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Analysis of a First-Order Delay Model under a History Function with Discontinuity

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Abstract: This paper analyzes the first-order delay equation $y'(t) = \alpha y(t) + \beta y(t-\tau)$ subject to a history function in addition to an initial condition that assumes discontinuity at $t = 0$. The method of steps is successfully applied to derive the exact solution in an explicit form. In addition, a unified formula is provided to describe the solution in any finite sub-interval of the problem's domain. The characteristics and properties of the solution are theoretically investigated and then confirmed through several plots. The behavior of the solution and its derivative are examined and interpreted. The results show that the method of steps is an effective method of solution to treat the current delay model. The present successful analysis can be used to investigate other delay models with complex initial conditions. Furthermore, the present approach can be generalized to include the inhomogeneous version of the current model without using numerical methods.

Keywords: delay differential equations; initial value problems; exact solution; method of step

MSC: 34K06; 34K07; 65L03

1. Introduction

The field of delay differential equations (DDEs) is fundamental for describing many real world problems in several areas of applied and life sciences [1,2]. There are two types of delay parameters in a given DDE, namely, pure delay parameters and proportional delay parameters. Usually, we refer to τ in the DDE $y'(t) = \alpha y(t) + \beta y(t-\tau)$ as a pure delay parameter, while γ arises in the pantograph equation (PE) $y'(t) = \alpha y(t) + \beta y(\gamma t)$, $0 < \gamma < 1$ and is often called a proportional delay parameter [3–5]. PE is used in practical applications, including railway electrification [6–8], the dynamic behaviour of contact systems in electric railways [9], and current collection in electric locomotive [10] (see also Refs. [11–13] for some studies on PE). Another proportional delay parameter $1/q$ arises in the Ambartsumian equation (AE) $y'(t) = -y(t) + \frac{1}{q} y\left(\frac{t}{q}\right)$, $(q > 1)$, which has been used to describe the surface brightness in the Milky Way [14–18].

There are many essential differences between DDEs and ordinary differential equations (ODEs). The main difference between the initial value problems (IVPs) described by DDEs and ODEs lies in the type of initial conditions (ICs) provided along with the method of solution.The ICs for IVPs with ODEs are usually provided at initial points, while the ICs for IVPs with DDEs are defined in certain intervals in terms of the delay parameter. Also, there are many well-known standard methods of solution to solve linear IVPs governed by ODEs of first order or higher orders. For such types of IVPs, the exact solutions can be determined using various standard methods. In addition, such standard methods are capable of obtaining the exact solutions that are valid in the entire domains of the given IVPs. However, the situation is different when analyzing IVPs described by DDEs, where the solution of a given problem is to be determined in a finite number of sub-intervals, as will be demonstrated later. The method of steps (MOS) and the method of characterization (MOC) are two familiar standard methods to solve IVPs governed by DDEs. In order to

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apply the MOS on a DDE, the domain of the problem must be divided into sub-intervals so that the solution in a certain interval facilitates the solution in the subsequent interval. Thus, MOS must be applied to solve the given DDE in a finite number of sub-intervals. In some limited cases, the solution can be described by a unified formula in these sub-intervals.

It is important to refer to a solution where a DDE is very sensitive to the given ICs. This means that the solution of the DDE $y'(t) = \alpha y(t) + \beta y(t-\tau)$ under the IC $y(t) = \phi(t),\ -\tau \le t \le 0$ is completely different than the solution of the same DDE under the ICs $y(t) = \phi(t),\ -\tau \le t < 0$ and $y(0) = \lambda$, where $\phi(t)$ is a given function.

The former problem has been well studied in many text books and several papers for different choices of the function $\phi(t)$. However, the second problem defined above still needs some efforts to treat it analytically or numerically, even for the simplest case $\phi(t) = 0$. So, the main motivation and objective of this paper is to provide the exact solution of the delay model:

$$y'(t) = \alpha y(t) + \beta y(t-\tau),\ y(t) = 0\ \forall\ -\tau \le t < 0,\ y(0) = \lambda,\ \tau \ge 0, \tag{1}$$

where α, β, and λ are real constants. The Adomian decomposition method (ADM) [19–23], the homotopy perturbation method [24,25], and the Laplace transform (LT) [26–29] may perhaps encounter some difficulties when solving the current model.

Although ADM has been effectively applied in Ref. [30] to solve several delay problems, the structure of these models differs compared with the present one. From Ref. [30], one can see that the considered examples ignored the history functions as initial data, in contrast with the current paper. Also, it can be noticed from the above published work that ADM is an efficient method for solving DDEs when only the initial conditions are considered. For declaration, the examples from [30] have been taken in the forms $y'(t) = \frac{1}{2}e^{t/2}y(t/2) + \frac{1}{2}y(t),\ y(0) = 1,\ y''(t) = \frac{3}{4}y(t) + y(t/2) - t^2 + 2,\ y(0) = 0,$ $y'(0) = 0,\ y'''(t) = -y(t) - y(t-0.3) + e^{-t+0.3},\ y(0) = 1,\ y'(0) = -1,\ y''(0) = 1$, and $y'(t) = 1 - 2y^2(t/2),\ y(0) = 0$. As can be seen from these examples, only initial conditions have been taken into account, while the history functions were ignored. This simply explains the main differences in both the structure and the initial data between the present delay model and those examples in Ref. [30]. In this context, the effectiveness and applicability of ADM to handle DDEs, subjected to history functions, were not really checked in this reference. On the other hand, MOS is a well-known standard method for solving DDEs governed by initial conditions including history functions.

Recently, Lessard et al. [31] developed a rigorous implicit Chebyshev integrator to analyze the first-order nonlinear systems of DDEs $y'(t) = f(y(t), y(t-\tau))$ with the aide of MOS, where f is continuous in both variables. Their approach was based on Chebyshev series expansions under a past history in the interval $[-\tau, 0]$. They implemented Chebyshev series to discretize the problem and approximately solve it using a standard numerical scheme corrected via Newton's method. The numerical method applied by Lessard et al. [31] was found to be effective and efficient, especially for solving the Mackey–Glass equation. In addition, Mayorga et al. [32] introduced an exact numerical scheme to solve the first-order linear neutral DDE $y'(t) - \gamma y'(t-\tau) = \alpha y(t) + \beta y(t-\tau),\ t > 0$, subject to a general initial function $\phi(t)$ in the interval $[-\tau, 0]$. They expressed the numerical value of $y(t+h)$ as a function of $y(t)$ and previous values. The resulting expression included all previous τ-lagged values, i.e., $y(t-k\tau),\ k \ge 1$, and an integral term with the initial function $\phi(t)$, (see Equstion (18) in Ref. [32]). Despite the distinct efforts made by the above authors to numerically solve DDE (1), no available explicit analytical solutions have been presented. In view of the above discussion, the difference between the current work and Refs. [31,32] is obvious regarding the provided initial condition. This difference includes the discontinuity at $t = 0$ for $\lambda \ne 0$. However, it will be declared later that only the trivial solution is continuous at $t = 0$ if and only if $\lambda = 0$. Further, a collocation approach to solve a neutral fractional delay stochastic differential equation was presented by Banihashemi et al. [33], while the numerical treatment of a fractional order system of nonlinear stochastic delay differential equations was analyzed by He et al. [34].

Obtaining an exact solution for a delay equation is not an easy task. Also, the exact solution for any mathematical model is optimal when available. So, the solution for problem (1) will be directly obtained in this paper by applying MOS through a clear/simple analysis. Moreover, a unified explicit formula is to be constructed for the exact/analytic solution in any sub interval of the domain of the problem. Also, the properties of the solution are to be discussed theoretically and graphically.

2. Analysis

The method of steps (MOS) is used in this section to derive the solution of the present delay model. Normally, MOS is based on dividing the domain of the problem into a finite number of intervals where the solution in any interval depends mainly on the solution in the previous interval. Accordingly, the solution is to be obtained sequentially through a unified formula. The following theorem determines the explicit form of the solution in any sub-interval. Then, some properties about the behavior of the solution and its derivative will be theoretically proven and discussed in a subsequent section.

Theorem 1. *The solution $y_n(t)$ in the interval $I_n = [(n-1)\tau, n\tau)$ of problem (1) is provided by*

$$y_n(t) = \lambda \sum_{k=0}^{n-1} \frac{\beta^k}{k!}(t-k\tau)^k e^{\alpha(t-k\tau)}, \quad n \geq 1, \tag{2}$$

or

$$y_n(t) = \lambda e^{\alpha t} \sum_{k=0}^{n-1} \frac{\left(\beta e^{-\alpha\tau}\right)^k}{k!}(t-k\tau)^k,\ t \in I_n,\ n \geq 1. \tag{3}$$

Proof. Assume that $y_0(t)$ is the solution in the interval $I_0 = [-\tau, 0)$. Let us define $I_1 = [0, \tau)$, then $-\tau \leq t-\tau < 0$, i.e., $t-\tau \in I_0$. This yields $y(t-\tau) = y_0(t-\tau) = 0$ $\forall\, t \in I_1 = [0, \tau)$. Accordingly, the solution $y_1(t)$ in the interval I_1 is governed by the IVP:

$$y_1'(t) = \alpha y_1(t), \quad y_1(0) = \lambda, \quad t \in I_1 = [0, \tau), \tag{4}$$

which has the solution:

$$y_1(t) = \lambda e^{\alpha t}, \quad t \in I_1 = [0, \tau). \tag{5}$$

In the interval $I_2 = [\tau, 2\tau)$, we have $0 \leq t-\tau < \tau$, which implies $t-\tau \in I_1$. Therefore, $y(t-\tau) = y_1(t-\tau) = \lambda e^{\alpha(t-\tau)}$ $\forall\, t \in I_2$. Thus, the solution $y_2(t)$ in the interval I_2 is subjected to the IVP:

$$y_2'(t) = \alpha y_2(t) + \beta\lambda e^{\alpha(t-\tau)}, \quad y_2(\tau) = y_1(\tau) = \lambda e^{\alpha\tau}, \quad t \in I_2 = [\tau, 2\tau). \tag{6}$$

One can easily solve problem (6) to find that

$$y_2(t) = \lambda e^{\alpha t} + \beta\lambda(t-\tau)e^{\alpha(t-\tau)}, \quad t \in I_2 = [\tau, 2\tau). \tag{7}$$

Let $I_3 = [2\tau, 3\tau)$, then $\tau \leq t-\tau < 2\tau$, i.e., $t-\tau \in I_2$. Hence, $y(t-\tau) = y_2(t-\tau) = \lambda e^{\alpha(t-\tau)} + \beta\lambda(t-2\tau)e^{\alpha(t-2\tau)}$ $\forall\, t \in I_3$. So, the solution $y_3(t)$ in the interval I_3 is governed by the IVP:

$$y_3'(t) = \alpha y_3(t) + \beta y_2(t-\tau), \quad y_3(2\tau) = y_2(2\tau) = \lambda e^{2\alpha\tau} + \beta\lambda\tau e^{\alpha\tau}, \quad t \in I_3 = [2\tau, 3\tau), \tag{8}$$

or

$$y_3'(t) = \alpha y_3(t) + \beta\lambda e^{\alpha(t-\tau)} + \beta^2\lambda(t-2\tau)e^{\alpha(t-2\tau)}, \quad y_3(2\tau) = \lambda e^{2\alpha\tau} + \beta\lambda\tau e^{\alpha\tau}, \quad t \in I_3 = [2\tau, 3\tau). \tag{9}$$

Problem (9) has the solution:

$$y_3(t) = \lambda e^{\alpha t} + \beta\lambda(t-\tau)e^{\alpha(t-\tau)} + \frac{1}{2}\beta^2\lambda(t-2\tau)^2 e^{\alpha(t-2\tau)}, \quad t \in I_3 = [2\tau, 3\tau). \tag{10}$$

Through induction, one can obtain the n-component of the solution in the interval $I_n = [(n-1)\tau, n\tau)$, i.e., $y_n(t)$, as

$$y_n(t) = \lambda \sum_{k=0}^{n-1} \frac{\beta^k}{k!}(t-k\tau)^k e^{\alpha(t-k\tau)}, \quad n \geq 1, \tag{11}$$

which can be placed in form (3) and, hence, the proof is completed. □

Lemma 1. *At $\alpha = 0$, the solution reduces to a polynomial of degree $n-1$, provided as*

$$y_n(t) = \lambda \sum_{k=0}^{n-1} \frac{1}{k!}(\beta(t-k\tau))^k, \ t \in I_n = [(n-1)\tau, n\tau), \ n \geq 1, \tag{12}$$

for the reduced DDE:

$$y'(t) = \beta y(t-\tau), \ y(t) = 0 \ \forall \ -\tau \leq t < 0, \ y(0) = \lambda, \ \tau \geq 0. \tag{13}$$

Proof. The proof follows immediately by setting $\alpha = 0$ in the result of Theorem 1. □

3. Properties of the Solution

This section is devoted to introducing some properties for the solution and its derivatives. Before launching to the main purpose of this section, it may be reasonable to rewrite the solution $y_n(t)$ in interval I_n in the following form:

$$y_n(t) = \lambda \sum_{k=0}^{n-1} \frac{\beta^k}{k!}(t-t_k)^k e^{\alpha(t-t_k)}, \quad n \geq 1, \tag{14}$$

where $t_k = k\tau$.

Theorem 2. *Assuming that $\lambda \neq 0$, the derivative of the solution for problem (1) is discontinuous at $t = t_n = n\tau$ for $n = 0, 1$ and continuous $\forall \ n \geq 2$.*

Proof. At $t = 0$, the left derivative is determined from (1) as $y'_-(0) = 0$, while the right derivative can be evaluated from the solution in the interval $I_1 = [0, \tau)$, i.e., Equation (5), as $y'_+(0) = \alpha\lambda$. Hence, $y'_-(0) \neq y'_+(0)$ and, thus. the derivative is discontinuous at $t = t_0 = 0$ if either $\alpha \neq 0$ or $\lambda \neq 0$. Similarly at $t = \tau$, one can find that $y'_-(\tau) = \alpha\lambda e^{\alpha\tau}$ and $y'_+(\tau) = \alpha\lambda e^{\alpha\tau} + \beta\lambda$. Accordingly, $y'_-(\tau) \neq y'_+(\tau)$ and, consequently, the derivative is discontinuous at $t = t_1 = \tau$ if either $\beta \neq 0$ or $\lambda \neq 0$. The above analysis reveals that the derivative is discontinuous at $t = t_n$ for $n = 0, 1$.

From (14), we have

$$y'_n(t) = \lambda \sum_{k=0}^{n-1} \frac{\beta^k}{k!}\left(\alpha(t-t_k)^k e^{\alpha(t-t_k)} + k(t-t_k)^{k-1} e^{\alpha(t-t_k)}\right), \quad n \geq 1, \tag{15}$$

or

$$y'_n(t) = \lambda \sum_{k=0}^{n-1} \frac{\beta^k}{k!}\alpha(t-t_k)^k e^{\alpha(t-t_k)} + \lambda \sum_{k=1}^{n-1} \frac{\beta^k}{(k-1)!}(t-t_k)^{k-1} e^{\alpha(t-t_k)}, \tag{16}$$

which is equivalent to

$$y'_n(t) = \frac{\lambda\alpha\beta^{n-1}}{(n-1)!}(t-t_{n-1})^{n-1}e^{\alpha(t-t_{n-1})} + \lambda\sum_{k=0}^{n-2}\frac{\beta^k}{k!}\Big[\alpha(t-t_k)^k e^{\alpha(t-t_k)} + \beta(t-t_{k+1})^k e^{\alpha(t-t_{k+1})}\Big]. \tag{17}$$

Also, we have

$$y'_{n+1}(t) = \frac{\lambda\alpha\beta^{n}}{n!}(t-t_{n})^{n}e^{\alpha(t-t_{n})} + \lambda\sum_{k=0}^{n-1}\frac{\beta^k}{k!}\Big[\alpha(t-t_k)^k e^{\alpha(t-t_k)} + \beta(t-t_{k+1})^k e^{\alpha(t-t_{k+1})}\Big]. \tag{18}$$

From (17) and (18), one can obtain the left and right derivative at $t = t_n$ as

$$y'_-(t_n) = y'_n(t_n) = \frac{\lambda\alpha\beta^{n-1}}{(n-1)!}(t_n-t_{n-1})^{n-1}e^{\alpha(t_n-t_{n-1})} + \lambda\sum_{k=0}^{n-2}\frac{\beta^k}{k!}\Big[\alpha(t_n-t_k)^k e^{\alpha(t_n-t_k)} + \beta(t_n-t_{k+1})^k e^{\alpha(t_n-t_{k+1})}\Big], \tag{19}$$

and

$$y'_+(t_n) = y'_{n+1}(t_n) = \lambda\sum_{k=0}^{n-1}\frac{\beta^k}{k!}\Big[\alpha(t_n-t_k)^k e^{\alpha(t_n-t_k)} + \beta(t_n-t_{k+1})^k e^{\alpha(t_n-t_{k+1})}\Big], \tag{20}$$

respectively. The last two equations lead to

$$y'_-(t_n) = y'_+(t_n), \quad \forall\, n \geq 2, \tag{21}$$

which indicates that the derivative is continuous at every point $t = t_n = n\tau\ \forall\, n \geq 2$, this completes the proof. □

4. Behavior of the Solution

This section aims to explore the behavior of the solution and its derivative through graphical representation. It will be shown that the preceding theoretical results for the properties of the solution and its derivative can be validated/confirmed via several plots in the domain of a finite number of intervals. To achieve this target, the first five intervals for the domain of the problem will be considered as an example. However, the plots of the solution and its derivative can be represented in any finite number of intervals, as desired through the obtained analytical solution in the previous sections.

At $\alpha = 0$, Figures 1 and 2 display the representation of the solution $y(t)$ in the first five intervals $[(n-1)\tau, n\tau)$, $n = 1,2,3,4,5$ at $\lambda = 1$, $\beta = 1$, $\tau = 1$ and at $\lambda = 1$, $\beta = -1$, $\tau = 1$, respectively. In these figures, the dots in black connect the considered five intervals. The continuity of the solution $y(t)$ at $t = \tau, 2\tau, 3\tau, 4\tau$ is obvious in Figures 1 and 2. However, it should be noted that $y(t)$ is discontinuous at $t = 0$ due to the initial conditions provided.

At $\alpha \neq 0$ ($\alpha = 2$, positive value), Figures 3 and 4 show the curves of $y(t)$ at $\lambda = 1$, $\beta = 1$, $\tau = 1$ and at $\lambda = 1$, $\beta = -1$, $\tau = 1$, respectively. These figures indicate that the behavior of the solution is of exponential growth. For $\alpha \neq 0$ ($\alpha = -2$, negative value), Figures 5 and 6 represent the curves of $y(t)$ at $\lambda = 1$, $\beta = 1$, $\tau = 1$ and at $\lambda = 1$, $\beta = -1$, $\tau = 1$, respectively. These figures reveal that the behavior of the solution follows oscillation with an observable decay.

Regarding the continuity of the derivative $y'(t)$, Figures 7–10 depict the behavior of the derivative at different sets of parameters values. Moreover, these figures confirm our theoretical results, where $y'(t)$ is discontinuous at $t = \tau = 1$, while $y'(t)$ is continuous at $t = n\tau = n\ \forall\, n \geq 2$.

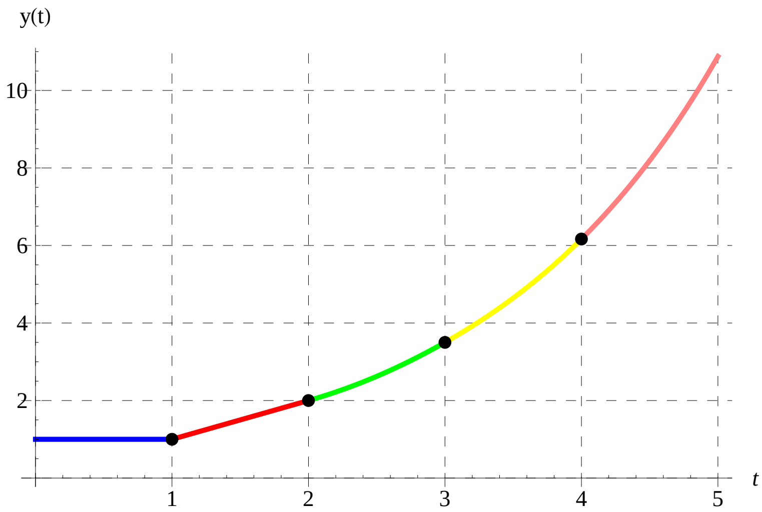

Figure 1. Representation of $y(t)$ in the first five intervals $[(n-1)\tau, n\tau)$, $n = 1,2,3,4,5$ at $\lambda = 1$, $\alpha = 0$, $\beta = 1$, and $\tau = 1$.

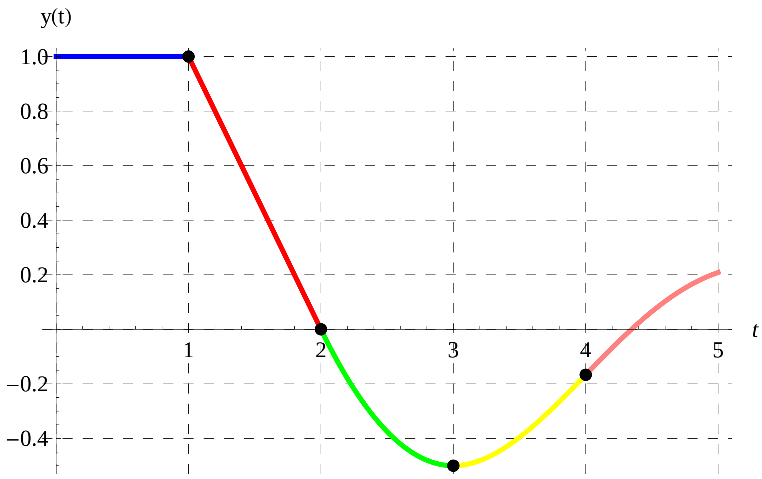

Figure 2. Representation of $y(t)$ in the first five intervals $[(n-1)\tau, n\tau)$, $n = 1,2,3,4,5$ at $\lambda = 1$, $\alpha = 0$, $\beta = -1$, and $\tau = 1$.

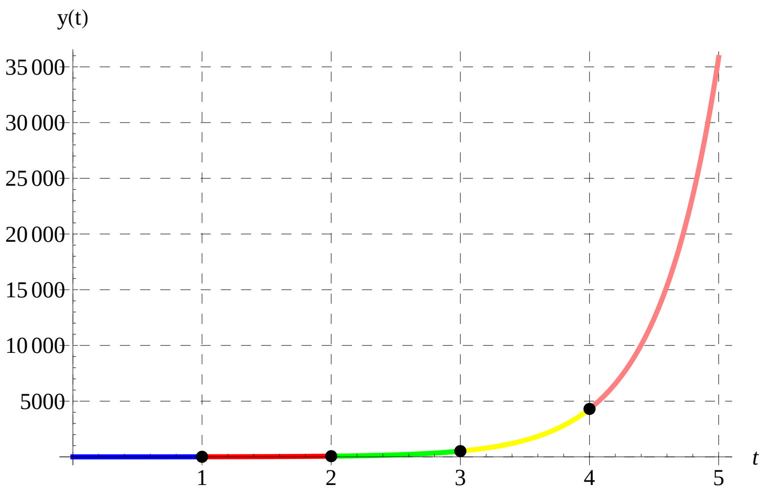

Figure 3. Representation of $y(t)$ in the first five intervals $[(n-1)\tau, n\tau)$, $n = 1,2,3,4,5$ at $\lambda = 1$, $\alpha = 2$, $\beta = 1$, and $\tau = 1$.

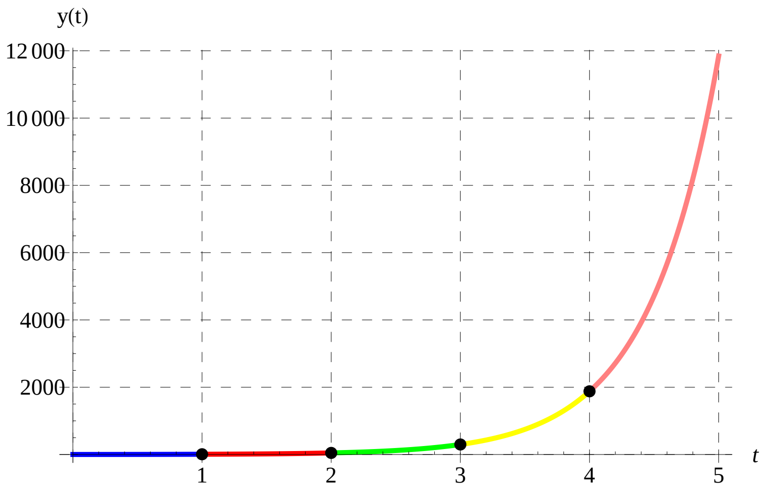

Figure 4. Representation of $y(t)$ in the first five intervals $[(n-1)\tau, n\tau)$, $n = 1, 2, 3, 4, 5$ at $\lambda = 1$, $\alpha = 2$, $\beta = -1$, and $\tau = 1$.

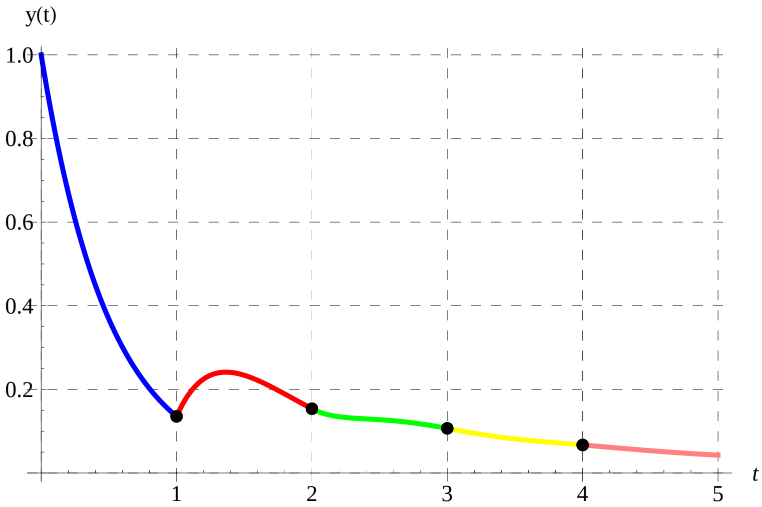

Figure 5. Representation of $y(t)$ in the first five intervals $[(n-1)\tau, n\tau)$, $n = 1, 2, 3, 4, 5$ at $\lambda = 1$, $\alpha = -2$, $\beta = 1$, and $\tau = 1$.

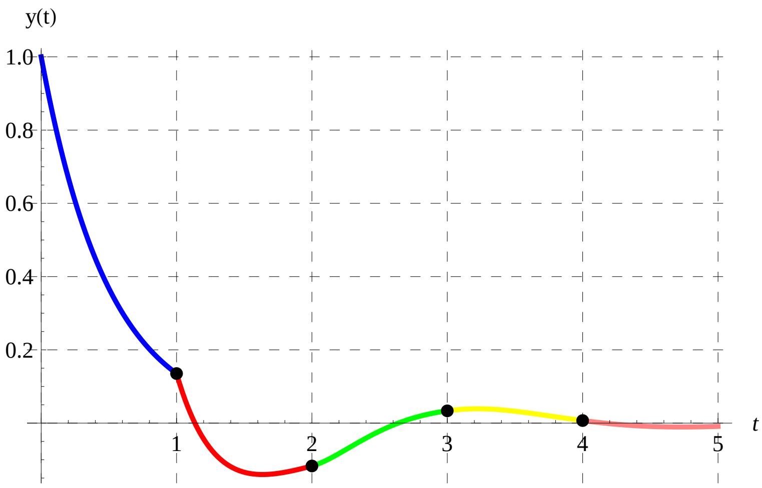

Figure 6. Representation of $y(t)$ in the first five intervals $[(n-1)\tau, n\tau)$, $n = 1, 2, 3, 4, 5$ at $\lambda = 1$, $\alpha = -2$, $\beta = -1$, and $\tau = 1$.

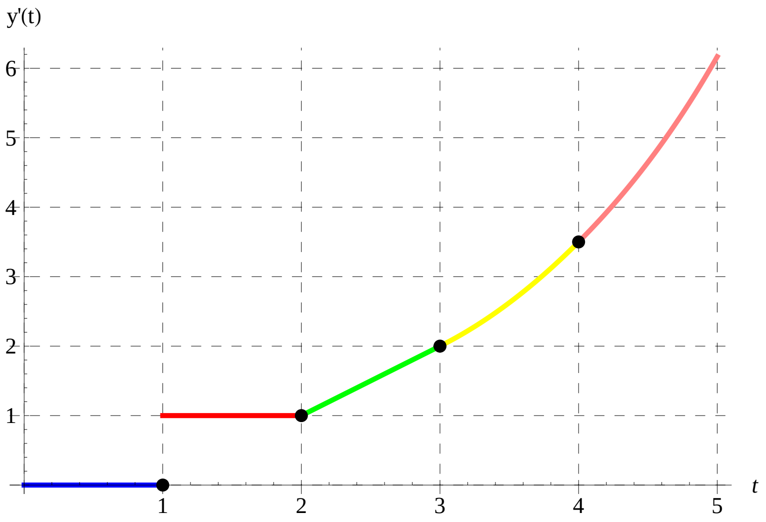

Figure 7. Representation of $y'(t)$ in the first five intervals $[(n-1)\tau, n\tau)$, $n = 1, 2, 3, 4, 5$ at $\lambda = 1$, $\alpha = 0$, $\beta = 1$, and $\tau = 1$.

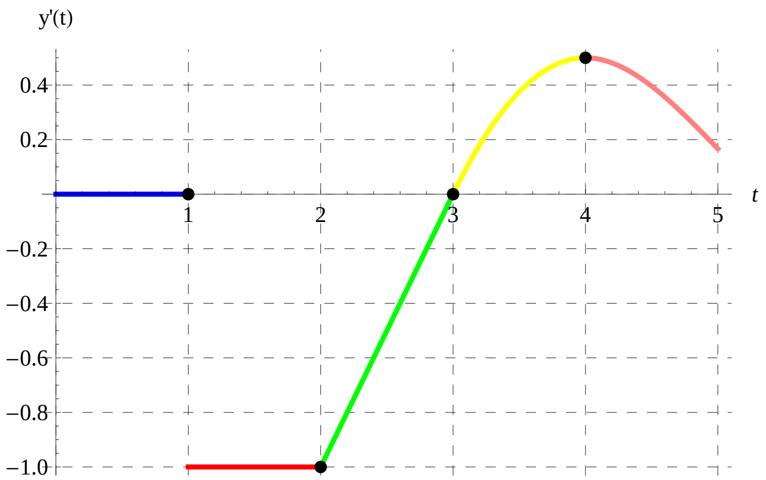

Figure 8. Representation of $y'(t)$ in the first five intervals $[(n-1)\tau, n\tau)$, $n = 1, 2, 3, 4, 5$ at $\lambda = 1$, $\alpha = 0$, $\beta = -1$, and $\tau = 1$.

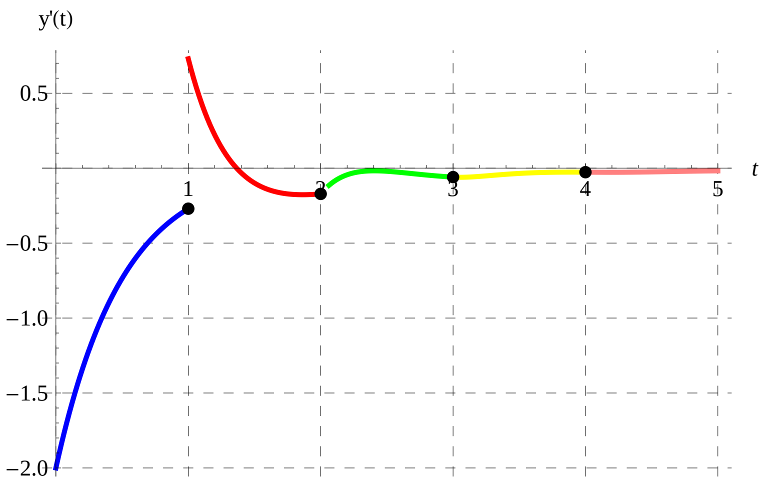

Figure 9. Representation of $y'(t)$ in the first five intervals $[(n-1)\tau, n\tau)$, $n = 1, 2, 3, 4, 5$ at $\lambda = 1$, $\alpha = -2$, $\beta = 1$, and $\tau = 1$.

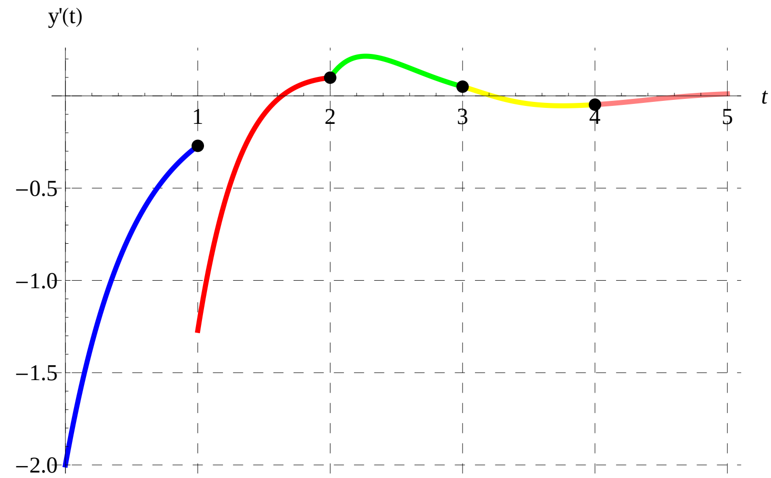

Figure 10. Representation of $y'(t)$ in the first five intervals $[(n-1)\tau, n\tau)$, $n = 1, 2, 3, 4, 5$ at $\lambda = 1$, $\alpha = -2$, $\beta = -1$, and $\tau = 1$.

5. Conclusions

A delay model with prescribed initial conditions is investigated in this paper. The present model was precisely solved in an explicit form by means of the method of steps (MOS). MOS was found to be effective and efficient at treating the problem. Furthermore, a unified formula was successfully obtained that provided the solution in any sub-interval for the domain of the problem. A theoretical analysis was introduced to examine the characteristics/properties of the solution. Such characteristics/properties were confirmed through several graphs. Further, the behavior of the solution and its derivative were analyzed and interpreted. In view of the obtained results, MOS can be viewed as a direct, simple, and effective approach to deal with the current delay model. Indeed, the proposed analysis can be generalized by including other delay models containing complex initial conditions or by addressing the general inhomogeneous class $y'(t) = \alpha y(t) + \beta y(t-\tau) + h(t)$, $y(t) = \phi(t)\ \forall\ -\tau \le t < 0,\ y(0) = \lambda,\ \tau \ge 0$, where $\phi(t)$ and $h(t)$ are arbitrary functions. This suggested class can be effectively analyzed in future work.

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References

1. Kuang, Y. *Delay Differential Equations: With Applications in Population Dynamics*; Academic Press: Cambridge, MA, USA, 1993.
2. Smith, H. *Introduction to Delay Differential Equations with Applications to the Life Sciences*; Springer Science+Business Media: Berlin, Germany, 2010; Volume 57.
3. El-Zahar, E.R.; Ebaid, A. Analytical and Numerical Simulations of a Delay Model: The Pantograph Delay Equation. *Axioms* **2022**, *11*, 741. [CrossRef]
4. Alenazy, A.H.S.; Ebaid, A.; Algehyne, E.A.; Al-Jeaid, H.K. Advanced Study on the Delay Differential Equation $y'(t) = ay(t) + by(ct)$. *Mathematics* **2022**, *10*, 4302. [CrossRef]
5. Albidah, A.B.; Kanaan, N.E.; Ebaid, A.; Al-Jeaid, H.K. Exact and Numerical Analysis of the Pantograph Delay Differential Equation via the Homotopy Perturbation Method. *Mathematics* **2023**, *11*, 944. [CrossRef]
6. Andrews, H.I. Third paper: Calculating the behaviour of an overhead catenary system for railway electrification. *Proc. Inst. Mech. Eng.* **1964**, *179*, 809–846. [CrossRef]
7. Gilbert, G.; Davtcs, H.E.H. Pantograph motion on a nearly uniform railway overhead line. *Proc. Inst. Electr. Eng.* **1966**, *113*, 485–492. [CrossRef]
8. Caine, P.M.; Scott, P.R. Single-wire railway overhead system. *Proc. Inst. Electr. Eng.* **1969**, *116*, 1217–1221. [CrossRef]
9. Abbott, M.R. Numerical method for calculating the dynamic behaviour of a trolley wire overhead contact system for electric railways. *Comput. J.* **1970**, *13*, 363–368. [CrossRef]
10. Ockendon, J.; Tayler, A.B. The dynamics of a current collection system for an electric locomotive. *Proc. R. Soc. Lond. A Math. Phys. Eng. Sci.* **1971**, *322*, 447–468.

11. Fox, L.; Mayers, D.; Ockendon, J.R.; Tayler, A.B. On a functional differential equation. *IMA J. Appl. Math.* **1971**, *8*, 271–307. [CrossRef]
12. Kato, T.; McLeod, J.B. The functional-differential equation $y'(x) = ay(\lambda x) + by(x)$. *Bull. Am. Math. Soc.* **1971**, *77*, 891–935.
13. Iserles, A. On the generalized pantograph functional-differential equation. *Eur. J. Appl. Math.* **1993**, *4*, 1–38. [CrossRef]
14. Ambartsumian, V.A. On the fluctuation of the brightness of the milky way. *Dokl. Akad Nauk USSR* **1994**, *44*, 223–226.
15. Patade, J.; Bhalekar, S. On analytical solution of Ambartsumian equation. *Natl. Acad. Sci. Lett.* **2017**, *40*, 291–293. [CrossRef]
16. Bakodah, H.O.; Ebaid, A. Exact solution of Ambartsumian delay differential equation and comparison with Daftardar-Gejji and Jafari approximate method. *Mathematics* **2018**, *6*, 331. [CrossRef]
17. Khaled, S.M.; El-Zahar, E.R.; Ebaid, A. Solution of Ambartsumian delay differential equation with conformable derivative. *Mathematics* **2019**, *7*, 425. [CrossRef]
18. Kumar, D.; Singh, J.; Baleanu, D.; Rathore, S. Analysis of a fractional model of the Ambartsumian equation. *Eur. Phys. J. Plus* **2018**, *133*, 133–259. [CrossRef]
19. Adomian, G. *Solving Frontier Problems of Physics: The Decomposition Method*; Kluwer Acad: Boston, MA, USA, 1994.
20. Wazwaz, A.M. Adomian decomposition method for a reliable treatment of the Bratu-type equations. *Appl. Math. Comput.* **2005**, *166*, 652–663. [CrossRef]
21. Duan, J.S.; Rach, R. A new modification of the Adomian decomposition method for solving boundary value problems for higher order nonlinear differential equations. *Appl. Math. Comput.* **2011**, *218*, 4090–4118. [CrossRef]
22. Alshaery, A.; Ebaid, A. Accurate analytical periodic solution of the elliptical Kepler equation using the Adomian decomposition method. *Acta Astronaut.* **2017**, *140*, 27–33. [CrossRef]
23. Li, W.; Pang, Y. Application of Adomian decomposition method to nonlinear systems. *Adv. Differ. Equ.* **2020**, *2020*, 67. [CrossRef]
24. Ebaid, A. Remarks on the homotopy perturbation method for the peristaltic flow of Jeffrey fluid with nano-particles in an asymmetric channel. *Comput. Math. Appl.* **2014**, *68*, 77–85. [CrossRef]
25. Ayati, Z.; Biazar, J. On the convergence of Homotopy perturbation method. *J. Egypt. Math. Soc.* **2015**, *23*, 424–428. [CrossRef]
26. Khaled, S.M. The exact effects of radiation and joule heating on Magnetohydrodynamic Marangoni convection over a flat surface. *Therm. Sci.* **2018**, *22*, 63–72. [CrossRef]
27. Khaled, S.M. Exact solution of the one-dimensional neutron diffusion kinetic equation with one delayed precursor concentration in Cartesian geometry. *AIMS Math.* **2022**, *7*, 12364–12373. [CrossRef]
28. Alrebdi, R.; Al-Jeaid, H.K. Accurate Solution for the Pantograph Delay Differential Equation via Laplace Transform. *Mathematics* **2023**, *11*, 2031. [CrossRef]
29. Alrebdi, R.; Al-Jeaid, H.K. Two Different Analytical Approaches for Solving the Pantograph Delay Equation with Variable Coefficient of Exponential Order. *Axioms* **2024**, *13*, 229. [CrossRef]
30. Evans, D.J.; Raslan, K.R. The Adomian decomposition method for solving delay differential equation. *Int. J. Comput. Math.* **2005**, *82*, 49–54. [CrossRef]
31. Lessard, J.P.; James, J.M. A Rigorous implicit C^1 Chebyshev integrator for delay equations. *J. Dyn. Differ. Equ.* **2021**, *33*, 1959–1988. [CrossRef]
32. Mayorga, C.J.; Castro, M.Á.; Sirvent, A.; Rodríguez, F. On the construction of exact numerical schemes for linear delay models. *Mathematics* **2023**, *11*, 1836. [CrossRef]
33. Banihashemi, S.; Jafari, H.; Babaei, A. A stable collocation approach to solve a neutral delay stochastic differential equation of fractional order. *J. Comput. Appl. Math.* **2022**, *403*, 113845. [CrossRef]
34. He, L.; Banihashemi, S.; Jafari, H.; Babaei, A. Numerical treatment of a fractional order system of nonlinear stochastic delay differential equations using a computational scheme. *Chaos Solitons Fractals* **2021**, *149*, 111018. [CrossRef]