^2 + V_x^2 + W_x^2)\right). \tag{46}$$

Further, the conserved vector fields $\mathbf{J}_5, \mathbf{J}_6$ and $\mathbf{J}_7$ become

$$\mathbf{J}_5 = (-F_1 U_x\,,\, F_{1,t}U + F_1(U_t - 2K_1))\,, \tag{47}$$
$$\mathbf{J}_6 = (-F_2 V_x\,,\, F_{2,t}V + F_2(V_t - 2K_2))\,, \tag{48}$$
$$\mathbf{J}_7 = (-F_3 W_x\,,\, F_{3,t}W + F_3(W_t - 2K_3))\,, \tag{49}$$

where K_1, K_2, and K_3 are functions depending on the time t.

3.1.2. Noether Symmetries for $\Gamma = 1$

When $\Gamma = 1$ and $\Delta \neq 0$, there are *seven* Noether symmetries, of which the six ones are the same as $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, $\mathbf{X}_5$, $\mathbf{X}_6$, and $\mathbf{X}_7$ in Equations (30) and (32)–(34). However, the Noether symmetry $\mathbf{X}_4$ differs as follows:

$$\mathbf{X}_4 = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial U} \quad \text{with} \quad f^0 = U\,, \quad f^1 = 0\,, \tag{50}$$

which includes the boost symmetry along the x direction and the dilatation symmetry along U direction, respectively. Then, the conserved vector for $\mathbf{X}_4$ provided in (50) becomes

$$\mathbf{J}_4 = t\,\mathbf{J}_2 - x\,\mathbf{J}_0\,. \tag{51}$$

The remaining conserved vectors for $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, $\mathbf{X}_5$, $\mathbf{X}_6$, and $\mathbf{X}_7$ are the same as in Section 3.1.1, taking $\Gamma = 1$ in the conserved densities and fluxes.

For $\Gamma = 1$ and $\Delta = 0$, there are in total *nine* Noether symmetries, such as

$$\mathbf{X}_4 = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial U} \quad \text{with} \quad f^0 = U\,, \quad f^1 = 0\,, \tag{52}$$
$$\mathbf{X}_8 = 2\,t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}\,, \tag{53}$$
$$\mathbf{X}_9 = t^2\frac{\partial}{\partial t} + x\,t\,\frac{\partial}{\partial x} + \frac{1}{2}x^2\frac{\partial}{\partial U} \quad \text{with} \quad f^0 = x\,U\,, \quad f^1 = 0\,, \tag{54}$$

in addition to $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}_5, \mathbf{X}_6$, and $\mathbf{X}_7$. Here, the vector field $\mathbf{X}_8$ is the scaling symmetry in the t and x directions, while the vector field $\mathbf{X}_9$ represents a conformal symmetry. The conserved vector $\mathbf{J}_4$ for the above $\mathbf{X}_4$ is the same with the relation (51). Additionally, the Noether symmetries $\mathbf{X}_8$ and $\mathbf{X}_9$ have respective conserved vectors

$$\mathbf{J}_8 = 2\,t\,\mathbf{J}_1 + x\,\mathbf{J}_2\,, \tag{55}$$

$$\mathbf{J}_9 = t^2\mathbf{J}_1 + x\,t\,\mathbf{J}_2 - \frac{1}{2}x^2\mathbf{J}_0\,. \tag{56}$$

We note here that both $\mathbf{J}_8$ and $\mathbf{J}_9$ are the conserved vectors directly related to conservation of energy and momentum, respectively.

3.2. Lie Symmetries

The vector field $\mathbf{Y} = \xi^\mu \partial_{x^\mu} + \zeta^i\,\partial_{q^i}$ is called a Lie point symmetry of the differential equations E_i's ($i = 1,2,3$) provided in (8)–(10) if the following condition holds:

$$\mathbf{Y}^{[2]} E_i\,|_{E_i=0} = 0 \tag{57}$$

where

$$\mathbf{Y}^{[2]} = \mathbf{Y} + \zeta^i_\alpha\,\partial_{q^i_{,\alpha}} + \zeta^i_{\alpha\beta}\,\partial_{q^i_{,\alpha\beta}}$$

is the second prolongation of the vector field $\mathbf{Y}$. The coefficients ζ^i_α and $\zeta^i_{\alpha\beta}$ are provided by the expressions $\zeta^i_\alpha = D_\alpha \zeta^i - q^i_{,\beta} D_\alpha \xi^\beta$ and $\zeta^i_{\alpha\beta} = D_\beta \zeta^i_\alpha - q^i_{,\alpha\mu} D_\beta \xi^\mu$. For the Lie point symmetry $\mathbf{Y}$ of the differential equations E_i ($i = 1,2,3$), we can define the Lagrange system

$$\frac{dx^\mu}{\xi^\mu} = \frac{dq^i}{\zeta^i} = \frac{dq^i_{,\alpha}}{\zeta^i_{,\alpha}} = \frac{dq^i_{,\alpha\beta}}{\zeta^i_{,\alpha\beta}}\,, \tag{58}$$

the solution of which provides the characteristic functions that can be applied to reduce the number of dependent variables.

In this section, we briefly consider Lie point symmetries of the dispersive TDNLS system (8)–(10). For $K_1 = K_2 = K_3 = 0$, the Lie point symmetries of the latter system have been determined and studied extensively in [2]. It is obviously seen that if the $F_i(t)$s ($i = 1,2,3$) in (32)–(34) are constants, then the Lie and Noether symmetries coincide with each other for $K_1 = K_2 = K_3 = 0$. Furthermore, it is not necessary to take the K_is to be vanishing. In the following, we consider the vector fields $\mathbf{X}_1, \ldots, \mathbf{X}_7$ provided in (30)–(34) with $F_1 = F_2 = F_3 = 1$. For the first example of non-vanishing K_is, taking

$$K_1(t) = \frac{k_1}{t - t_0}\,, \quad K_2(t) = \frac{k_2}{t - t_0}\,, \quad K_3(t) = \frac{k_3}{t - t_0}\,, \tag{59}$$

we find the following Lie symmetries:

$$\mathbf{Y}_1 = \mathbf{X}_1 + \frac{1}{t - t_0}\left(k_1 \frac{\partial}{\partial U} + k_2 \frac{\partial}{\partial V} + k_3 \frac{\partial}{\partial W}\right), \qquad \mathbf{Y}_2 = \mathbf{X}_2\,, \tag{60}$$

$$\mathbf{Y}_3 = \mathbf{X}_3 + \ln(t - t_0)\left(-k_3 \frac{\partial}{\partial V} + k_2 \frac{\partial}{\partial W}\right), \tag{61}$$

$$\mathbf{Y}_4 = \mathbf{X}_4 + \frac{t_0}{t - t_0}\left(k_1 \frac{\partial}{\partial U} + k_2 \frac{\partial}{\partial V} + k_3 \frac{\partial}{\partial W}\right), \tag{62}$$

$$\mathbf{Y}_5 = \mathbf{X}_5\,, \qquad \mathbf{Y}_6 = \mathbf{X}_6\,, \qquad \mathbf{Y}_7 = \mathbf{X}_7\,, \tag{63}$$

where k_1, k_2, k_3 are the constant parameters. Second, for K_1, K_2 and K_3 provided by

$$K_1(t) = \frac{k_1}{t}\,, \quad K_2(t) = \frac{1}{t}[k_2 \cos(\Omega \ln t) + k_3 \sin(\Omega \ln t)]\,,\; K_3(t) = \frac{1}{t}[k_3 \cos(\Omega \ln t) - k_2 \sin(\Omega \ln t)]\,, \tag{64}$$

it is found that the Lie symmetries are

$$\mathbf{Y}_1 = \mathbf{X}_1 + K_1(t)\frac{\partial}{\partial U} + K_2(t)\frac{\partial}{\partial V} + K_3(t)\frac{\partial}{\partial W}, \qquad \mathbf{Y}_2 = \mathbf{X}_2, \tag{65}$$

$$\mathbf{Y}_3 = \mathbf{X}_3 + \frac{t}{\Omega}\left(K_2(t)\frac{\partial}{\partial V} + K_3(t)\frac{\partial}{\partial W}\right), \tag{66}$$

$$\mathbf{Y}_4 = \mathbf{X}_4 - 4\sin(\frac{\Omega}{2}\ln t)\left(\left[k_2\sin(\frac{\Omega}{2}\ln t) - k_3\cos(\frac{\Omega}{2}\ln t)\right]\frac{\partial}{\partial V} + \left[k_3\sin(\frac{\Omega}{2}\ln t) + k_2\cos(\frac{\Omega}{2}\ln t)\right]\frac{\partial}{\partial W}\right), \tag{67}$$

$$\mathbf{Y}_5 = \mathbf{X}_5, \qquad \mathbf{Y}_6 = \mathbf{X}_6, \qquad \mathbf{Y}_7 = \mathbf{X}_7, \tag{68}$$

where k_1, k_2, k_3 and Ω are the constant parameters. Third, if we take the functions K_1, K_2, and K_3 as

$$K_1(t) = k_1, \quad K_2(t) = k_2\cos(\Omega t) - k_3\sin(\Omega t), \; K_3(t) = k_3\cos(\Omega t) + k_2\sin(\Omega t), \tag{69}$$

it is found that the Lie symmetries are

$$\mathbf{Y}_1 = \mathbf{X}_1 + K_2(t)\frac{\partial}{\partial V} + K_3(t)\frac{\partial}{\partial W}, \qquad \mathbf{Y}_2 = \mathbf{X}_2, \quad \mathbf{Y}_3 = \mathbf{X}_3 + \frac{1}{\Omega}\left(K_2(t)\frac{\partial}{\partial V} + K_3(t)\frac{\partial}{\partial W}\right), \tag{70}$$

$$\mathbf{Y}_4 = \mathbf{X}_4 + 2t\left(k_1\frac{\partial}{\partial U} + K_2(t)\frac{\partial}{\partial V} + K_3(t)\frac{\partial}{\partial W}\right), \quad \mathbf{Y}_5 = \mathbf{X}_5, \qquad \mathbf{Y}_6 = \mathbf{X}_6, \qquad \mathbf{Y}_7 = \mathbf{X}_7. \tag{71}$$

Finally, when the functions K_1, K_2, and K_3 have the forms

$$K_1(t) = \frac{k_1}{(t-t_0)^n}, \quad K_2(t) = k_2\sin^n(\Omega t), \quad K_3(t) = k_3\cos^n(\Omega t), \tag{72}$$

we obtained the Lie point symmetries as follows:

$$\mathbf{Y}_1 = \mathbf{X}_1 + K_1(t)\frac{\partial}{\partial U} + n\,\Omega\left(\int K_2(t)\cot(\Omega t)dt\right)\frac{\partial}{\partial V} - \left(\int K_3(t)\tan(\Omega t)dt\right)\frac{\partial}{\partial W}, \tag{73}$$

$$\mathbf{Y}_2 = \mathbf{X}_2, \qquad \mathbf{Y}_3 = \mathbf{X}_3 - \left(\int K_3(t)dt\right)\frac{\partial}{\partial V} + \left(\int K_2(t)dt\right)\frac{\partial}{\partial W}, \tag{74}$$

$$\mathbf{Y}_4 = \mathbf{X}_4 + 2\,t\,K_1(t)\frac{\partial}{\partial U} + 2\left[\int K_2(t)(n\,\Omega\cot(\Omega t) + 1)dt\right]\frac{\partial}{\partial V} - 2\left[\int K_3(t)(n\,\Omega\tan(\Omega t) - 1)dt\right]\frac{\partial}{\partial W}, \tag{75}$$

$$\mathbf{Y}_5 = \mathbf{X}_5, \qquad \mathbf{Y}_6 = \mathbf{X}_6, \qquad \mathbf{Y}_7 = \mathbf{X}_7. \tag{76}$$

One can obviously find other examples of the Lie point symmetries for different non-vanishing functions $K_1(t), K_2(t)$, and $K_3(t)$ than those considered above.

4. Conclusions

This study uses the Noether symmetry approach directly with a gauge term to search Noether symmetries for the dispersive TDNLS system provided by (8)–(10). In Section 3.1.1 of Section 3, for $\Gamma \neq 1$ we obtain *seven* Noether symmetries for the latter system of dispersive TDNLS equations, which include the arbitrary functions $F_1(t), F_2(t)$, and $F_3(t)$. Therefore, while Noether symmetries $\mathbf{X}_1, \mathbf{X}_2$, and $\mathbf{X}_3$ are respectively associated with the energy conservation (translation in time), momentum conservation (translation in space), and helicity conservation (rotational invariance about the magnetic field), the invariance property of Noether symmetries $\mathbf{X}_5, \mathbf{X}_6$, and $\mathbf{X}_7$ depends on these arbitrary functions, such as translations in the U, V, and W directions if $F_i(t)$ ($i = 1,2,3$) are nonzero constants, or boost symmetries along the U, V, and W directions if the $F_i(t)$s are linear polynomial functions of time. Characterization of the symmetry $\mathbf{X}_4$ yields the finding that it is a combination of scaling symmetry in the x and t directions, boost symmetry in the x direction, and dilatation symmetry along the U direction with respect to the variable x.

In Section 3.1.2 we study Noether symmetries for $\Gamma = 1$, finding that there are seven Noether symmetries when $\Delta \neq 0$, with the Noether symmetry $\mathbf{X}_4$ differing from those in Section 3.1.1. For $\Gamma = 1$ and $\Delta = 0$, we find a total of nine Noether symmetries in Section 3.1.2. It should be noted that we computed the conserved densities J_a^0 and fluxes J_a^1 corresponding to the Noether symmetries using the Noether symmetry approach, unlike the conserved quantities using Lie point symmetries in [2]. The conserved quantities obtained in this study differ from those in [2] due to the existence of boundary (or gauge) terms f^μ for the Noether symmetries, which is the main goal of this study. Furthermore, in Section 3.2 we find some new Lie point symmetries of the dispersive TDNLS system (8)–(10) for non-vanishing functions $K_i(t)$'s ($i = 1, 2, 3$), in which there are seven Lie point symmetries in each case.

Considering the obtained Noether symmetries, the dispersive TDNLS system of equations in (8)–(10) plays an important role in the derivation of similarity solutions, which are represented in [2,3]. Therefore, we did not study these solutions in this paper.

Funding: This research received no external funding.

Data Availability Statement: The author confirms that the data supporting the findings of this study are available within the article.

Conflicts of Interest: The author declares no conflicts of interest.

Abbreviations

The following abbreviations are used in this manuscript:

MHD	Magnetohydrodynamic
DNLS	Discrete Nonlinear Schrödinger
TDNLS	Triple Degenerate DNLS System

References

1. Cramer, N.F. *The Physics of Alfvén Waves*; Wiley: Berlin, Germany, 2001.
2. Webb, G.M.; Brio, M.; Zank, G.P. Symmetries of the triple degenerate DNLS equations for weakly nonlinear dispersive MHD waves. *J. Plasma Phys.* **1995**, *54*, 201–244. [CrossRef]
3. Webb, G.M.; Brio, M.; Zank, G.P. Lie-Backlund symmetries of dispersionless, magnetohydrodynamic model equations near the triple umbilic point. *J. Phys. A Math. Gen.* **1996**, *29*, 5209–5240. [CrossRef]
4. Biskamp, D. *Nonlinear Magnetohydrodynamics*; Cambridge University Press: Cambridge, UK, 1993.
5. Qadir, A.; Camci, U. Physical Significance of Noether Symmetries. *Symmetry* **2022**, *14*, 476. [CrossRef]
6. Anderson, R.L.; Ibragimov, N.H. *Lie-Bäcklund Transformations in Applications*; SIAM: Philadelphia, PA, USA, 1979; pp. 81–82.
7. Hada, T. Evolution of large amplitude Alfvén waves in the solar wind with $\beta \sim 1$. *Geophys. Res. Lett.* **1993**, *20*, 2415. [CrossRef]