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FREENESS CONDITIONS OF TENSORS AND COPRODUCTS OF GROUPS

Ali Mutlu

Department of Mathematics, Faculty of Science and Arts, University of Celal Bayar
Muradiye Campus 45035 Manisa, Turkey. ali.mutlu@bayar.edu.tr

Abstract-Using a free simplicial group with given a CW-basis, we give an alternative description of the top group of the free square complex and 2-crossed complex of groups on 2-construction data in terms of tensors and coproducts of crossed module of groups. Therefore the functors are given between these categorical constructions.

Keywords- Free crossed square, square complex, 2-crossed complex, free simplicial group.

1. INTRODUCTION

Ellis [7] gave a description of a free crossed square of groups of a CW-complex using topological methods. In recent work we use Ellis construction of free crossed square of 2-skeleton data of free simplicial group with given CW-basis using algebraic methods in [9, 10]. For groups, 2-crossed modules have been defined by Conduché in [5].

Combining earlier work [8] of the author with our joint paper [9, 10] and [11], one starts to see how a study of links between simplicial groups and classical construction of homotopy group can be strengthened by interposing crossed algebraic models for homotopy types of simplicial groups. In this note, we continue this process using these methods to give description in terms of tensor products, of the top corner of a free crossed squared of groups and the top term of the corresponding free 2-crossed complex. The methods are slightly different, but these results are 2-dimensional analogues of the description of the free crossed module on a 'presentation' of a group.

We end up this article looking at consequence of two functors which amongst simplicial group and square complex and 2-crossed complexes of the structures of algebraic topology.

It will help to have the notion of simplicial resolution second dimensional Peiffer normal subgroup. We will present a precise description of the construction of a free crossed square by using the second dimensional Peiffer normal subgroup. To do this we need to recall the 2-dimensional construction for a free simplicial group in Ellis notation. This 1-dimensional form can be illustrated by the diagram

$\mathbf{F}^{(1)}$:

$$\ldots\; F(s_1s_0(\mathbf{B_1}) \cup s_0(\mathbf{B_2}) \cup s_1(\mathbf{B_2})) \underset{s_1, s_0}{\overset{d_0, d_1, d_2}{\rightrightarrows}} F(s_0(\mathbf{B_1}) \cup (\mathbf{B_2})) \underset{s_0}{\overset{d_1, d_0}{\rightrightarrows}} F(\mathbf{B_1})$$

and 2- dimensional form can be pictured by the diagram

$\mathbf{F}^{(2)}$:

$$\ldots F(s_1s_0(\mathbf{B_1}) \cup s_0(\mathbf{B_2}) \cup s_1(\mathbf{B_2}) \cup \mathbf{B_3}) \underset{s_1,s_0}{\overset{d_0,d_1,d_2}{\rightrightarrows}} F(s_0(\mathbf{B_1}) \cup (\mathbf{B_2})) \underset{s_0}{\overset{d_1,d_0}{\rightrightarrows}} F(\mathbf{B_1})$$

with the simplicial identities given in [10]. Here $\mathbf{B}_2 = \{B_i\}$ and $\mathbf{B}_3 = \{B'_i\}$ are finite sets and take F to be the free group and $H = F(\mathbf{B}_1)/\{b_i\}$ as an $F(\mathbf{B}_1)$-free group. Through in this paper the category *SimpGrp* of simplicial group will be denoted.

2. SQUARED COMPLEXES

The author, Arvasi and Porter have defined n-crossed complexes in [1]. In this paper, we will only need the case $n = 2$, which had already been defined by Ellis in [7]. There is clearly a category *SqComp* of squared complexes. This exists in both group and groupoid based versions.

By a *(totally) free squared complex*, we will mean one in which the crossed square is (totally) free, and in which each C_n is free as a π_0-module for $i \geq 3$.

Proposition 2.1. *There is a functor* $C(\ \ ,2): SimpGrp \longrightarrow SqComp$ *such that free simplicial groups are sent to totally free squared complexes.*

Proof: Let $\mathbf{F}$ is a free simplicial group or groupoid with given CW-basis. We will define a squared complex $C(\mathbf{F},2)$ by specifying $C(\mathbf{F},2)_A$ for each $A \subseteq <2>$ and for $n \geq 3$, $C(\mathbf{F},2)_n$. As usual, (cf. the other papers in this series, [9, 10, 11 and 12]), we will denote by D_n the subgroup or subgroupoid of NF_n generated by the degenerate elements.

For $A \subset <2>$, we define

$$C(\mathbf{F},2)_A = M(\mathbf{F},2)_A = \frac{\cap\{\mathrm{Ker}d_i^2 : i \in A\}}{d_3(\mathrm{Ker}d_0^3 \cap \bigcap\{\mathrm{Ker}d_{i+1}^3 : i \in A\} \cap D_3)}.$$

We do not need to define μ_i and the h-maps relative to these groups as they are already defined in the crossed square $M(\mathbf{F},2)$.

For $n \geq 3$, we set

$$C(\mathbf{F},2)_n = \frac{NF_n}{(NF_n \cap D_n)d_{n+1}(NF_{n+1} \cap D_{n+1})}.$$

As this is part of the crossed complex associated to $\mathbf{F}$, we can take the structure maps to be those of that crossed complex, (cf. [6,10]). The terms are all modules over the corresponding π_0 as is easily checked. The final missing piece, ∂_3 of the structure is induced by the differential ∂_3 of NF.

The axioms for a squared complex can now be verified using the known results for crossed squares and for crossed complexes with a direct verification of those axioms relating to the interaction of the two parts of the structure, much as in [6] and [10].

Now suppose the simplicial group is free. The proof above of the freeness of

$M(\mathbf{F},2)$ together with the freeness of the crossed complex of a free simplicial group, [10], now completes the proof. +

Suppose that ρ is a general squared complex. The *homotopy groups* $\pi_n(\rho)$, $n \geq 0$ of ρ are defined cf. [7], to be the homology groups of the complex

$$\cdots \xrightarrow{\partial_5} C_4 \xrightarrow{\partial_4} C_3 \xrightarrow{\partial_2} L \xrightarrow{\partial_2} M \rtimes N \xrightarrow{\partial_1} P \longrightarrow 1$$

with $\partial_2(l) = (\lambda' l^{-1}, \lambda l)$ and $\partial_1(m,n) = \mu(m)\mu'(n)$. The axioms of a crossed square guarantee that ∂_2 and ∂_1 are homomorphism with $\partial_3(C_3)$ normal in $\text{Ker}(\partial_2), \partial_2(L)$ normal in $\text{Ker}(\partial_1)$, and $\partial_1(MN)$ normal in P.

Proposition 2.2. *The homotopy groups of $C(\mathbf{F},2)$ are isomorphic to those of* $\mathbf{F}$ *itself.*

Proof: Again this is a consequence of well-known results on the two parts of the structure. +

3. ALTERNATIVE FREENESS CONDITION

In the context of CW-complexes, Ellis in [7] gave a neat description of the top group L in a (totally) free crossed square derived from that data. A simplicial group with a given CW-basis is the algebraic analogue of a CW-complex so one would expect a similar result to hold in that setting. Ellis uses the generalised van Kampen theorem of Brown and Loday, [4] and in the algebraic setting no such tool is available, but in fact its use is not needed.

Ellis description is in terms of tensor products and coproducts. For completeness we recall the background definitions of these constructions (see [3, 4, 7 and 11]).

Proposition 3.1. ([7]) *Let $(L, M, \overline{M}, M \circ F)$ be a (totally) free crossed square on the 2-dimensional construction data or on functions (f_2, f_3) as described above (see [7, 11]). Let $\partial : C \to M \circ F$ be the free crossed module on the function $\mathbf{B}_3 \to M \circ F$ given by $y \mapsto (f_3 y, 1)$. Note that the image of ∂ lies in $M \cap \overline{M}$. From the crossed module $M \otimes \overline{M} \to M \circ F$, then L is isomorphic to the coproduct $(M \otimes \overline{M}) \circ C$ factored by the relations*

1) $i(\partial c \otimes \overline{m}) = j(c)\, j({}^{\overline{m}}c^{-1})$

2) $i(m \otimes \partial c) = j({}^{m}c)\, j(c^{-1})$

for $c \in C, m \in M$ and *$\overline{m} \in \overline{M}$.*

The homomorphisms $L \to M$ and $L \to \overline{M}$ are given by the homomorphisms $\lambda : M \otimes \overline{M} \to M$ and *$\lambda' : M \otimes \overline{M} \to \overline{M}$ and $\partial : C \to M \cap \overline{M}$. The h-map of the crossed square is given by $h(m, \overline{n}) = i(m \otimes \overline{n})$ for $m \in M$ and $\overline{n} \in \overline{M}$.*

Proof: This comes by direct verification using the universal properties of tensors and coproducts. +

Remark:

For future applications it is again important to note that the result is not dependent on the crossed square being *totally* free, although this is the form proved and used by Ellis in [7]. If $M \to F$ is any pre-crossed module, one can form the 'corner'

$$\begin{array}{ccc} & & M \\ & & \downarrow \\ \overline{M} & \longrightarrow & M \rtimes F, \end{array}$$

complete it to a crossed square via $M \otimes \bar{M}$ and then add in $\mathbf{B}_3 \to M$. Nowhere does this use freeness of $M \to F$.

Corollary 3.2. *Let* $\mathbf{F}^{(1)}$ *be the* 1*-skeleton of a free simplicial group with given CW-basis. Given the free crossed square* $M(\mathbf{F}^{(1)},2)$ *described above, then*

$$NF_2^{(1)}/\partial_3 NF_3^{(1)} \cong \mathrm{Ker} d_1^{(1)} \otimes \mathrm{Ker} d_0^{(1)}.$$

Proof: In the 1-skeleton of a free simplicial group with given CW-basis $\mathbf{F}^{(1)}$, the set $\mathbf{B}_3$ is empty. Thus it is clear from the previous proposition. +

Remark:

If we set $M = \mathrm{Ker} d_0^{(1)} = NF_1^{(1)}$ and $\bar{M} = \mathrm{Ker} d_1^{(1)}$, then the identification given by Corollary 2.2 gives $NF_2^{(1)}/\partial_3 NF_3^{(1)} \cong M \otimes \bar{M}$. This uses the fact that $\mathrm{Ker} d_0^{(1)}$ and $\mathrm{Ker} d_1^{(1)}$ are linked via the map sending m to $m s_0 d_1 m^{-1}$ for $m \in \mathrm{Ker} d_0^{(1)}$. The h-map $h: M \times \bar{M} \to NF_2^{(1)}/d_3^3 NF_3^{(1)}$ is $h(x,y) = [s_1 x, s_1 y s_0 y^{-1}]\, d_3^3 NF_3^{(1)}$, but this is also $h(x,y) = x \otimes y$. Thus $x \otimes y = [s_1 x, s_1 y s_0 y^{-1}]\, d_3^3 NF_3^{(1)}$ under the identification via the isomorphism of Corollary 3.2.

4. APPLICATIONS TO 2-CROSSED COMPLEXES

Of course there are similar results for free squared complexes. What is less obvious is the way in which these results can be applied to the situation that we studied in our earlier papers, [9, 10 and 11]. There we considered the alternative model for 3-types given by Conduché's 2-crossed modules and also looked at the corresponding 2-crossed complexes.

We refer the reader to [5] or [11] for the exact meaning of 2-crossed module. Note that for any 2-crossed module, $L \xrightarrow{\partial_2} M \xrightarrow{\partial_1} N$, $K = \mathrm{Ker}\partial_2$ is abellian, since $L \xrightarrow{\partial_2} M$ is a crossed module. The notion of morphism for 2-crossed complexes should be clear. Morphism will be a morphism of graded groups restricting to a morphism of 2-crossed modules on the bottom three terms and compatible with the action. This will give us a category, $X_2 - Comp$ of 2-crossed complexes and morphisms between them.

A 2-crossed complex C will be said to be *free* if for $n \geq 3$, the $C_0/\partial C_1$ – modules, C_n are free and the 2-crossed module at the base is a free 2-crossed module. It will be *totally free* if in addition the base 2-crossed module is totally free.

Proposition 4.1. *There is a functor* $C^{(2)}: SimpGrp \to X_2 - Comp$.

Proof: Given a free simplicial group(oid) with given CW-basis, $\mathbf{F}$, define

$$C_n = \begin{cases} NF_n & \text{for } n = 0,1 \\ NF_2 / d_3(NF_3 \cap D_3) & \text{for } n = 2 \\ NF_n /(NF_n \cap D_n) d_{n+1}(NF_{n+1} \cap D_{n+1}) & \text{for } n \geq 3 \end{cases}$$

with ∂_n induced by the differential of $\mathbf{NF}$. Note that the bottom three terms (for $n = 0,1$ and 2) form a 2-crossed module considered in [5] or [11] and that for $n \geq 3$, the groups are all $\pi_0(F)$-modules, since in these dimensions C_n is the same as the corresponding crossed complex term (cf. Ehlers and Porter [6] for instance). The only thing remaining is to check that $\partial_2 \partial_3$ is trivial which is straightforward. +

Proposition 4.2. ([11]) *With the above structure (C_n, ∂_n) is a 2-crossed complex, which will be denoted $C(\mathbf{F})$.* +

Here we note in particular that the term $C^{(2)}(\mathbf{F})_2$ is $NF_2 / d_3(NF_3 \cap D_3)$ and so is the same as $C(\mathbf{F},2)_{<2>}$. Thus if $\mathbf{F}$ is a simplicial group, we obtain the following result.

Corollary 4.3. *Let $\mathbf{F}^{(1)}$ be the 1-skeleton of a simplicial group with given CW-basis. The 2-crossed complex of $\mathbf{F}^{(1)}$ satisfies $C^{(2)}(\mathbf{F}^{(1)})_2 \cong \mathrm{Kerd}_1^{(1)} \otimes \mathrm{Kerd}_0^{(1)}$.* +

We also get in general a description of $C(\mathbf{F}^{(2)})_2$ as a quotient of the form $(\mathrm{Kerd}_1^{(1)} \otimes \mathrm{Kerd}_0^{(1)}) \circ C / \square$ where as in Proposition 3.1, this C is a free crossed module on the 'new cells' in dimension 2.

Lemma 4.4. *If $\mathbf{F}$ is a simplicial resolution of D then $C^{(2)}(\mathbf{F}^{(2)})_k$ is a free D-group on the given data.*

To summarise we have the following theorem.

Theorem 4.5. *The step-by-step construction of a simplicial resolution of a group with given CW-basis, D gives a step-by-step construction of a 2-crossed resolution of D via the 2-crossed construction $C^{(2)}$.*

We recall from [13] the functor $M(-,n): SimpGrp \to Crs^n$ from the category *SimpGrp* of simplicial group to that of crossed $n-$cubes. If for $n = 2$ we apply this functor to the 1-skeleton $\mathbf{F}^{(1)}$ of free simplicial group with given CW-basis $\mathbf{F}$, we get $M(\mathbf{F}^{(1)}, 2)$

this is the free crossed square

$$\begin{array}{ccc} NF_2^{(1)}/\partial_3 NF_3^{(1)} & \xrightarrow{\partial_2} & \mathrm{Kerd}_0^{(1)} \\ \downarrow{\scriptstyle \partial_2'} & & \downarrow{\scriptstyle \mu} \\ \mathrm{Kerd}_1^{(1)} & \xrightarrow[\mu']{} & F_1^{(1)} \end{array}$$

together with the $h-$map

$$h: \mathrm{Kerd}_1^{(1)} \times \mathrm{Kerd}_0^{(1)} \to NF_2^{(1)} / \partial_3 NF_3^{(1)}$$

given by

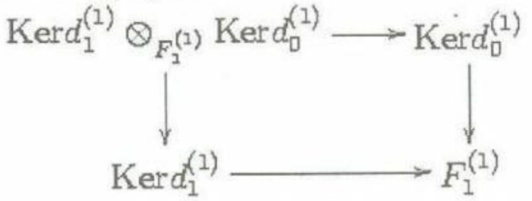
$$x \otimes y = h(x,y) = [s_1 x, s_1 y s_0 y^{-1}]\, \partial_3 NF_3^{(1)}.$$

Corollary 4.6. *For any simplicial group $\mathbf{F}$, if $\mathbf{F}$ equals to its 1-skeleton, so that $\mathbf{F} = \mathbf{F}^{(1)}$ then*

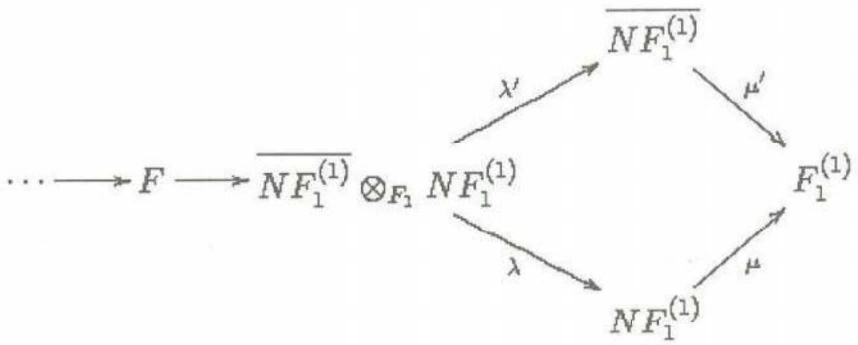
$$\pi_2(\mathbf{F}) = \mathrm{Ker}\,(\mathrm{Ker}d_1^{(1)} \otimes_{F_1^{(1)}} \mathrm{Ker}d_0^{(1)} \to F_0^{(1)}).$$

Proof: In [11], we proved that for $k \geq 1$ we have that if $\mathbf{F}^{(k)}$ is $k-$skeleton of the free simplicial group $\mathbf{F}$ then $\pi_k(\mathbf{F}^{(k)}) = \mathrm{Ker}\,(NF_k^{(k)} / \partial_{k+1}(NF_{k+1}^{(k)} \cap D_{k+1}^{(k)}) \to F_{k+1}^{(k)})$. Taking $k=1$ by the previous corollary, we get the result. +

Let $\mathbf{F}^{(1)}$ be the 1-skeleton of a free simplicial group with given CW-basis. By Corollary 3.2, we know the following square



is free square. Thus the free squared complex is



where $NF_1^{(1)} = \mathrm{Ker}d_0^{(1)}$ $\overline{NF}_1^1 = \mathrm{Ker}d_1^{(1)}$, $F = \dfrac{NF_3^{(1)}}{(NF_3^{(1)} \cap D_3)d_4(NF_4^{(1)} \cap D_4)}$,

$F_3^{(1)} = F(s_2s_1s_0(X_0) \cup s_1s_0(Y_1) \cup s_2s_0(Y_1)s_2s_1(Y_1))$, $NF_3^{(1)} = < s_2s_1(Y_1) > \cap < Z > \cap < Z_1 >$,

$< Z > = \mathrm{Ker}d_1^{(1)}$,

$$Z = \begin{cases} s_1(y)^{-1}s_0(y) \\ \qquad\qquad\qquad\qquad . \quad y \in Y_1 \\ s_2s_1(y)^{-1}s_2s_0(y) \end{cases}$$

$< Z_1 > = \mathrm{Ker}d_2^{(1)}$, $\qquad Z_1 = \{s_2(y)^{-1}s_1(y) : y \in Y_1\}$,

$F_4^{(1)} = F(s_3s_2s_1s_0(Y_1) \cup s_2s_1s_0(Y_1) \cup s_3s_1s_0(Y_1) \cup s_3s_2s_0(Y_1)s_3s_2s_1(Y_1))$,

$NF_4^{(1)} = < s_3s_2s_1(Y_1) > \cap < Z > \overline{\cap < Z_1 >} \cap < Z_2 >$, $< Z > = \mathrm{Ker}d_1^{(1)}$,

$$Z = \begin{cases} s_1(y)^{-1}s_0(y) \\ s_2s_1(y)^{-1}s_2s_0(y) \qquad y \in Y_1 \\ s_3s_2s_1(y)^{-1}s_3s_2s_0(y) \end{cases}$$

$Z_1 = Kerd_2^{(1)}$,

$$Z_1 = \begin{cases} s_2(y)^{-1}s_1(y) \\ & y \in Y_1 \\ s_3s_2(y)^{-1}s_3s_1(y) \end{cases}$$

and $< Z_2 > = \mathrm{Ker}d_3^{(3)}$, $Z_2 = \{s_3(y)^{-1}s_2(y) : y \in Y_1\}$ so where, by a *free crossed complex* we mean one in which the crossed square is free, and in which each C_n is free group for $n \geq 4$. Also

$$\cdots \longrightarrow F \longrightarrow \overline{NF_1^{(1)}} \otimes_{F_1} NF_1^{(1)} \longrightarrow NF_1^{(1)} \longrightarrow F_1^{(1)}$$

is a free 2-crossed complex because

$$\overline{NF_1^{(1)}} \otimes_{F_1} NF_1^{(1)} \longrightarrow NF_1^{(1)} \longrightarrow F_1^{(1)}$$

is a free 2-crossed module (see [9]).

So we thus have a functor $\rho : 1-skSimpGrp \rightarrow FreeSqComp$ where $1-skSimpGrp$ is the category of 1-skeleton of free simplicial group and $FreeSqComp$ is the category of the free squared complexes.

The *homotopy group* $\pi_n(\rho)$, $n \geq 1$ of the squared complex is defined to be the homology group of the complex

$$\cdots \xrightarrow{\partial_6} C_5 \xrightarrow{\partial_5} C_4 \xrightarrow{\partial_4} L \xrightarrow{\partial_3} M \rtimes N \xrightarrow{\partial_2} P \longrightarrow 1$$

with $\partial_3(l) = (\lambda' l^{-1}, \lambda l)$ and $\partial_2(m,n) = \mu(m)\mu'(n)$. The axioms of a crossed square guarantee that ∂_3 and ∂_2 are homomorphisms with $\partial_4(C_4)$ normal subgroup in $\mathrm{Ker}(\partial_3)$, $\partial_3(L)$ normal subgroup in $\mathrm{Ker}(\partial_2)$ and $\partial_2(M \rtimes N)$ normal subgroup in P. Clearly $\pi_n(\rho) = \mathrm{Ker}\partial_n / \mathrm{Im}\,\partial_{n+1}$.

Note that the homotopy group $\pi_n(\rho(\mathbf{F}^{(1)}))$ of the squared complex

$$\rho\left(\mathbf{F}^{(1)})\right) = \left(C_n, \left(\begin{array}{ccc} \overline{NF_1^{(1)}} \otimes_{F_1^{(1)}} NF_1^{(1)} & \rightarrow & NF_1^{(1)} \\ \downarrow & & \downarrow \\ \overline{NF_1^{(1)}} & \longrightarrow & F_1^{(1)} \end{array} \right) \right)$$

are the homology group of the complex

$$\cdots \xrightarrow{\partial_6} C_5 \xrightarrow{\partial_5} C_4 \xrightarrow{\partial_4} \overline{NF_1^{(1)}} \otimes_{F_1^{(1)}} NF_1^{(1)} \xrightarrow{\lambda} NF_1^{(1)} \xrightarrow{\partial} F_1^{(1)} \longrightarrow 1$$

5. CONCLUSION

In this final section we give some results of algebraic topology structure as above. We take the 1-skeleton of simplicial group with given by CW-basis and see how it relates to other algebraic construction such as homology and a square complex and 2-crossed complex of tensor product and coproduct.

From Corollary 3.2 we have the free crossed square

$$\begin{array}{ccc} J & \longrightarrow & < \mathbf{B}_2 > \\ \downarrow & & \downarrow \\ < Z_2 > & \longrightarrow & F_1^{(2)} \end{array}$$

where $NF_1 = \mathrm{Ker}d_0 =< \mathbf{B}_2 >$ and $NF_2 = \mathrm{Ker}d_0 \cap \mathrm{Ker}d_1 =< s_1(\mathbf{B}_2) \cup \mathbf{B}_3 > \cap < Z \cap \mathbf{B}_3 >$ here P_2 is the second dimensional Peiffer normal subgroup in $< s_1(\mathbf{B}_2) \cup \mathbf{B}_3 > \cap < Z \cap \mathbf{B}_3 >$, $< Z_2 >$ is $< Z \cup \mathbf{B}_3 >$ and J is $< s_1(\mathbf{B}_2) \cup \mathbf{B}_3 > \cap < Z \cap \mathbf{B}_3 > / P_2$ so using Corollary 3.2 there is following isomorphism: $J \cong < Z_2 > \otimes < \mathbf{B}_2 >$. Thus the free crossed square becomes

$$\begin{array}{ccc} < Z_2 > \otimes < \mathbf{B}_2 > & \longrightarrow & < \mathbf{B}_2 > \\ \downarrow & & \downarrow \\ < Z_2 > & \longrightarrow & F_1^{(2)}. \end{array}$$

In his section 4 we saw that there is a 2-crossed module is defined by D. Conduché unpublished work determines that there exits an equivalence (up to homotopy) between the category of crossed squares of groups and that of 2-crossed modules of groups. (for see the details [8, 9, 10 and 11]) Therefore we have

$$\mathbb{X} :< Z_2 > \otimes < \mathbf{B}_2 > \xrightarrow{\partial_2} < Z_2 > \rtimes < \mathbf{B}_2 > \xrightarrow{\partial_1} F_1^{(1)}$$

where $\partial_2(x \otimes y) = (\lambda(x \otimes y)^{-1}, \lambda'(x \otimes y))$ and $\partial_1(x, y) = \mu(x)\mu'(x)$. The axioms of square complex ensure that ∂_2 and ∂_1 are homomorphisms and ∂_2 is a crossed module.

The 2-crossed complex $C^{(2)}(\mathbf{F}^{(2)})$ has a smaller 2- crossed module at its base namely

$$\mathbb{Y} :< Z_2 > \otimes < \mathbf{B}_2 > \xrightarrow{\partial_2} < Z_2 > \xrightarrow{\partial_1} F_1^{(1)}$$

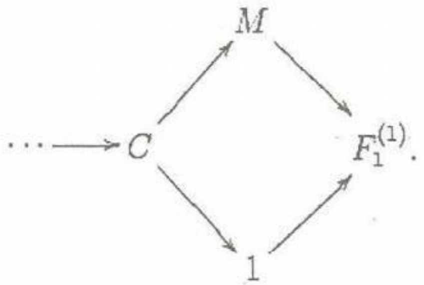
and it is important to compare these two. In fact there is a split epimorphism from Ξ to Ψ with kernel $1 \to < Z_2 > \xrightarrow{=} < Z_2 >$ which has, of course trivial homotopy. Thus Ξ and Ψ encode the same information about presentation of $F(\mathbf{B}_1)/\mathrm{Im}\,\partial_2$. So we get a functor from the category of simplicial group to the category of 2-crossed complex, that is

$$C : SimpGrp \to X_2 - Comp$$

and the category of square complex is equivalent to the category of 2-crossed complex, i.e., there is a functor

$$D : SrqComp \to X_2 - Comp.$$

Crossed complexes from a category Crs which can be considered as a full subcategory of both the categories of 2-crossed complexes and of square complexes. In the case of 2-crossed complexes and any crossed complex $\mathrm{C} : C \to M \to F_1^{(1)}$ yields a 2-crossed complex with the same terms at each level and trivial Peiffer commutator mapping $\{\otimes\} : M \otimes M \to C$ whilst considered as a square complex we have X verifies



In both cases higher dimensional terms are left unchanged. Both inclusions have adjoints i.e. embedding give reflective subcategories. The proofs are quite easy (and will be given elsewhere).

The functors from *SimpGrp* to X_2-Comp and *SimpGrp* to *SqComp* used above, when composed with the reflection to *Crs* yield the associated crossed complex functor mentioned earlier. (see [9 and 10])

Finally the category of chain complexes over $F(\mathbf{B}_1)$ embeds as a reflexive subcategory of *Crs* and the reflection sends a crossed resolution to the intermediate stage of the cotangent complex (see [9 and 10]). Thus given a simplicial resolution of $F(\mathbf{B}_1)$, constructed as in [10] by step-by-step method the 2-crossed and squares resolution it gives can be considered as "quadratic" analogues of the cotangent complex. Here we are using quadratic in the analogues to way to that used by Baues [2] for the group based theory.

Given this it is of interest to study the complex X (or equivalently Y) and their analogues when $\mathbf{B}_3$ information is added in. Here we have no definitive results, only problems.

The idea will be to try to provide algorithms for calculating, the kernel of ∂_2 in X (or equivalently Y). We know there give $\pi_3(\mathbf{F}^{(1)})$ and it is hoped that if there algorithms works, they would allow an analysis of $\pi_3(\mathbf{F}^{(2)})$ and thus to study the effect of adding in $\mathbf{B}_3$ information to the higher terms of the simplicial resolution. As we are not sure yet if a general analysis is possible or whether it is necessary to limit ourselves to specific classes of example, using, for instance methods from group base theory.

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