photonics

MDPI

Article

A Systematic Summary and Comparison of Scalar Diffraction Theories for Structured Light Beams

Fuping Wu [1], Yi Luo [1] and Zhiwei Cui [1,2,*]

[1] School of Physics, Xidian University, Xi'an 710071, China; fpwu@xidian.edu.cn (F.W.); yluo_2@stu.xidian.edu.cn (Y.L.)
[2] Guangzhou Institute of Technology, Xidian University, Guangzhou 510555, China
* Correspondence: zwcui@mail.xidian.edu.cn

Abstract: Structured light beams have recently attracted enormous research interest for their unique properties and potential applications in optical communications, imaging, sensing, etc. Since most of these applications involve the propagation of structured light beams, which is accompanied by the phenomenon of diffraction, it is very necessary to employ diffraction theories to analyze the obstacle effects on structured light beams during propagation. The aim of this work is to provide a systematic summary and comparison of the scalar diffraction theories for structured light beams. We first present the scalar fields of typical structured light beams in the source plane, including the fundamental Gaussian beams, higher-order Hermite–Gaussian beams, Laguerre–Gaussian vortex beams, non-diffracting Bessel beams, and self-accelerating Airy beams. Then, we summarize and compare the main scalar diffraction theories of structured light beams, including the Fresnel diffraction integral, Collins formula, angular spectrum representation, and Rayleigh–Sommerfeld diffraction integral. Finally, based on these theories, we derive in detail the analytical propagation expressions of typical structured light beams under different conditions. In addition, the propagation of typical structured light beams is simulated. We hope this work can be helpful for the efficient study of the propagation of structured light beams.

Keywords: scalar diffraction theory; structured light beams; propagation

check for updates

Citation: Wu, F.; Luo, Y.; Cui, Z. A Systematic Summary and Comparison of Scalar Diffraction Theories for Structured Light Beams. *Photonics* **2023**, *10*, 1041. https://doi.org/10.3390/photonics10091041

Received: 14 August 2023
Revised: 10 September 2023
Accepted: 11 September 2023
Published: 13 September 2023

1. Introduction

It is well known that light, as a most important information carrier, has a wide range of applications in communications, sensing, imaging, etc. Traditionally, light fields with simple spatial distribution, such as plane waves and fundamental Gaussian beams, were employed in these applications. In recent years, light fields with strong spatial inhomogeneity of amplitude, phase, polarization, and other parameters are gaining attention as novel information carriers for these applications. Such light fields are usually called structured light beams, including the Hermite–Gaussian beams [1], the Laguerre–Gaussian beams [2], the Bessel beams [3], the Airy beams [4], and so on. Compared with conventional plane waves and fundamental Gaussian beams, structured light beams exhibit a variety of novel physical effects and phenomena, e.g., strong spatial inhomogeneity, phase singularity, diffraction-free propagation, transverse acceleration, and so on [5]. Owing to their fascinating properties and promising potential applications, structured light beams have become a research hotspot in optics and optoelectronics [6–13]. Because most of the applications of structured light beams involve their propagation, which is accompanied by the phenomenon of diffraction, the research on the obstacle effects of such beams during propagation has important significance. In most cases, the propagation effects of the structured light beams can be examined by the scalar diffraction theories.

There have been extensive studies on the propagation effects of structured light beams using various scalar diffraction theories, which can be divided into four categories: the

Fresnel diffraction integral [14–17], the Collins formula [18–24], the angular spectrum representation [25–30], and the Rayleigh–Sommerfeld diffraction integral [31–40]. The Fresnel diffraction integral, which is an approximation of the Kirchhoff diffraction formula, is usually used to study the propagation of paraxial light beams in a homogeneous medium. The Collins formula, which is a generalized Fresnel diffraction integral formula expressed in terms of ray transfer matrix elements, is usually applied to analyze the propagation of paraxial light beams passing through the ABCD optical system. The angular spectrum representation, whose basic idea is to expand any light beams in terms of lots of plane waves by Fourier transform, is not only suitable for describing the propagation of light beams in a homogeneous medium but also suitable for describing the reflection and refraction of structured light beams in a layered medium. Moreover, under the paraxial approximation, the angular spectrum representation is identical to the framework of the Fresnel diffraction integral, which extends its importance even further. The Rayleigh–Sommerfeld diffraction integral, which degenerates into the Fresnel diffraction integral under paraxial approximation, is usually adopted to study the non-paraxial propagation of structured light beams. As mentioned earlier, these scalar diffraction theories have been extensively applied to study the propagation of structured light beams, including the Hermite–Gaussian beams [41–44], Laguerre–Gaussian beams [45–53], Bessel beams [54–57], and Airy beams [58–63]. However, most of the previous works only focused on studying the propagation of one type of structured light beam using one of the scalar diffraction theories. In addition, some analytical formulas for the propagation of structured light beams presented in previous works are not concise, explicit, and systematic enough. The purpose of this work is to provide a summary of the main scalar diffraction theories of structured light beams and systematically derive concise and explicit analytical expressions for the propagation of typical structured light beams under different conditions.

The remainder of this paper is composed of five parts. In Section 2, the scalar fields of typical structured light beams in the source plane are presented. In Section 3, the main scalar diffraction theories of light beams are summarized. In Section 4, the analytical propagation expressions of typical structured light beams described by various scalar diffraction theories are derived in detail. In Section 5, some numerical simulations are performed and analyzed. Finally, the conclusion is drawn in Section 6.

2. Scalar Fields of Typical Structured Light Beams in the Source Plane

As is well known, the scalar diffraction theories of light beams give the relationship between the scalar expressions of light beams in the source plane and that in the observation plane. In what follows, the source plane is denoted as $(x_0, y_0, 0)$ in Cartesian coordinates and $(r_0, \varphi_0, 0)$ in cylindrical coordinates, and the corresponding scalar fields of light beams are represented as $E_0(x_0, y_0, 0)$ and $E_0(x_0, y_0, 0)$, respectively.

2.1. Fundamental Gaussian Beams

We start with the simplest fundamental Gaussian beams that are the solutions of the scalar paraxial wave equation. In Cartesian coordinates, the scalar field of the fundamental Gaussian beams in the source plane reads as

$$E_0(x_0, y_0, 0) = \exp\left[-\frac{(x_0^2 + y_0^2)}{w_0^2}\right], \tag{1}$$

where w_0 is the beam waist radius.

2.2. Hermite–Gaussian Beams

Hermite–Gaussian beams, a class of structured light beams with higher-order modes, are the solutions of the paraxial wave equation in Cartesian coordinates. The scalar field of such beams in the source plane reads as

$$E_0(x_0, y_0, 0) = H_m\left(\frac{\sqrt{2}}{w_0}x_0\right)H_n\left(\frac{\sqrt{2}}{w_0}y_0\right)\exp\left[-\frac{(x_0^2+y_0^2)}{w_0^2}\right], \tag{2}$$

where w_0 is the beam waist radius and $H_m(.)$ and $H_n(.)$ are Hermite polynomials of order m and n, respectively. Note that Equation (2) reduces to Equation (1) when $m = n = 0$.

2.3. Laguerre–Gaussian Beams

Laguerre–Gaussian beams, a class of structured light beams with vortex phase fronts, are the solutions of the paraxial wave equation in cylindrical coordinates. The scalar field of such beams in the source plane reads as

$$E_0(r_0, \varphi_0, 0) = \left(\frac{\sqrt{2}r_0}{w_0}\right)^l L_p^l\left(\frac{2r_0^2}{w_0^2}\right)\exp\left(-\frac{r_0^2}{w_0^2}\right)\exp(il\varphi_0), \tag{3}$$

where w_0 is the beam waist radius, $\varphi_0 = \tan^{-1}(y_0/x_0)$ is the phase angle, and $L_p^l(.)$ is the associated Laguerre polynomial, with p and l being the radial and azimuthal mode numbers, respectively.

2.4. Bessel Beams

Bessel beams, a class of structured light beams with non-diffracting characteristics, are the exact solutions of the scalar Helmholtz wave equation in cylindrical coordinates. The scalar field of such beams in the source plane reads as

$$E_0(r_0, \varphi_0, 0) = J_m(k_r r_0)\exp(im\varphi_0), \tag{4}$$

where $J_m(.)$ is the mth-order Bessel function of the first kind, $\varphi_0 = \tan^{-1}(y_0/x_0)$ is the phase angle, and $k_r = k\sin\theta_0$ is the transverse component of the wavenumber k, with θ_0 being the half-cone angle of the Bessel beams.

2.5. Airy Beams

Airy beams, a class of structured light beams with self-accelerating characteristics, are the solutions of the scalar paraxial wave equation or its quantum mechanics analog, potential-free Schrödinger equation. In Cartesian coordinates, the scalar field of such beams in the source plane reads as

$$E_0(x_0, y_0, 0) = Ai\left(\frac{x_0}{w_x}\right)\exp\left(\frac{a_0x_0}{w_x}\right)Ai\left(\frac{y_0}{w_y}\right)\exp\left(\frac{a_0y_0}{w_y}\right), \tag{5}$$

where $Ai(\cdot)$ is the Airy function, $w_x(w_y)$ is the transverse scaled parameter in the $x(y)$ direction, and a_0 is the decay parameter.

3. Main Scalar Diffraction Theories of Light Beams

To proceed, we make a summarization of the main scalar diffraction theories of light beams, including the Fresnel diffraction integral, the Collins formula, the angular spectrum representation, and the Rayleigh–Sommerfeld diffraction integral. Without loss of generality, the light beams are assumed to propagate parallel to the positive z-axis, and the scalar fields of the light beams in the observation plane are represented as $E(x, y, z)$ in Cartesian coordinates and $E(r, \varphi, z)$ in cylindrical coordinates.

3.1. Fresnel Diffraction Integral

As mentioned previously, the Fresnel diffraction integral is usually used to characterize the propagation of paraxial light beams in a homogeneous medium. In Cartesian coordinates, the Fresnel diffraction integral is expressed as [14–17]

$$\begin{aligned} E(x,y,z) = \left(-\tfrac{ik}{2\pi z}\right)\exp(ikz)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_0(x_0,y_0,0) \\ \times\exp\left\{\tfrac{ik}{2z}\left[(x-x_0)^2+(y-y_0)^2\right]\right\}dx_0dy_0, \end{aligned} \tag{6}$$

where $k = 2\pi/\lambda$ is the wavenumber, with λ being the wavelength of the light beams. Clearly, once the scalar field $E_0(x_0, y_0, 0)$ in the source plane is known, the scalar field $E(x, y, z)$ in the observation plane can be obtained by using the Fresnel diffraction integral. In circular cylindrical coordinates, Equation (6) takes the form

$$\begin{aligned} E(r,\varphi,z) = \left(-\tfrac{ik}{2\pi z}\right)\exp(ikz)\int_0^{\infty}\int_0^{2\pi} E_0(r_0,\varphi_0,0) \\ \times\exp\left\{\tfrac{ik}{2z}\left[{r_0}^2+r^2-2r_0r\cos(\varphi_0-\varphi)\right]\right\}r_0dr_0d\varphi_0, \end{aligned} \tag{7}$$

where $\varphi_0 = \tan^{-1}(y_0/x_0)$ and $\varphi = \tan^{-1}(y/x)$ are the phase angles of the beam in the source plane and observation plane, respectively.

3.2. Collins Formula

The Collins formula is a generalized Fresnel diffraction integral formula expressed in terms of the ABCD transfer matrix for the paraxial optical system. In Cartesian coordinates, the Collins formula is expressed as [18–24]

$$\begin{aligned} E(x,y,z) = \left(-\tfrac{ik}{2\pi B}\right)\exp(ikz)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_0(x_0,y_0,0) \\ \times\exp\left\{\tfrac{ik}{2B}\left[A\left(x_0^2+y_0^2\right)+D\left(x^2+y^2\right)-2(x_0x+y_0y)\right]\right\}dx_0dy_0, \end{aligned} \tag{8}$$

and in circular cylindrical coordinates, the Collins formula has the form

$$\begin{aligned} E(r,\varphi,z) = \left(-\tfrac{ik}{2\pi B}\right)\exp(ikz)\int_0^{\infty}\int_0^{2\pi} E_0(r_0,\varphi_0,0) \\ \times\exp\left\{\tfrac{ik}{2B}\left[Ar_0^2+Dr^2-2r_0r\cos(\varphi_0-\varphi)\right]\right\}r_0dr_0d\varphi_0, \end{aligned} \tag{9}$$

where A, B, C, and D are elements of the transfer matrix. For the sake of free space, the ABCD transfer matrix can be written as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix}. \tag{10}$$

Obviously, substituting Equation (10) into Equations (8) and (9) yields Equations (6) and (7), i.e., the Collins formula reduces to the Fresnel diffraction integral.

3.3. Angular Spectrum Representation

The angular spectrum representation is a powerful approach to characterize the propagation of structured light beams in a homogeneous medium as well as the reflection and refraction of structured light beams in a layered medium. The basic idea of such an approach is to expand structured light beams in terms of lots of plane waves by Fourier transform. Specifically, the scalar field of the structured light beam to be considered in the observation plane is expressed as [25–30]

$$E(x,y,z) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \widetilde{E}(k_x,k_y)\exp\left[i\left(k_xx+k_yy\right)\right]\exp(ik_zz)dk_xdk_y, \tag{11}$$

where $\widetilde{E}(k_x, k_y)$ is the angular spectrum of the beam, k_x and k_y are the transverse components of the wave vector k, and $k_z = \sqrt{k^2 - k_x^2 - k_y^2}$ refers to the beam propagating into the half-space $z > 0$. The angular spectrum $\widetilde{E}(k_x, k_y)$ can be obtained by evaluating the two-dimensional Fourier transform of the scalar field of the structured light beam in the source plane as follows:

$$\widetilde{E}(k_x, k_y) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E_0(x_0, y_0, 0) \exp\left[-i\left(k_x x_0 + k_y y_0\right)\right] dx_0 dy_0. \tag{12}$$

Under the paraxial approximation, we can expand k_z in a series as

$$k_z = \sqrt{k^2 - \left(k_x^2 + k_y^2\right)} \approx k - \frac{k_x^2 + k_y^2}{2k}. \tag{13}$$

Substituting Equation (13) into Equation (11), we obtain

$$\begin{aligned} E(x, y, z) = \exp(ikz) \textstyle\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \widetilde{E}(k_x, k_y) \\ \times \exp\left[i\left(k_x x + k_y y - \tfrac{k_x^2 + k_y^2}{2k} z\right)\right] dk_x dk_y. \end{aligned} \tag{14}$$

Equations (11), (12), and (14) are the formulae of angular spectrum representation for describing the propagation of structured light beams in Cartesian coordinates. For the case of structured light beams described in circular cylindrical coordinates, the angular spectrum representation of the scalar field in the observation plane can be expressed as

$$\begin{aligned} E(r, \varphi, z) = \textstyle\int_0^{\pi} \int_0^{2\pi} \widetilde{E}(\theta, \phi) \exp[ikr \sin\theta \cos(\phi - \varphi)] \\ \times \exp(ik \cos\theta z) k^2 \cos\theta \sin\theta d\theta d\phi, \end{aligned} \tag{15}$$

$$\begin{aligned} E(r, \varphi, z) = \exp(ikz) \textstyle\int_0^{\pi} \int_0^{2\pi} \widetilde{E}(\theta, \phi) \exp[ikr \sin\theta \cos(\phi - \varphi)] \\ \times \exp\left(-ik \sin^2\theta z/2\right) k^2 \cos\theta \sin\theta d\theta d\phi, \end{aligned} \tag{16}$$

where

$$\widetilde{E}(\theta, \phi) = \frac{1}{4\pi^2} \int_0^{\infty} \int_0^{2\pi} E_0(r_0, \varphi_0, 0) \exp[-ikr_0 \sin\theta \cos(\phi - \varphi_0)] r_0 dr_0 d\varphi_0. \tag{17}$$

3.4. Rayleigh–Sommerfeld Diffraction Integral

The Rayleigh–Sommerfeld diffraction integral is an approximation method to characterize the non-paraxial propagation of structured light beams. In Cartesian coordinates, the Rayleigh–Sommerfeld diffraction integral is expressed as [31–40]

$$\begin{aligned} E(x, y, z) = \left(-\tfrac{ikz}{2\pi}\right) \tfrac{\exp(ik\rho)}{\rho^2} \textstyle\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} E_0(x_0, y_0, 0) \\ \times \exp\left[\tfrac{ik}{2\rho}\left(x_0^2 + y_0^2 - 2xx_0 - 2yy_0\right)\right] dx_0 dy_0, \end{aligned} \tag{18}$$

where $\rho = \sqrt{x^2 + y^2 + z^2}$. It is worth noting that the Rayleigh–Sommerfeld diffraction integral reduces to the Fresnel diffraction integral by replacing ρ of the exponential part in Equation (18) with $\rho = z + \left(x^2 + y^2\right)/2z$ and other terms with z. In circular cylindrical coordinates, Equation (18) takes the form

$$\begin{aligned} E(r, \varphi, z) = \left(-\tfrac{ikz}{2\pi}\right) \tfrac{\exp(ik\rho)}{\rho^2} \textstyle\int_0^{\infty} \int_0^{2\pi} E_0(r_0, \varphi_0, 0) \\ \times \exp\left\{\tfrac{ik}{2\rho}\left[r_0^2 - 2r_0 r \cos(\varphi_0 - \varphi)\right]\right\} r_0 dr_0 d\varphi_0, \end{aligned} \tag{19}$$

where $\rho = \sqrt{r^2 + z^2}$.

4. Analytical Propagation Expressions of Typical Structured Light Beams Described by Various Scalar Diffraction Theories

4.1. Fundamental Gaussian Beams

4.1.1. Fresnel Diffraction Integral

Substituting Equation (1) into Equation (6) and making use of the integral formula

$$\int_{-\infty}^{\infty} \exp\left(-ax^2 + ibx\right)dx = \sqrt{\frac{\pi}{a}} \exp\left(-\frac{b^2}{4a}\right), \tag{20}$$

we obtain the analytical propagation expression of the fundamental Gaussian beams under the paraxial approximation as follows:

$$E(x, y, z) = \frac{1}{1 + iz/z_R} \exp\left[-\frac{\left(x^2 + y^2\right)/w_0^2}{1 + iz/z_R}\right] \exp(ikz), \tag{21}$$

where $z_R = kw_0{}^2/2$ is the Rayleigh range.

4.1.2. Collins Formula

Substituting Equation (1) into Equation (8) and recalling the integral formula given by Equation (20), we obtain the analytical propagation expression of paraxial fundamental Gaussian beams passing through an ABCD optical system as follows:

$$E(x, y, z) = \frac{q_0}{Aq_0 + B} \exp\left[\frac{ik\left(x^2 + y^2\right)}{2}\left(\frac{Cq_0 + D}{\mathrm{A}q_0 + B}\right)\right] \exp(ikz), \tag{22}$$

where $q_0 = -iz_R$, with $z_R = kw_0{}^2/2$. Notably, by substituting Equation (10) into Equation (22), we obtain Equation (21), which indicates that the Collins formula is a generalized form of the Fresnel diffraction integral.

4.1.3. Angular Spectrum Representation

Substituting Equation (1) into Equation (12) and using the integral formula given by Equation (20), we obtain the angular spectrum of the fundamental Gaussian beams as follows:

$$\widetilde{E}\left(k_x, k_y\right) = \frac{w_0^2}{4\pi} \exp\left[-\left(k_x^2 + k_y^2\right)\frac{w_0^2}{4}\right]. \tag{23}$$

Further substituting Equation (23) into Equation (14) and performing the integration with the help of Equation (20), we obtain the analytical propagation expression of paraxial fundamental Gaussian beams, which coincides with Equation (21) derived by the Fresnel diffraction integral.

4.1.4. Rayleigh–Sommerfeld Diffraction Integral

Substituting Equation (1) into Equation (18) and applying the integral formula given by Equation (20), we obtain the analytical propagation expression of the fundamental Gaussian beams beyond the paraxial approximation as follows:

$$E(x, y, z) = \left(-\frac{ikz}{2a\rho^2}\right) \exp\left(-\frac{b_x^2 + b_y^2}{4a}\right) \exp(ik\rho), \tag{24}$$

where

$$a = \frac{1}{w_0^2} - \frac{ik}{2\rho}, b_x = -\frac{kx}{\rho}, b_y = -\frac{ky}{\rho}, \rho = \sqrt{x^2 + y^2 + z^2}. \tag{25}$$

Replacing ρ of the exponential part in Equation (24) by $\rho = z + (x^2 + y^2)/(2z)$ and other terms by z, we obtain the analytical propagation expression of fundamental Gaussian beams under the paraxial approximation, which coincides with Equation (21).

4.2. Hermite–Gaussian Beams

4.2.1. Fresnel Diffraction Integral

Substituting Equation (2) into Equation (6) and making use of the integral formula

$$\int_{-\infty}^{+\infty} \exp\left[-\frac{(x-a)^2}{b}\right] H_m(cx)dx = \sqrt{\pi b}\left(1-c^2 b\right)^{m/2} H_m\left(\frac{ca}{\sqrt{1-c^2 b}}\right), \tag{26}$$

we obtain the analytical propagation expression of the Hermite–Gaussian beams under the paraxial approximation as follows:

$$\begin{aligned} E(x,y,z) = {} & H_m\left(\tfrac{\sqrt{2}}{w}x\right) H_n\left(\tfrac{\sqrt{2}}{w}y\right)\left[\tfrac{1-iz/z_R}{\sqrt{1+(z/z_R)^2}}\right]^{m+n} \\ & \times \tfrac{1}{1+iz/z_R} \exp\left[-\tfrac{(x^2+y^2)/w_0^2}{1+iz/z_R}\right] \exp(ikz), \end{aligned} \tag{27}$$

where $w(z) = w_0\sqrt{1+(z/z_R)^2}$ and $z_R = kw_0{}^2/2$. Letting $m = n = 0$, Equation (27) reduces to Equation (21), which corresponds to the propagation expression of fundamental Gaussian beams.

4.2.2. Collins Formula

Substituting Equation (2) into Equation (8) and recalling the integral formula given by Equation (26), we obtain the analytical propagation expression of paraxial Hermite–Gaussian beams passing through an ABCD optical system as follows:

$$\begin{aligned} E(x,y,z) = {} & H_m\left(\tfrac{\sqrt{2}}{w\prime}x\right) H_n\left(\tfrac{\sqrt{2}}{w\prime}y\right)\left(\sqrt{\tfrac{Aq_0-B}{Aq_0+B}}\right)^{m+n} \\ & \times \tfrac{q_0}{Aq_0+B} \exp\left[\tfrac{ik(x^2+y^2)}{2}\left(\tfrac{Cq_0+D}{Aq_0+B}\right)\right] \exp(ikz), \end{aligned} \tag{28}$$

where $w\prime = w_0\sqrt{A^2 - B^2/q_0^2}$ and $q_0 = -iz_R$, with $z_R = kw_0{}^2/2$.

4.2.3. Angular Spectrum Representation

Substituting Equation (2) into Equation (12) and using the integral formula given by Equation (26), we obtain the angular spectrum of the Hermite–Gaussian beams as follows:

$$\widetilde{E}(k_x,k_y) = i^{m+n} H_m\left(-\frac{w_0 k_x}{\sqrt{2}}\right) H_m\left(-\frac{w_0 k_y}{\sqrt{2}}\right)\frac{w_0^2}{4\pi}\exp\left[-\frac{\left(k_x^2+k_y^2\right)w_0^2}{4}\right]. \tag{29}$$

Further substituting Equation (29) into Equation (14) and performing the integration with the help of Equation (26), we obtain the analytical propagation expression of paraxial Hermite–Gaussian beams, which coincides with Equation (27) derived by the Fresnel diffraction integral.

4.2.4. Rayleigh–Sommerfeld Diffraction Integral

Substituting Equation (2) into Equation (18) and applying the integral formula given by Equation (26), we obtain the analytical propagation expression of the Hermite–Gaussian beams beyond the paraxial approximation as follows:

$$\begin{aligned} E(x,y,z) = {} & H_m\left(\frac{\sqrt{2}}{w}x\right)H_n\left(\frac{\sqrt{2}}{w}y\right)\left[\frac{1-i\rho/z_R}{\sqrt{1+(\rho/z_R)^2}}\right]^{m+n} \\ & \times\left(-\frac{ikz}{2a\rho^2}\right)\exp\left(-\frac{b_x^2+b_y^2}{4a}\right)\exp(ik\rho), \end{aligned} \tag{30}$$

where a, b_x, b_y, and ρ are defined by Equation (25).

4.3. Laguerre–Gaussian Beams

4.3.1. Fresnel Diffraction Integral

Substituting Equation (3) into Equation (7) and utilizing the integral formulas [5]

$$\int_0^{2\pi}\exp[ix\cos(\varphi_0-\varphi)]\exp(il\varphi_0)d\varphi_0 = 2\pi i^l J_l(x)\exp(il\varphi), \tag{31}$$

$$\begin{aligned} & \int_0^{\pi}(x)^{l+\frac{1}{2}}\exp(-ax^2)L_p^l(bx^2)J_l(cx)\sqrt{cx}dx \\ & = 2^{-l-1}a^{-l-p-1}(a-b)^p c^{l+\frac{1}{2}}\exp\left(-\frac{c^2}{4a}\right)L_p^l\left[\frac{bc^2}{4a(b-a)}\right], \end{aligned} \tag{32}$$

we obtain the analytical propagation expression of the paraxial Laguerre–Gaussian beams

$$\begin{aligned} E(r,\varphi,z) = {} & \left(\sqrt{2}\frac{r}{w}\right)^l L_p^l\left(2\frac{r^2}{w^2}\right)\left[\frac{1-iz/z_R}{\sqrt{1+(z/z_R)^2}}\right]^{2p+l}\exp(il\varphi) \\ & \times\frac{1}{1+iz/z_R}\exp\left(-\frac{r^2/w_0^2}{1+iz/z_R}\right)\exp(ikz), \end{aligned} \tag{33}$$

where $w(z) = w_0\sqrt{1+(z/z_R)^2}$ and $z_R = k{w_0}^2/2$.

4.3.2. Collins Formula

Substituting Equation (3) into Equation (9) and recalling the integral formulas given by Equations (31) and (32), we obtain the analytical propagation expression of paraxial Laguerre–Gaussian beams passing through an ABCD optical system as follows:

$$\begin{aligned} E(r,\varphi,z) = {} & \left(\frac{\sqrt{2}r}{w_0}\frac{q_0}{Aq_0+B}\right)^l\left(\frac{Aq_0-B}{Aq_0+B}\right)^p L_p^l\left(2\frac{r^2}{w'^2}\right)\exp(il\varphi) \\ & \times\frac{q_0}{Aq_0+B}\exp\left[\frac{ikr^2}{2}\left(\frac{Cq_0+D}{Aq_0+B}\right)\right]\exp(ikz), \end{aligned} \tag{34}$$

where $w' = w_0\sqrt{A^2-B^2/q_0^2}$ and $q_0 = -iz_R$ with $z_R = k{w_0}^2/2$.

4.3.3. Angular Spectrum Representation

Substituting Equation (3) into Equation (17) and using the integral formulas given by Equations (31) and (32), we obtain the angular spectrum of the Laguerre–Gaussian beams as follows [5]:

$$\begin{aligned} \widetilde{E}(\theta,\phi) = {} & \left(-\frac{iw_0k\sin\theta}{\sqrt{2}}\right)^l(-1)^p L_p^l\left(\frac{w_0^2k^2\sin^2\theta}{2}\right) \\ & \times\exp(il\phi)\frac{w_0^2}{4\pi}\exp\left(-\frac{w_0^2k^2\sin^2\theta}{4}\right). \end{aligned} \tag{35}$$

Further substituting Equation (35) into Equation (16) and performing the integration with the help of Equations (31) and (32), we obtain the analytical propagation expression of paraxial Laguerre–Gaussian beams, which coincides with Equation (33) derived by the Fresnel diffraction integral.

4.3.4. Rayleigh–Sommerfeld Diffraction Integral

Substituting Equation (3) into Equation (19) and applying the integral formulas given by Equations (31) and (32), we derive that the analytical propagation expression of the Laguerre–Gaussian beams beyond the paraxial approximation as follows:

$$\begin{aligned} E(r,\varphi,z) = {} & \left(\tfrac{\sqrt{2}r}{w}\right)^{l} L_p^l\left(2\tfrac{r^2}{w^2}\right)\left(\tfrac{1-i\rho/z_R}{\sqrt{1+\rho^2/z_R^2}}\right)^{2p+l} \\ & \times \exp(il\varphi)\left(-\tfrac{ikz}{2a\rho^2}\right)\exp\left(-\tfrac{b^2}{4a}\right)\exp(ik\rho), \end{aligned} \tag{36}$$

where

$$a = \frac{1}{w_0^2} - \frac{ik}{2\rho}, b = -\frac{k}{\rho}r, w = w_0\sqrt{1+(\rho/z_R)^2}, r = \sqrt{x^2+y^2}, \rho = \sqrt{r^2+z^2}. \tag{37}$$

Replacing ρ of the exponential part in Equation (37) by $\rho = z + r^2/(2z)$ and other terms by z, we obtain the analytical propagation expression of fundamental Gaussian beams under the paraxial approximation, which coincides with Equation (33).

4.4. Bessel Beams

4.4.1. Fresnel Diffraction Integral

Substituting Equation (4) into Equation (7), employing the integral formula given by Equation (31) and the following integral formula

$$\int_0^\infty J_m(ar)J_m(br)\exp\left(-cr^2\right)rdr = \frac{i^{-m}}{2c}\exp\left[-\frac{1}{4c}\left(a^2+b^2\right)\right]J_m\left(\frac{iab}{2c}\right), \tag{38}$$

we obtain the analytical propagation expression of the paraxial Bessel beams

$$E(r,\varphi,z) = \exp\left(-\frac{ik_r^2 z}{2k}\right)J_m(k_r r)\exp(im\varphi)\exp(ikz), \tag{39}$$

where $k_r = k\sin\theta_0$ with θ_0 being the half-cone angle of the Bessel beams.

4.4.2. Collins Formula

Substituting Equation (4) into Equation (9) and employing the integral formulas given by Equations (31) and (38), we obtain the analytical propagation expression of paraxial Bessel beams passing through an ABCD optical system

$$E(r,\varphi,z) = \frac{1}{A}\exp\left(\frac{ik^2Cr^2 - ik_r^2B}{2kA}\right)J_m\left(\frac{k_r r}{A}\right)\exp(im\varphi)\exp(ikz). \tag{40}$$

4.4.3. Angular Spectrum Representation

Substituting Equation (4) into Equation (17), employing the integral formula given by Equation (31), and recalling the properties of the Dirac delta function [5]

$$\delta(b-a) = b\int_0^\infty rJ_l(br)J_l(ar)dr, \tag{41}$$

$$\delta(ax) = \frac{\delta(x)}{|a|}, \delta[g(x)] = \frac{\delta(x-x_0)}{|g'(x_0)|}, \tag{42}$$

we obtain the angular spectrum of the Bessel beams as follows [5]:

$$\widetilde{E}(\theta,\phi) = \frac{1}{2\pi i^m}\frac{\delta(\theta-\theta_0)}{k^2\sin\theta_0\cos\theta_0}\exp(im\phi), \tag{43}$$

where θ_0 is the half-cone angle of the Bessel beam.

After substituting Equation (43) into Equation (15) and performing the integrations, we can obtain the analytical propagation expression of the Bessel beams that satisfy the scalar Helmholtz wave equation as follows:

$$E(\rho,\varphi,z) = J_m(k_r r)\exp(im\varphi)\exp(ik_z z), \tag{44}$$

where $k_z = k\cos\theta_0$ is the longitudinal component of the wavenumber k.

The analytical propagation expression of the Bessel beams that satisfy the paraxial wave equation can be calculated by substituting Equations (43) and (16). The calculation result is the same as Equation (39).

4.4.4. Rayleigh–Sommerfeld Diffraction Integral

Substituting Equation (4) into Equation (19) and applying the integral formulas given by Equations (31) and (38), we obtain the analytical propagation expression of the non-paraxial Bessel beams

$$E(r,\varphi,z) = \left(\frac{z}{\rho}\right)\exp\left(-\frac{ikr^2}{2\rho}\right)\exp\left(-\frac{ik_r^2\rho}{2k}\right)J_m(k_r r)\exp(im\varphi)\exp(ik\rho), \tag{45}$$

where $k_r = k\sin\theta_0$ and $\rho = \sqrt{r^2+z^2}$.

4.5. Airy Beams

4.5.1. Fresnel Diffraction integral

Substituting Equation (5) into Equation (6) and making use of the integral formula

$$\begin{aligned}&\int_{-\infty}^{+\infty} Ai(x)\exp\left(bx^2+cx\right)dx\\ &= \sqrt{-\tfrac{\pi}{b}}\exp\left(-\tfrac{c^2}{4b}+\tfrac{c}{8b^2}-\tfrac{1}{96b^3}\right)Ai\left(\tfrac{1}{16b^2}-\tfrac{c}{2b}\right),\end{aligned} \tag{46}$$

we obtain the analytical propagation expression of the paraxial Airy beams

$$E(x,y,z) = Ai(T_x)\exp(M_x)\mathrm{Ai}(T_y)\exp(M_y)\exp(ikz), \tag{47}$$

with

$$T_x = \frac{x}{w_x} - \frac{z^2}{4k^2w_x^4} + \frac{ia_0 z}{kw_x^2}, \tag{48}$$

$$T_y = \frac{y}{w_y} - \frac{z^2}{4k^2w_y^4} + \frac{ia_0 z}{kw_y^2}, \tag{49}$$

$$M_x = \frac{a_0 x}{w_x} - \frac{a_0 z^2}{2k^2w_x^4} - \frac{iz^3}{12k^3w_x^6} + \frac{ia_0^2 z}{2kw_x^2} + \frac{izx}{2kw_x^3}, \tag{50}$$

$$M_y = \frac{a_0 y}{w_y} - \frac{a_0 z^2}{2k^2w_y^4} - \frac{iz^3}{12k^3w_y^6} + \frac{ia_0^2 z}{2kw_y^2} + \frac{iyz}{2kw_y^3}, \tag{51}$$

where $z_R = k{w_0}^2/2$ is the Rayleigh range.

4.5.2. Collins Formula

Substituting Equation (5) into Equation (8) and utilizing the integral formulas given by Equation (46), we obtain the analytical propagation expression of paraxial Airy beams passing through an ABCD optical system

$$E(x,y,z) = \frac{1}{A}\mathrm{Ai}(T'_x)\exp(M'_x)\mathrm{Ai}\left(T'_y\right)\exp\left(M'_y\right)\exp(ikz), \tag{52}$$

with

$$T'_x = \frac{x}{Aw_x} - \frac{B^2}{4A^2k^2w_x^4} + \frac{ia_0B}{Akw_x^2}, T'_y = \frac{y}{Aw_y} - \frac{B^2}{4A^2k^2w_y^4} + \frac{ia_0B}{Akw_y^2}, \tag{53}$$

$$M'_x = \frac{ikC}{2A}x^2 + \frac{a_0x}{Aw_x} - \frac{a_0B^2}{2A^2k^2w_x^4} - \frac{iB^3}{12A^3k^3w_x^6} + \frac{ia_0^2B}{2Akw_x^2} + \frac{iBx}{2A^2kw_x^3}, \tag{54}$$

$$M'_y = \frac{ikC}{2A}y^2 + \frac{a_0y}{Aw_y} - \frac{a_0B^2}{2A^2k^2w_y^4} - \frac{iB^3}{12A^3k^3w_y^6} + \frac{ia_0^2B}{2Akw_y^2} + \frac{iBy}{2A^2kw_y^3}. \tag{55}$$

4.5.3. Angular Spectrum Representation

Substituting Equation (5) into Equation (12) and using the integral formula given by Equation (46), we obtain the angular spectrum of the Airy beams [63]

$$\begin{aligned}\widetilde{E}(k_x,k_y) = {}& \tfrac{w_xw_y}{4\pi^2}\exp\left[-a_0\left(k_x^2w_x^2 + k_y^2w_y^2\right) + \tfrac{2}{3}a_0^3\right] \\ & \times\exp\left\{\tfrac{i}{3}\left[\left(k_x^3w_x^3 + k_y^3w_y^3\right) - 3a_0^2(k_xw_x + k_yw_y)\right]\right\}.\end{aligned} \tag{56}$$

After substituting Equation (56) into Equation (14) and performing the integration, we can obtain the analytical propagation expression of paraxial Airy beams, which coincides with Equation (47) derived by the Fresnel diffraction integral.

4.5.4. Rayleigh–Sommerfeld Diffraction Integral

Substituting Equation (5) into Equation (18) and applying the integral formula given by Equation (46), we obtain the analytical propagation expression of the non-paraxial Airy beams

$$\begin{aligned}E(x,y,z) = {}& \left(\tfrac{z}{\rho}\right)\exp\left[-\tfrac{ik(x^2+y^2)}{2\rho}\right] \\ & \times Ai(T''_x)\exp(M''_x)Ai(T''_y)\exp(M''_y)\exp(ik\rho),\end{aligned} \tag{57}$$

with

$$T''_x = \frac{x}{w_x} - \frac{\rho^2}{4k^2w_x^4} + \frac{ia_0\rho}{kw_x^2}, T''_y = \frac{y}{w_y} - \frac{\rho^2}{4k^2w_y^4} + \frac{ia_0\rho}{kw_y^2}, \tag{58}$$

$$M''_x = \frac{a_0x}{w_x} - \frac{a_0\rho^2}{2k^2w_x^4} - \frac{i\rho^3}{12k^3w_x^6} + \frac{ia_0^2\rho}{2kw_x^2} + \frac{ix\rho}{2kw_x^3}, \tag{59}$$

$$M''_y = \frac{a_0y}{w_y} - \frac{a_0\rho^2}{2k^2w_y^4} - \frac{i\rho^3}{12k^3w_y^6} + \frac{ia_0^2\rho}{2kw_y^2} + \frac{iy\rho}{2kw_y^3}, \tag{60}$$

where $\rho = \sqrt{x^2 + y^2 + z^2}$.

5. Numerical Results and Discussion

Based on the above derived analytical expressions, we perform some numerical simulations to illustrate the propagation of typical structured light beams under different conditions. In what follows, besides the parameters given below every figure, the common parameters are chosen as follows: the free space wavelength of all the structured light beams $\lambda = 632.8$ nm, the orders of the Hermite–Gaussian beams $m = n = 1$, the mode numbers of the Laguerre–Gaussian beams $p = l = 2$, the order of the Bessel beams $m = 2$, and the decay parameter of the Airy beams $a_0 = 0.15$.

Firstly, the paraxial propagation of typical structured light beams in the free space is simulated and analyzed. In the simulations, the beam waist radius of the Gaussian-type beams, which include the fundamental Gaussian beams, the Hermite–Gaussian beams, and the Laguerre–Gaussian beams, is set as $w_0 = 2.0\lambda$, the half-cone angle of the Bessel beams is set as $\theta_0 = 5^o$, and the transverse scaled parameters of the Airy beams are chosen as $w_x = w_y = 2.0\lambda$. In Figure 1, we display the transverse intensity distributions of typical structured light beams under the paraxial approximation at different propagation distances. As we can see, with increasing propagation distance, the distributions of intensity in the transverse plane for the fundamental Gaussian beam, the Hermite–Gaussian beam, and the Laguerre–Gaussian beam gradually go away from the center due to diffraction effects. In contrast, the transverse intensity distributions of the Bessel beam and the Airy beam remain almost invariable during propagation. This arises from the fact that both the Bessel beam and the Airy beam have the characteristic of diffraction-free propagation under the paraxial approximation.

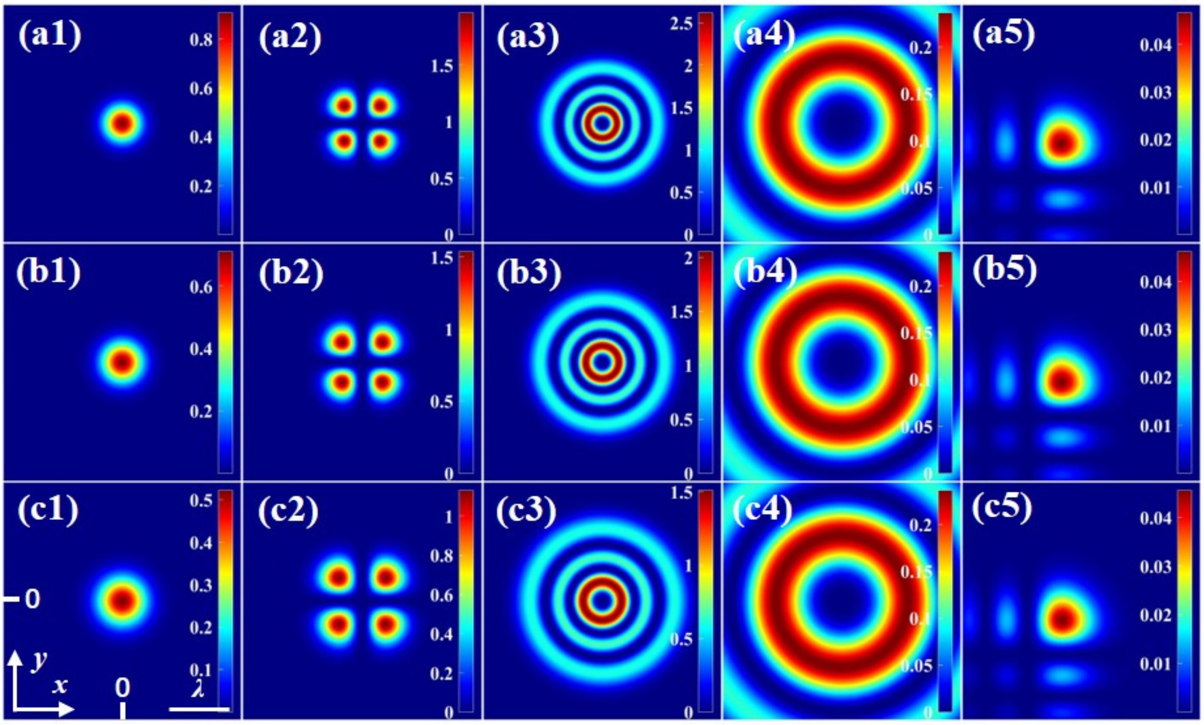

Figure 1. Transverse intensity distributions of typical structured light beams under the paraxial approximation at different propagation distances. (**a1**–**a5**) $z = 4\lambda$, (**b1**–**b5**) $z = 8\lambda$, and (**c1**–**c5**) $z = 12\lambda$. Shown from left to right are the cases of the fundamental Gaussian beam, the Hermite–Gaussian beam, the Laguerre–Gaussian beam, the Bessel beam, and the Airy beam, respectively.

Next, we examine the non-paraxial propagation of typical structured light beams in the free space. The non-paraxial parameters used in simulations are set as follows: the beam waist radius of the fundamental Gaussian beams, the Hermite–Gaussian beams, and the Laguerre–Gaussian beams $w_0 = 0.6\lambda$, the half-cone angle of the Bessel beams $\theta_0 = 40^o$, and the transverse scaled parameters of the Airy beams $w_x = w_y = 0.6\lambda$. Figure 2 shows the transverse intensity distributions of typical structured light beams beyond the paraxial approximation at different propagation distances. It can be observed that the distributions of intensity in the transverse plane for the fundamental Gaussian beam, the Hermite–Gaussian beam, and the Laguerre–Gaussian beam rapidly go away from the center with increasing propagation distance, which indicates that the diffraction phenomenon of non-paraxial beams is more pronounced than that of paraxial beams. It is also observed that the non-paraxial Airy beam no longer exhibits the characteristics of diffraction-free propagation, and its peak intensity distribution will become wider with the propagation distance increasing, which is completely different from the paraxial Airy beam. Most notably, the Bessel beam still has the characteristic of diffraction-free propagation beyond the paraxial approximation, as exhibited in Figure 2a4–c4.

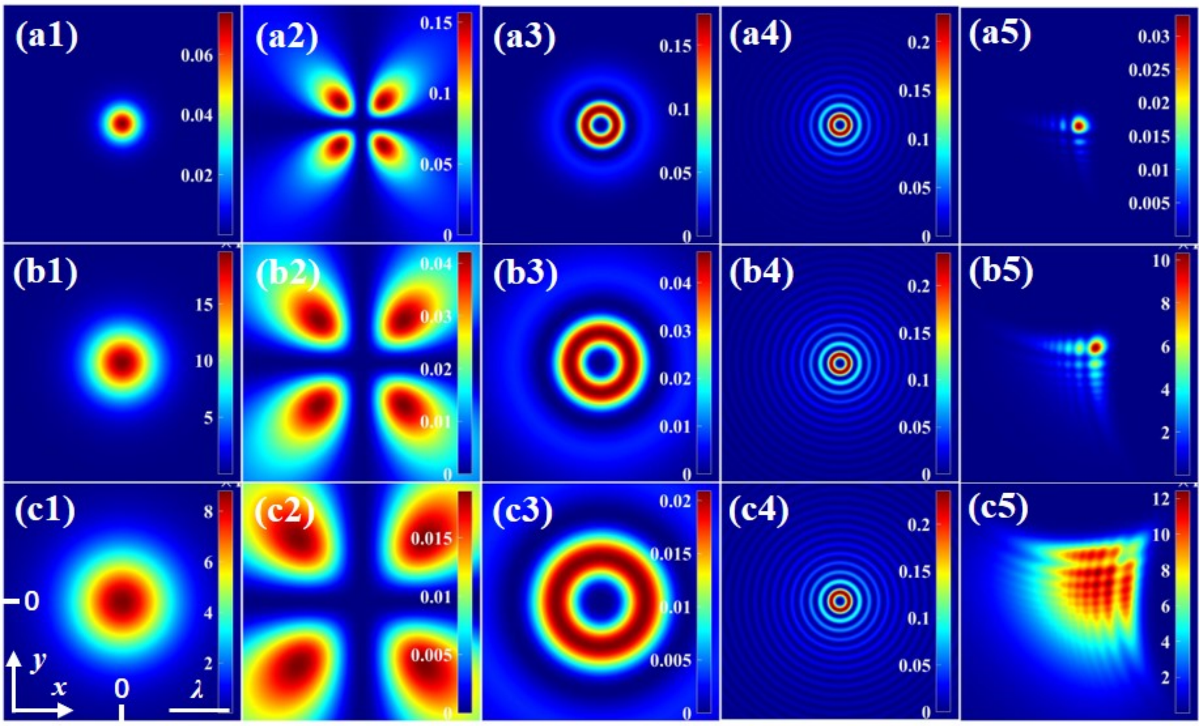

Figure 2. Transverse intensity distributions of typical structured light beams beyond the paraxial approximation at different propagation distances. (**a1**–**a5**) $z = 4\lambda$, (**b1**–**b5**) $z = 8\lambda$, and (**c1**–**c5**) $z = 12\lambda$. Shown from left to right are the cases of the fundamental Gaussian beam, the Hermite–Gaussian beam, the Laguerre–Gaussian beam, the Bessel beam, and the Airy beam, respectively.

Finally, based on the analytical propagation expressions of typical structured light beams described with the Collins formula, we simulate their propagation in a gradient-index medium with the following refractive index [64–66]:

$$n = n_0\left(1 - \frac{r^2}{2\beta^2}\right), \tag{61}$$

where n_0 is the refractive index on the symmetry axis, $r = \sqrt{x^2 + y^2}$ is the radial distance from the symmetry axis, and β is the distribution factor involved in the determination of the gradient-index distribution. For such a gradient-index medium, the ABCD transfer matrix can be expressed as [56,67]

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \cos(z/\beta) & \frac{\beta \sin(z/\beta)}{n_0} \\ \frac{-n_0 \sin(z/\beta)}{\beta} & \cos(z/\beta) \end{bmatrix}, \tag{62}$$

where z is the propagation distance of the beam in the gradient-index medium.

Figure 3 illustrates the propagation of typical structured light beams in the gradient-index medium described above, where the parameters used in simulations are set as $n_0 = 1.594$, $\beta = 0.05$, $w_0 = 2.0\lambda$, $\theta_0 = 5^o$, and $w_x = w_y = 2.0\lambda$. Since the intensity pattern reconstructs itself repeatedly, only one propagation period $L = 2\pi\beta$ is shown. It can be seen from Figure 3 that the Fundamental Gaussian beam, the Hermite–Gaussian beam, and the Laguerre–Gaussian beam converge and increase sharply in intensity at $z = L/2$, while the Bessel beam focuses at $z = L/4$ and $z = 3L/4$. In addition, the Airy beam exhibits two singularities at $z = L/4$ and $z = 3L/4$, respectively. The sidelobe of the intensity distribution of the Airy beam weakens as it approaches the singularities and reconstructs gradually when close to maximum intensity, demonstrating the acceleration and self-healing nature of the Airy beam.

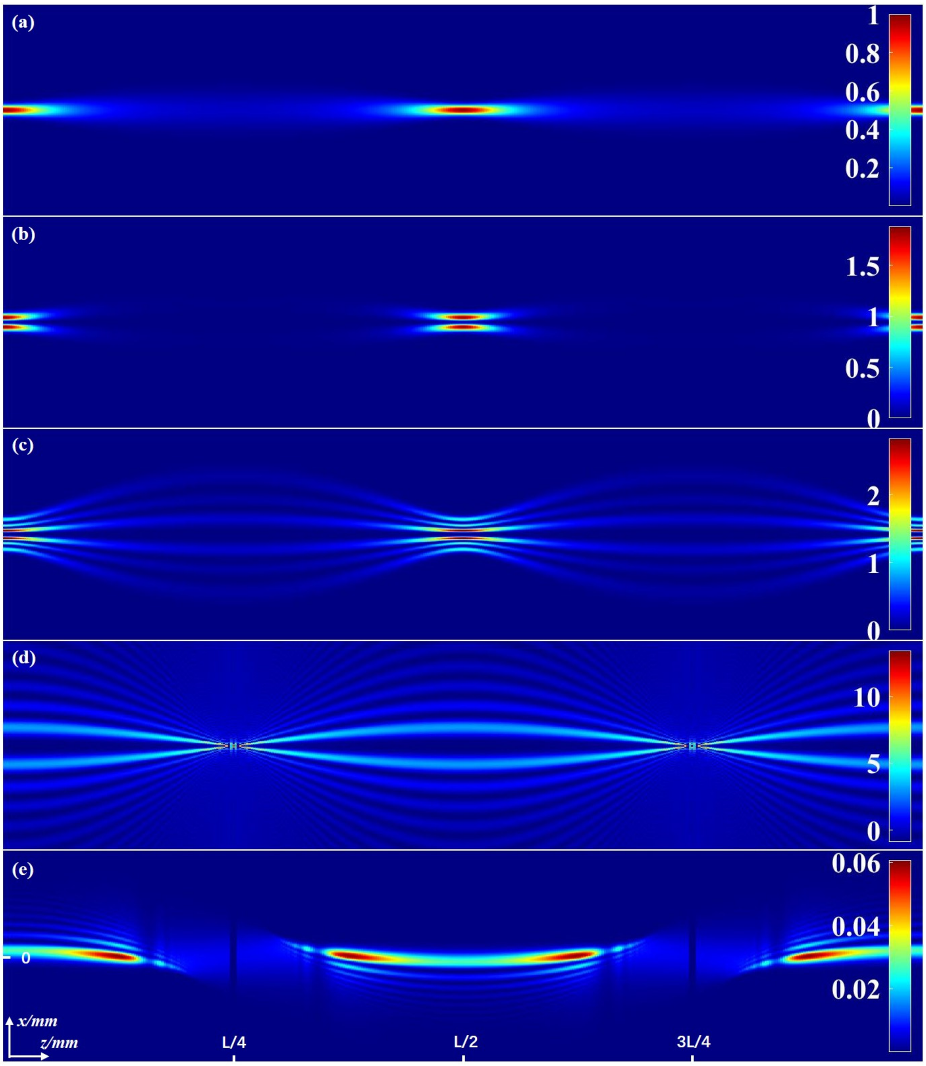

Figure 3. Illustrations of the propagation of typical structured light beams in a gradient-index medium. (**a**) Fundamental Gaussian beam, (**b**) Hermite–Gaussian beam, (**c**) Laguerre–Gaussian beam, (**d**) Bessel beam, and (**e**) Airy beam.

6. Conclusions

In conclusion, a summary of the main scalar diffraction theories of structured light beams is provided. The main scalar diffraction theories of light beams, including the Fresnel diffraction integral, Collins formula, angular spectrum representation, and Rayleigh–Sommerfeld diffraction integral, are summarized and compared. On the base of such theories, the concise and explicit analytical expressions for the propagation of typical structured light beams under different conditions are systematically derived, and the consistency between these expressions is analyzed. Some numerical simulations are performed to illustrate the paraxial and non-paraxial propagation of typical structured light beams. This work is expected to be beneficial to the study of the propagation effects of structured light beams by using various scalar diffraction theories.

Author Contributions: Conceptualization, F.W. and Z.C.; methodology, Y.L. and Z.C.; software, Y.L.; validation, Y.L.; investigation, F.W. and Z.C.; writing—original draft preparation, F.W. and Y.L.; writing—review and editing, Z.C.; supervision, Z.C.; Funding acquisition, F.W. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the Natural Science Basic Research Program of Shaanxi, grant number 2023-JC-YB-536, and the Guangdong Basic and Applied Basic Research Foundation, grant number 2022A1515011138.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank the editor and anonymous reviewers who handled our paper.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Siegman, A.E. Hermite–Gaussian Functions of Complex Argument as Optical-Beam Eigenfunctions. *J. Opt. Soc. Am.* **1973**, *63*, 1093–1094. [CrossRef]
2. Allen, L.; Beijersbergen, M.W.; Spreeuw, R.J.C.; Woerdman, J.P. Orbital Angular Momentum of Light and the Transformation of Laguerre-Gaussian Laser Modes. *Phys. Rev. A* **1992**, *45*, 8185. [CrossRef]
3. Durnin, J. Exact Solutions for Nondiffracting Beams. I. The Scalar Theory. *J. Opt. Soc. Am. A* **1987**, *4*, 651–654. [CrossRef]
4. Siviloglou, G.A.; Broky, J.; Dogariu, A.; Christodoulides, D.N. Observation of Accelerating Airy Beams. *Phys. Rev. Lett.* **2007**, *99*, 213901. [CrossRef] [PubMed]
5. Cui, Z.; Wang, J.; Ma, W.; Zhao, W. Concise and Explicit Expressions for Typical Spatial-Structured Light Beams beyond the Paraxial Approximation. *J. Opt. Soc. Am. A* **2022**, *39*, 1794–1804. [CrossRef]
6. Won, R. Structured Light Spiralling Up. *Nat. Photonics* **2017**, *11*, 619–622. [CrossRef]
7. Rubinsztein-Dunlop, H.; Forbes, A.; Berry, M.V.; Dennis, M.R.; Andrews, D.L.; Mansuripur, M.; Denz, C.; Alpmann, C.; Banzer, P.; Bauer, T.; et al. Roadmap on Structured Light. *J. Opt.* **2016**, *19*, 013001. [CrossRef]
8. Forbes, A. Structured Light from Lasers. *Laser Photonics Rev.* **2019**, *13*, 1900140. [CrossRef]
9. Angelsky, O.V.; Bekshaev, A.Y.; Hanson, S.G.; Zenkova, C.Y.; Mokhun, I.I.; Jun, Z. Structured Light: Ideas and Concepts. *Front. Phys.* **2020**, *8*, 114. [CrossRef]
10. Forbes, A.; de Oliveira, M.; Dennis, M.R. Structured Light. *Nat. Photonics* **2021**, *15*, 253–262. [CrossRef]
11. Cox, M.A.; Mphuthi, N.; Nape, I.; Mashaba, N.; Cheng, L.; Forbes, A. Structured Light in Turbulence. *IEEE J. Sel. Top. Quant.* **2020**, *27*, 1–21. [CrossRef]
12. Yang, Y.; Ren, Y.; Chen, M.; Arita, Y.; Rosales-Guzmán, C. Optical Trapping with Structured Light: A Review. *Adv. Photon.* **2021**, *3*, 034001. [CrossRef]
13. Wang, J.; Liang, Y. Generation and Detection of Structured Light: A Review. *Front. Phys.* **2021**, *9*, 688284. [CrossRef]
14. Jun, Y.; Dian-Yuan, F.; Shi-Ji, W.; Yuan, G. Generalized Fresnel Diffraction Integral and Its Applications. *Chin. Phys.* **2000**, *9*, 119. [CrossRef]
15. Ding, G.; Lü, B. Generalized Huygens–Fresnel Diffraction Integral for Misaligned Asymmetric First-Order Optical Systems and Decentered Anisotropic Gaussian Schell-Model Beams. *J. Opt. Soc. Am. A* **2002**, *19*, 485–490. [CrossRef]
16. Cywiak, M.; Cywiak, D.; Yáñez, E. Finite Gaussian Wavelet Superposition and Fresnel Diffraction Integral for Calculating the Propagation of Truncated, Non-Diffracting and Accelerating Beams. *Opt. Commun.* **2017**, *405*, 132–142. [CrossRef]
17. Feshchenko, R.M.; Vinogradov, A.V.; Artyukov, I.A. Propagation of Waves from an Arbitrary Shaped Surface—A Generalization of the Fresnel Diffraction Integral. *Opt. Commun.* **2018**, *413*, 291–294. [CrossRef]
18. Liu, Z.; Wu, X.; Fan, D. Collins Formula in Frequency-Domain and Fractional Fourier Transforms. *Opt. Commun.* **1998**, *155*, 7–11. [CrossRef]
19. Zhao, D.M. Collins Formula in Frequency-Domain Described by Fractional Fourier Transforms or Fractional Hankel Transforms. *Optik* **2000**, *111*, 9–12.
20. Zhao, D.; Wang, S. Collins Formula in Spatial-Domain Written in Terms of Fractional Fourier Transform or Fractional Hankel Transform. *Optik* **2000**, *111*, 371–374.
21. Lin, Q.; Wang, L. Collins Formula and Tensor ABCD Law in Spatial-Frequency Domain. *Opt. Commun.* **2000**, *185*, 263–269. [CrossRef]

22. Cai, L.Z.; Yang, X.L. Collins Formulae in Both Space and Frequency Domains for ABCD Optical Systems with Small Deformations. *J. Mod. Opt.* **2001**, *48*, 1389–1396. [CrossRef]
23. Li, J.; Li, C. Algorithm Study of Collins Formula and Inverse Collins Formula. *Appl. Opt.* **2008**, *47*, A97–A102. [CrossRef] [PubMed]
24. Chuan-Mei, X.; Hong-Yi, F. A Generalized Collins Formula Derived by Virtue of the Displacement-Squeezing Related Squeezed Coherent State Representation. *Chin. Phys. B* **2010**, *19*, 064207. [CrossRef]
25. Fagerholm, J.; Friberg, A.T.; Huttunen, J.; Morgan, D.P.; Salomaa, M.M. Angular-Spectrum Representation of Nondiffracting X Waves. *Phy. Rev. E* **1996**, *54*, 4347. [CrossRef]
26. Tervo, J.; Turunen, J. Angular Spectrum Representation of Partially Coherent Electromagnetic Fields. *Opt. Commun.* **2002**, *209*, 7–16. [CrossRef]
27. Borghi, R. On the Angular-Spectrum Representation of Multipole Wave Fields. *J. Opt. Soc. Am. A* **2004**, *21*, 1805–1810. [CrossRef] [PubMed]
28. Arnoldus, H.F. Angular Spectrum Representation of the Electromagnetic Multipole Fields, and Their Reflection at a Perfect Conductor. *Surf. Sci.* **2005**, *590*, 101–116. [CrossRef]
29. Gbur, G.; Korotkova, O. Angular Spectrum Representation for the Propagation of Arbitrary Coherent and Partially Coherent Beams through Atmospheric Turbulence. *J. Opt. Soc. Am. A* **2007**, *24*, 745–752. [CrossRef]
30. Korotkova, O.; Gbur, G. Angular Spectrum Representation for Propagation of Random Electromagnetic Beams in a Turbulent Atmosphere. *J. Opt. Soc. Am. A* **2007**, *24*, 2728–2736. [CrossRef]
31. Veerman, J.A.; Rusch, J.J.; Urbach, H.P. Calculation of the Rayleigh–Sommerfeld Diffraction Integral by Exact Integration of the Fast Oscillating Factor. *J. Opt. Soc. Am. A* **2005**, *22*, 636–646. [CrossRef] [PubMed]
32. Shen, F.; Wang, A. Fast-Fourier-Transform Based Numerical Integration Method for the Rayleigh-Sommerfeld Diffraction Formula. *Appl. Opt.* **2006**, *45*, 1102–1110. [CrossRef]
33. Nascov, V.; Logofătu, P.C. Fast Computation Algorithm for the Rayleigh-Sommerfeld Diffraction Formula Using a Type of Scaled Convolution. *Appl. Opt.* **2009**, *48*, 4310–4319. [CrossRef] [PubMed]
34. Sheppard, C.J.R.; Lin, J.; Kou, S.S. Rayleigh–Sommerfeld Diffraction Formula in k Space. *J. Opt. Soc. Am. A* **2013**, *30*, 1180–1183. [CrossRef] [PubMed]
35. Pozzi, G. A Computer Algebra Derivation of the Rayleigh–Sommerfeld Diffraction Formula by a Plane Screen. *Optik* **2015**, *126*, 4405–4409. [CrossRef]
36. Ochoa, N.A. Alternative Approach to Evaluate the Rayleigh-Sommerfeld Diffraction Integrals Using Tilted Spherical Waves. *Opt. Express* **2017**, *25*, 12008–12019. [CrossRef]
37. Mehrabkhani, S.; Schneider, T. Is the Rayleigh-Sommerfeld Diffraction Always an Exact Reference for High Speed Diffraction Algorithms? *Opt. Express* **2017**, *25*, 30229–30240. [CrossRef]
38. Buitrago-Duque, C.; Garcia-Sucerquia, J. Non-Approximated Rayleigh–Sommerfeld Diffraction Integral: Advantages and Disadvantages in the Propagation of Complex Wave Fields. *Appl. Opt.* **2019**, *58*, G11–G18. [CrossRef]
39. Gao, Y.; Xie, J.; Yu, X. Rayleigh-Sommerfeld Diffraction on a Subwavelength Scale: Theories and a Resolution Criterion. *Phys. Rew. A* **2019**, *99*, 023814. [CrossRef]
40. Chen, N.; Wang, C.; Heidrich, W. HTRSD: Hybrid Taylor Rayleigh-Sommerfeld Diffraction. *Opt. Express* **2022**, *30*, 37727–37735. [CrossRef]
41. Tao, X.; Zhou, N.; Lü, B. Recurrence Propagation Equation of Hermite-Gaussian Beams through a Paraxial Optical ABCD System with Hard-Edge Aperture. *Optik* **2003**, *114*, 113–117. [CrossRef]
42. Lu, B.; Wu, P. Analytical Propagation Equation of Astigmatic Hermite-Gaussian Beams through a 4x4 Paraxial Optical System and Their Symmetrizing Transformation. *Opt. Laser Technol.* **2003**, *35*, 497–504. [CrossRef]
43. Cai, Y.; Chen, C. Paraxial Propagation of a Partially Coherent Hermite-Gaussian Beam through Aligned and Misaligned ABCD Optical Systems. *J. Opt. Soc. Am. A* **2007**, *24*, 2394–2401. [CrossRef]
44. Ortiz-Acebedo, A.; Mata-Mendez, O.; Chavez-Rivas, F. Diffraction of Hermite-Gaussian Beams by Ronchi and Aperiodic Rulings. *Rev. Mex. Fis.* **2008**, *54*, 35–41.
45. Lü, B.; Peng, R. Relative Phase Shift in Laguerre-Gaussian Beams Propagating through an Apertured Paraxial ABCD System. *J. Mod. Optic.* **2003**, *50*, 857–865. [CrossRef]
46. Mei, Z.; Zhao, D.; Gu, J. Propagation of Elegant Laguerre–Gaussian Beams through an Annular Apertured Paraxial ABCD Optical System. *Opt. Commun.* **2004**, *240*, 337–343. [CrossRef]
47. Cai, Y.; He, S. Propagation of a Laguerre–Gaussian Beam through a Slightly Misaligned Paraxial Optical System. *Appl. Phys. B* **2006**, *84*, 493–500. [CrossRef]
48. Luo, H.; Ren, Z.; Shu, W.; Wen, S. Reversed Propagation Dynamics of Laguerre-Gaussian Beams in Left-Handed Materials. *Phys. Rev. A* **2008**, *77*, 023812. [CrossRef]
49. Xu, Y. Propagation of Elegant Laguerre Gaussian Beam through a Misaligned Optical System. *Opt. Commun.* **2013**, *308*, 237–242. [CrossRef]
50. Li, H.; Honary, F.; Wu, Z.; Bai, L. Reflection and Transmission of Laguerre-Gaussian Beams in a Dielectric Slab. *J. Quant. Spectrosc. Ra.* **2017**, *195*, 35–43. [CrossRef]

51. Volyar, A.V.; Abramochkin, E.G.; Razueva, E.V.; Bretsko, M.V.; Akimova, Y.E. Geometry of spiral beams: 3D curved structured vortex beams and optical currents. *J. Opt.* **2021**, *23*, 044003. [CrossRef]
52. Volyar, A.; Abramochkin, E.; Akimova, Y.; Bretsko, M. Destroying and recovering spiral vortex beams due to figured perturbations. *J. Opt. Soc. Am. A* **2021**, *38*, 1793–1802. [CrossRef]
53. Volyar, A.; Akimova, Y. Structural stability of spiral vortex beams to sector perturbations. *Appl. Opt.* **2021**, *60*, 8865–8874. [CrossRef]
54. Vahimaa, P.; Kettunen, V.; Kuittinen, M.; Turunen, J.; Friberg, A.T. Electromagnetic Analysis of Nonparaxial Bessel Beams Generated by Diffractive Axicons. *J. Opt. Soc. Am. A* **1997**, *14*, 1817–1824. [CrossRef]
55. El Halba, E.M.; Boustimi, M.; Ez-zariy, L.; Hennani, S.; Belafhal, A. Propagation Characteristics of Bessel-like Beams through ABCD Optical System. *Opt. Quant. Electron* **2017**, *49*, 269. [CrossRef]
56. Cao, Z.; Zhai, C.; Xu, S.; Chen, Y. Propagation of On-Axis and off-Axis Bessel Beams in a Gradient-Index Medium. *J. Opt. Soc. Am. A* **2018**, *35*, 230–235. [CrossRef]
57. Cao, Z.; Zhai, C. Propagation of Bessel Beam in a Strongly Nonlocal Nonlinear Media. *Optik* **2019**, *178*, 301–309. [CrossRef]
58. Torre, A. Airy Beams beyond the Paraxial Approximation. *Opt. Commun.* **2010**, *283*, 4146–4165. [CrossRef]
59. Lin, H.-C.; Pu, J.-X. Propagation of Airy Beams from Right-Handed Material to Left-Handed Material. *Chin. Phys. B* **2012**, *21*, 054201. [CrossRef]
60. Wen, W.; Lu, X.; Zhao, C.; Cai, Y. Propagation of Airy Beam Passing through the Misaligned Optical System with Hard Aperture. *Opt. Commun.* **2014**, *313*, 350–355. [CrossRef]
61. Wen, W.; Chu, X. Propagation of Symmetric Tunable Dual Airy Beam through ABCD Optical System. *Opt. Commun.* **2014**, *333*, 38–44. [CrossRef]
62. Li, H.; Wang, J.; Tang, M.; Cao, J.; Li, X. Propagation of Airy Beams in the Quadratic-Index Medium Based on Matrix Optics. *Optik* **2017**, *149*, 144–148. [CrossRef]
63. Hui, Y.; Cui, Z.; Zhao, M.; Han, Y. Vector Wave Analysis of Airy Beams upon Reflection and Refraction. *J. Opt. Soc. Am. A* **2020**, *37*, 1480–1489. [CrossRef]
64. Feng, L.; Zhang, J.; Pang, Z.; Wang, L.; Zhong, T.; Yang, X.; Deng, D. Propagation Properties of the Chirped Airy Beams through the Gradient-Index Medium. *Opt. Commun.* **2017**, *402*, 60–65. [CrossRef]
65. Zhao, R.; Deng, F.; Yu, W.; Huang, J.; Deng, D. Propagation Properties of Airy–Gaussian Vortex Beams through the Gradient-Index Medium. *J. Opt. Soc. Am. A* **2016**, *33*, 1025–1031. [CrossRef]
66. Zeng, Z.; Zhou, K.; Deng, D. Propagation of Cosine Complex Variable Function Airy–Gaussian Beams through the Gradient-Index Medium. *Appl. Phys. B* **2019**, *125*, 91. [CrossRef]
67. Pei, S.; Xu, S.; Cui, F.; Pan, Q.; Cao, Z. Propagation of a Bessel–Gaussian Beam in a Gradient-Index Medium. *Appl. Opt.* **2019**, *58*, 920–926. [CrossRef]