Mathematical and Computational Applications, Vol. 9, No. 3, pp. 409-416, 2004.

ON THE K_r -CORE OF COMPLEX SEQUENCES AND THE ABSOLUTE EQUIVALENCE OF SUMMABILITY MATRICES

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Abstract- The K_r-core of a complex valued sequence has been introduced in [5]. In this paper, we have determined a class of matrices such that K_r - core (Ax) $\subseteq K_r$ - core (x) hold for all $x \in l_\infty$. Also; we have defined a new type of absolute equivalence, r - absolute equivalence, and characterized these type of matrices.
Key words- Sequence spaces, absolute equivalence, matrix transformations and core theorems.

1. INTRODUCTION

Let A be an infinite matrix of complex entries a_{nk} ($n, k \in N$, the set of natural numbers) and $x = (x_k)$ be a sequence of complex numbers. Then $Ax = \{(Ax)_n\}$ is called the A transform of x, if $(Ax)_n = \sum_k a_{nk}x_k$ converges for each n. For two sequence spaces X and Y we say that $A \in (X, Y)$ if $Ax \in Y$ for each $x \in X$. If X and Y are equipped with the limits X-lim and Y-lim, respectively, and if $A \in (X, Y)$ and $Y\text{-}lim_n (Ax)_n = X\text{-}lim_k\, x_k$ for all $x \in X$, then we say A regularly transforms X into Y and write $A \in (X, Y)_{reg}$. The matrix $A \in (c, c)_{reg}$ is said to be regular and the conditions of regularity are well-known, [4, pp. 4], where c is the space of all convergent complex sequences.

The regular matrices A and B are said to be absolutely equivalent on l_∞, the space of all bounded complex sequences, [4, pp. 97] if $lim\,(Ax - Bx) = 0$, (i.e., Ax and Bx have the same limit or neither of them goes to a limit but their difference goes to zero). It is also well-known [4, pp. 105] that the regular matrices A and B are absolutely equivalent on l_∞ if and only if

$$\lim_n \sum_k | a_{nk} - b_{nk} | = 0 .$$

Let us define, for any real number r, the matrix $A^r = (a^r_{nk})$ by

$$a^r_{nk} = \begin{cases} \dfrac{1+r^k}{1+n}, k \le n \\ 0 \quad\quad , k > n. \end{cases}$$

In [3], it is shown that the matrix A^r is regular for $0< r <1$ and it is stronger than the Cesáro matrix defined by

$$c_{nk} = \begin{cases} \dfrac{1}{1+n}, k \le n \\ 0 \quad\quad , k > n. \end{cases}$$

The β–dual space of a sequence space X is defined by

$$X^{\beta} = \left\{ (a_k) : \sum_k a_k x_k \, converges \, for all x \in X \right\}$$

and by X_A we mean the set of sequences such that $Ax \in X$, i.e.,

$$X_A = \{ (x_k) : Ax \in X \}.$$

In [2], the new sequence space a_c^r is defined by c_{A^r} and it is shown that

$$(a_c^r)^{\beta} = \left\{ (a_k) : \sum_k \left| \Delta\left(\frac{a_k}{1+r^k} \right)(k+1) \right| < \infty \, and \, \left\{ \frac{a_k}{1+r^k} \right\} \in cs \right\},$$

where cs is the space of all convergent series. Note that if a sequence $x \in a_c^r$, then we write a_c^r-$lim\ x$ exists. Also, the sequence space a_∞^r is introduced as $(l_\infty)_{A^r}$ and it is established that

$$(a_\infty^r)^{\beta} = \left\{ (a_k) : \sum_k \left| \Delta\left(\frac{a_k}{1+r^k} \right)(k+1) \right| < \infty \, and \, \left\{ \frac{1+k}{1+r^k} a_k \right\} \in c_0 \right\},$$

where c_0 is the space of all null sequences.

Let us write

$$t_n^r(x) = A^r(x) = \frac{1}{1+n} \sum_{k=0}^{n} (1+r^k) x_k$$

and H_n be the least closed convex hull containing $t_n^r, t_{n+1}^r, t_{n+2}^r, \ldots$. In [5], K_r-core of a complex sequence x is defined by the intersection of all H_n. Also, it is shown that

$$K_r - core(x) = \bigcap_{z \in C} G_x(z)$$

for any $x \in l_\infty$, where $G_x(z) = \{ w \in C : | w - z | \leq \limsup_n | t_n^r(x) - z | \}$ and C is the set of all complex numbers.

In the present paper, we have determined the necessary and sufficient conditions on a matrix A for which K_r-$core\ (Ax) \subseteq K_r$-$core\ (x)$ for all $x \in l_\infty$. Also, we have introduced a new type of absolute equivalence, r-absolute equivalence, and characterized the r-absolutely equivalent matrices.

2. THE INCLUSION THEOREM

Firstly, we shall quote some lemmas which will be useful to our proofs.

Lemma 2.1 [5, Lemma 2.1]. $A \in (l_\infty, a_c^r)$ *if and only if*

(2.1) $$\| A \|_r = \sup_n \sum_k | \tilde{a}_{nk} | < \infty$$

(2.2) $$\lim_n \tilde{a}_{nk} = \alpha_k \ for\ each\ k,$$

(2.3) $$\lim_n \sum_k | \tilde{a}_{nk} - \alpha_k | = 0,$$

where

$$\widetilde{a}_{nk} = \frac{1}{1+n}\sum_{j=0}^{n}(1+r^k)a_{jk}\,,\ (n,\,k \in N).$$

Lemma 2.2. $A \in (a^r_\infty, c)$ *if and only if*

(2.4) $$\lim_k \frac{1+k}{1+r^k}a_{nk} = 0\,,\ \textit{for each } n,$$

(2.5) $$\sup_n \sum_k |\,t_{nk}\,| < \infty\,,$$

(2.6) $$\lim_n t_{nk} = \alpha_k \ \textit{for each } k,$$

(2.7) $$\lim_n \sum_k |\,t_{nk} - \alpha_k\,| = 0\,,$$

where $t_{nk} = \Delta\left(\dfrac{a_{nk}}{1+r^k}\right)(k+1) = \left(\dfrac{a_{nk}}{1+r^k} - \dfrac{a_{n,k+1}}{1+r^{k+1}}\right)(k+1)$.

Next lemma is a special case of the Corollary 5.5 of [2] for $s = 1$.

Lemma 2.3. $A \in (a^r_c, c)_{reg}$ *if and only if the conditions (2.5) and (2.6) holds for* $\alpha_k = 0$ *and*

(2.8) $$\left\{\frac{a_{nk}}{1+r^k}\right\}_{k\in N} \in cs\,,\ (\mathrm{n} \in \mathrm{N})$$

(2.9) $$\lim_n \sum_k t_{nk} = 1\,.$$

Lemma 2.4. $A \in (\mathrm{a}^r_c,\ \mathrm{a}^r_c)_{\mathrm{reg}}$ *if and only if the condition (2.8) of Lemma 2.3 holds and*

(2.10) $$\sup_n \sum_k \left|\Delta\widetilde{a}_{nk}\right| < \infty$$

(2.11) $$\lim_n \Delta\widetilde{a}_{nk} = 0 \ \textit{for each } k,$$

(2.12) $$\lim_n \sum_k \Delta\widetilde{a}_{nk} = 1$$

where

$$\Delta\widetilde{a}_{nk} = \Delta\left(\frac{\widetilde{a}_{nk}}{1+r^k}\right)(k+1) = \left(\frac{\widetilde{a}_{nk}}{1+r^k} - \frac{\widetilde{a}_{n,k+1}}{1+r^{k+1}}\right)(k+1)\,.$$

Proof. Let $x \in a^r_c$ and consider the equality

(2.13)
$$\frac{1}{1+n}\sum_{j=0}^{n}(1+r^j)\sum_{k=0}^{m}a_{jk}x_k = \sum_{k=0}^{m}\frac{1}{1+n}\sum_{j=0}^{n}(1+r^j)a_{jk}x_k = \sum_{k=0}^{m}d_{nk}x_k\,,\ (m, n \in N)$$

which yields for $m \to \infty$ that

$$\frac{1}{1+n}\sum_{j=0}^{n}(1+r^j)(Ax)_j = (Dx)_n\,,\ (n \in N)\cdot$$

where $D = (d_{nk})$ defined by

$$d_{nk} = \begin{cases} \frac{1}{1+n}\sum_{j=0}^{n}(1+r^j)a_{jk}\,, & (0 \le \mathrm{k} \le \mathrm{n}) \\ 0\,, & (\mathrm{k} > \mathrm{n}). \end{cases}$$

Furthermore, since the spaces a_c^r and c are linearly isomorphic (see [2]), we deduce from that $A \in (a_c^r, a_c^r)_{reg}$ if and only if $D \in (a_c^r, c)_{reg}$. Therefore, the necessary and sufficient conditions are obtained from the Lemma 2.3 by replacing the entries of matrix A by those of the matrix D.

Lemma 2.5. $A \in (a_\infty^r, a_c^r)$ *if and only if the conditions (2.4) and (2.10) hold and*

(2.14) $\lim_n \Delta\tilde{a}_{nk} = \alpha_k$ *for each k,*

(2.15) $\lim_n \sum_k |\Delta\tilde{a}_{nk} - \alpha_k| = 0.$

Proof. For $x \in a_\infty^r$, by (2.13), one can easily see that $A \in (a_\infty^r, a_c^r)$ if and only if $D \in (a_\infty^r, c)$. Hence, the proof follows from Lemma 2.2.

Following is a Steinhauss type theorem.

Lemma 2.6. *The classes* $(a_\infty^r,\ a_c^r)$ *and* $(\mathrm{a}_\mathrm{c}^\mathrm{r},\ \mathrm{a}_\mathrm{c}^\mathrm{r})_{\mathrm{reg}}$ *are disjoint.*

Proof. Suppose, if possible, there exists a matrix A belonging to the two classes. Then, Lemma 2.4 and 2.5 implies that

$$\lim_n \sum_k |\Delta\tilde{a}_{nk}| = 0\,.$$

But since

$$\left|\sum_k \Delta\tilde{a}_{nk}\right| \le \sum_k |\Delta\tilde{a}_{nk}|,$$

$\lim_n \sum_k \Delta\tilde{a}_{nk} = 0$ which contradicts to the condition (2.12). This completes the proof.

Now, we may give our main theorem.

Theorem 2.7. *Let* $A \in (a_c^r, a_c^r)_{reg}$. *Then,* K_r*-core* $(Ax) \subseteq K_r$*-core* (x) *for all* $x \in l_\infty$ *if and only if*

(2.16) $$\lim_n \sum_k |\Delta\tilde{a}_{nk}| = 1.$$

Proof (Necessity). Suppose that the condition (2. 16) does not hold. Then,

$$\lim_n \sum_k |\Delta\tilde{a}_{nk}| > 1.$$

The conditions (2.10)-(2.12) allow us to choose two strictly increasing sequences $\{n_i\}$ and $\{k(n_i)\}$ $(i = 1, 2, \ldots)$ of positive integers such that

$$\sum_{k=0}^{k(n_{i-1})} |\Delta\tilde{a}_{n_i,k}| < \frac{1}{4}, \qquad \sum_{k=k(n_{i-1})+1}^{k(n_i)} |\Delta\tilde{a}_{n_i,k}| > 1 + \frac{1}{2}$$

and

$$\sum_{k=k(n_i)+1}^{\infty} |\Delta\tilde{a}_{n_i,k}| < \frac{1}{4}.$$

Now, let us define a sequence $x = (x_k)$ by $x_k = sign\Delta\tilde{a}_{n_i,k}$, $k(n_{i-1}) + 1 \le k < k(n_i)$.

Then, since A_r is regular, $\limsup_k t_k^r(x) \le 1$. Therefore,

$$K_r - core(x) \subseteq \{w \in C : |w| \le 1\}.$$

Also,

$$|t_{n_i}^r(Ax)| = \left|\sum_k^{\infty} \Delta\tilde{a}_{n_i,k}\right|$$

$$\geq \sum_{k=k(n_{i-1})}^{k(n_i)} |\Delta\tilde{a}_{n_i,k}| - \sum_{k=0}^{k(n_{i-1})} |\Delta\tilde{a}_{n_i,k}| - \sum_{k=k(n_i)+1}^{\infty} |\Delta\tilde{a}_{n_i,k}|$$

$$> 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{4} = 1.$$

Since $A \in (\mathrm{a}_c^r, \mathrm{a}_c^r)_{reg}$, it follows that $\{t_{n_i}^r(Ax)\}$ is bounded and hence $\{t_n^r(Ax)\}$ has a subsequence whose a_c^r-limit cannot be in $\{w \in C : |w| \leq 1\}$. This is a contradiction with the fact that K_r-*core* $(Ax) \subseteq K_r$-*core* (x). Thus, (2.16) must be hold.

(Sufficiency). Let $w \in K_r$-*core* (Ax). Then, for any given $z \in C$, we can write

$$|w-z| \leq \limsup_n |t_n^r(Ax) - z| \tag{2.17}$$
$$= \limsup_n \left|\sum_k \tilde{a}_{nk} x_k - z\right|$$
$$\leq \limsup_n \left|\sum_k \tilde{a}_{nk}(x_k - z)\right| + \limsup_n |z| \left|\sum_k \tilde{a}_{nk} - 1\right|$$
$$= \limsup_n \left|\sum_k \tilde{a}_{nk}(x_k - z)\right|.$$

Now, let $\limsup_k |t_k^r(x) - z| = l$. Then, for any $\varepsilon > 0$ there exists an increasing sequence (k_s) of positive integers such that, $|t_{k_s}^r(x) - z| \leq l + \varepsilon$ whenever $k_s \geq k_0$. Hence, one can write

$$\left|\sum_k \tilde{a}_{nk}(x_k - z)\right| = \left|\sum_k \Delta\tilde{a}_{nk}(t_k^r(x) - z)\right| \tag{2.18}$$
$$= \left|\sum_{k<k_0} \Delta\tilde{a}_{nk}(t_k^r(x) - z) + \sum_{k \geq k_0} \Delta\tilde{a}_{nk}(t_k^r(x) - z)\right|$$
$$\leq \sup_k |z - t_k^r(x)| \sum_{k<k_0} |\Delta\tilde{a}_{nk}| + (l + \varepsilon) \sum_k |\Delta\tilde{a}_{nk}|.$$

Therefore, applying the operator $limsup_n$ to (2.18) and using the hypothesis with (2.17), we have

$$|w\text{-}z| \leq \limsup_n \left|\sum_k \tilde{a}_{nk}(x_k - z)\right| \leq l + \varepsilon.$$

This means that $w \in K_r\text{-}core(x)$ and the proof is completed.

Our next theorem is an application of the Lemma 2.6 and Theorem 2.7. In this theorem we consider the real bounded sequences. In that case, the K_r-*core* of a sequence x is the closed interval $[\liminf_k t_k^r(x),\ \limsup_k t_k^r(x)]$.

Theorem 2.8. *Let B be a matrix in the class* $(\mathrm{a}_\mathrm{c}^\mathrm{r},\ \mathrm{a}_\mathrm{c}^\mathrm{r})_{\mathrm{reg}}$ *satisfying the condition (2.16). Then, there is no matrix A such that* $\limsup_n t_n^r(Ax) \leq \liminf_n t_n^r(Bx)$ *for all* $x \in l_\infty$.

Proof. Suppose, if possible, there exists such a matrix A. Theorem 2.7 implies that

$$\limsup_n t_n^r(Bx) \leq \limsup_k t_k^r(x),$$

and so

$$\limsup_n t_n^r(Ax) \leq \limsup_k t_k^r(x),$$

whence $A \in (\mathrm{a}_\mathrm{c}^\mathrm{r},\ \mathrm{a}_\mathrm{c}^\mathrm{r})_{\mathrm{reg}}$. By the Lemma 2.6, there exists a $z \in l_\infty$ such that

$$\liminf_n t_n^r(Az) < \limsup_n t_n^r(Az).$$

On the other hand, since

$$\limsup_n t_n^r(Ax) \leq \limsup_n t_n^r(Bx) \text{ for all } x \in l_\infty,$$

we have

$$\liminf_n t_n^r(Bz) \leq \liminf_n t_n^r(Az).$$

Thus,

$$\liminf_n t_n^r(Bz) \leq \limsup_n t_n^r(Ax) \leq \liminf_n t_n^r(Bz)$$

contradicts to the fact $B \in (\mathrm{a}_\mathrm{c}^\mathrm{r},\ \mathrm{a}_\mathrm{c}^\mathrm{r})_{\mathrm{reg}}$. This proves the theorem.

3. *R* - ABSOLUTE EQUIVALENCE

In this section, we introduced and characterized r-absolutely equivalence matrices.

Definition 3.1. *The matrix A, B* $\in (\mathrm{a}_\mathrm{c}^\mathrm{r},\ \mathrm{a}_\mathrm{c}^\mathrm{r})_{\mathrm{reg}}$ *is said to be r-absolutely equivalent on* l_∞ *if* $\lim t_n^r(Ax - Bx) = 0$ *for all* $x \in l_\infty$.

Theorem 3.2. *The matrices A, B* $\in (\mathrm{a}_\mathrm{c}^\mathrm{r},\ \mathrm{a}_\mathrm{c}^\mathrm{r})_{\mathrm{reg}}$ *are r-absolutely equivalent on* l_∞ *if and only if*

(3.1)
$$\lim_n \sum_k |\tilde{a}_{nk} - \tilde{b}_{nk}| = 0.$$

Proof. (Necessity). Let A and B be r-absolutely equivalent on l_∞. Then, clearly the matrix $D = (d_{nk})$ defined by $d_{nk} = (a_{nk} - b_{nk})$ is in the class (l_∞, a_0^r). Therefore, the necessity of the condition (3.1) follows from a result of Lemma 2.1.

(Sufficiency). Let the condition (3.1) hold and $x \in l_\infty$. In this case, we have

$$|Ax - Bx| = \left|\sum_k (\tilde{a}_{nk} - \tilde{b}_{nk}) x_k\right|$$
$$\leq \|x\| \sum_k |\tilde{a}_{nk} - \tilde{b}_{nk}|,$$

which by (3.1) implies the r-absolute equivalence of A and B.

4. ACKNOWLEDGEMENT

We would like to express our great thanks to Professor Feyzi BAŞAR, İnönü University, Faculty of Education, Department of Mathematics Malatya- Turkey and the referee for their valuable suggestions improving the paper.

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