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Determination of Critical Reynolds Number for the Flow Near a Rotating Disk on the Basis of the Theory of Stochastic Equations and Equivalence of Measures †

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† This article is dedicated to the memory of Academician N. A. Anfimov.

Abstract: The determination of the flow regime of liquid and gas in power plants is the most important design task. Performing the calculations based on modern calculation methods requires a priori knowledge of the initial and boundary conditions, which significantly affect the final results. The purpose of the article is to present the solution for the critical Reynolds number for the flow near a rotating disk on the basis of the theory of stochastic equations of continuum laws and equivalence of measures between random and deterministic motions. The determination of the analytical dependence for the critical Reynolds number is essential for the study of flow regimes and the thermal state of disks and blades in the design of gas and steam turbines. The result of the calculation with using the new formula shows that for the flow near a wall of rotating disk, the critical Reynolds number is 325,000, when the turbulent Reynolds is 5 ÷ 10 and the degree of turbulence is 0.01 ÷ 0.02. Therefore, the result of solution shows a satisfactory correspondence of the obtained analytical dependence for the critical Reynolds number with the experimental data.

Keywords: stochastic equations; equivalence of measures; nature of turbulence; critical Reynolds number

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1. Introduction

The development of new physical and mathematical theories for phenomena occurring in the nature and technical devices requires the constant application of the theory for observed varieties of this phenomenon. Therefore, each application of the theory to a specific process also requires the theoretical comparison between the existing fundamental methods instead of only the mathematical solutions. It makes possible for scientists and specialists in various fields to understand the evolution of theoretical ideas as well as the difference, the essence, and the advantage of the new scientific methodology.

The theories on the nature of the turbulence were formulated in [1–10]. The main principles of the theory of measures of random stationary processes are described in the publications of Kolmogorov and Khinchin. These works underlie the statistical theory of turbulence. On the basis of statistical theory, Obukhov and, later, Heisenberg proposed the statistical theory for the process of generation of the turbulent field. It is also worth to mention the publications of J. Taylor in which one first tried to determine the critical Reynolds number in the function of initial parameters of the fluctuation. However, J. Taylor determined this dependence semi-empirically for one type of flow. The linear theory of turbulence led to a certain success. This is especially true for the Orr–Sommerfeld equation, which allowed calculating only the critical Reynolds numbers. A special place is occupied by the Landau theory, which has a qualitative character when describing the turbulence as a quasi-periodic process. However, even in this case, it is impossible to calculate the flow characteristics despite the qualitative description of the turbulence. Klimontovich investigated the Leontovich and Sato equations and presented the mathematical formulation of the entropy change, when the turbulence occurs. However, this theory only enables us to represent the turbulence process qualitatively without calculating the characteristics of the phenomenon.

The development of the theory of strange attractors and mathematical methods for obtaining a strict solution of the Navier–Stokes equation are presented in [11–32]. It is known that the theory of strange attractors is based on the measure theory, which allows one deducing the Kolmogorov–Sinai entropy. Somewhat later, a more general formula for the Renyi entropy made it possible to extend the application of attractor theory. However, this theory allows determining only the increase in the number of degrees of freedom in time. This theory does not allow explaining the spatial change in increasing number of degrees of freedom.

The theory of solitons was also considered to be useful for certain time for explaining the origin of the turbulence [10–12]. However, the results of the investigation of solutions to the Korteweg–de Vries equation provide no basis for determining and calculating the characteristics of the turbulence phenomenon.

The statistical and stochastic equations and the numerical methods for investigating the turbulent processes are presented in [33–53]. The above fundamentals of statistical hydrodynamics and the development of computer technology and numerical methods enabled us to implement the solution of moment equations for second- and higher-order correlations using the RANS method [39–50], and later the LES method [39–43].

The most powerful numerical method called DNS is represented by three methodologies that are fundamentally different from each other. Therefore, the study of the origin of turbulence by each of the methods has several features that require explanation. However, all DNS methods are very sensitive to the initial and boundary conditions for each type of the hydrodynamic flow.

In certain cases, the instantaneous Navier–Stokes equations undergo the artificial "stochasticization" by adding the left-side additional term. In this case, we need to do the same in the continuity and energy equations. However, it is necessary to be aware that if it is done for the instantaneous equations, then an open thermodynamic system is obtained. However, these methods failed in determining the unified physics and cause of the turbulence process [33–35].

The special attention was focused on the theoretical solutions for the critical Reynolds number. It should be noted that the most well-known ratio based on the theory of dimension was, as is known, determined with using the experimental data [37–42,52,53]. Therefore, on the basis of these experimental formulas, it was impossible to obtain the new theory for determining analytically the dependences for the critical Reynolds number of turbulence in different flows.

At the same time, it is known that in an arbitrary hydrodynamic flow, there are initial disturbances generated by various causes, both technical and natural. Therefore, it is obvious that whether or not the particular disturbance, which arises, exists and develops, depends on the interaction of the main undisturbed motion with this initial disturbance.

The stochastic theory of turbulence based on the stochastic equations and the theory of equivalent measures make it possible deriving the analytical dependences for the first and second critical Reynolds numbers in the cases of the isothermal and non-isothermal flows on the smooth flat plate and in the round tube [54–59]. The progress of this theory gives a new method for determining the analytical dependences for the profiles of averaged velocity and the temperature fields [60,61], the friction and heat transfer coefficients [62–64], the second-order correlations [58,65,66], the correlation dimension of the attractor in the boundary layer [67–70], the theoretical solutions for the spectral function of the turbulent medium [71–73], and the formula for the Reynolds analogy [74–76].

It should also be noted that from the results obtained on the basis of the theory of stochastic equations and the theory of equivalent measures, it was possible for the first time to investigate the analytical relations for calculating the spatial distribution of the number of degrees of freedom of a strange attractor. Such distributions of the attractor correlation dimension were calculated for the flow in a tube, on a flat plate, and in the Earth's atmosphere.

Also, as a result, it was determined that the spectrum $E(k)_j$ depends on the wave numbers k for the interval of generation of turbulence in the form $E(k)_j \sim k^n$, $n = 1.2 \div 1.5$. This formula was named the ratio of uncertainty in turbulence generation [62].

The uncertainty relation derived analytically determines the fact that in the turbulence generation region, there is the family of perturbations—the vortices, which have a space–energy similarity ($E{\cdot}L^{-a}$) = constant, and each of the perturbations of this family can interact with the main flow, which leads to the origin and development of turbulence [62]. Moreover, for each type of flow, whether or not it is the flow in a pipe or along a flat plate, the spatial–energy similarity has its own value of the indicator "a". Therefore, there is an uncertainty in both the geometric and energy parameters of the perturbation when determining the interaction with the main motion.

In accordance with [49–51], the essence of the discovery of the theory of equivalence of measures in the stochastic process determines the beginning of the interaction between the deterministic and random field. It is found that this interaction begins when the mass shift, the momentum shift, and the energy shift of the main undisturbed flow is equal to the mass fluctuation, momentum, and energy of the random field in the space edge, which is commensurate with the linear measure of the perturbation at a fixed time. As a result, the equivalence of substantial time derivatives is observed in the interaction domain.

It should be noted that the main part of publications [54–76] is devoted to such types of fluid flows as the flow in a round tube and the flow along a smooth flat plate for which there is a considerable experimental material, which allows calculating the parameters with using new formulas. Therefore, it is interesting to consider other types of fluid flows, which are also important for both theory and practice. In this connection, we presented here the solution for the critical Reynolds number for the motion near a rotating disk.

2. Conservation Equations for Stochastic Process

The equations derived in [54–59] take the form:

The equation of mass (continuity)

$$\frac{d(\rho)_{col_{st}}}{d\tau} = -\frac{(\rho)_{st}}{\tau_{cor}} - \frac{d(\rho)_{st}}{d\tau}, \tag{1}$$

the momentum equation

$$\frac{d\left(\rho\vec{U}\right)_{col_{st}}}{d\tau} = div(\tau_{i,j})_{col_{st}} + div(\tau_{i,j})_{st} - \frac{(\rho\vec{U})_{st}}{\tau_{cor}} - \frac{d(\rho\vec{U})_{st}}{d\tau} + F_{col_{st}} + F_{st} \tag{2}$$

and the energy equation

$$\begin{aligned} \frac{dE_{col_{st}}}{d\tau} = & div\left(\lambda\frac{\partial T}{\partial x_j} + u_i\tau_{i,j}\right)_{col_{st}} + div\left(\lambda\frac{\partial T}{\partial x_j} + u_i\tau_{i,j}\right)_{st} \\ & -\left(\frac{E_{st}}{\tau_{cor}}\right) - \left(\frac{dE_{st}}{d\tau}\right) + (u_iF)_{col_{st}} + (u_iF)_{st} \end{aligned} \tag{3}$$

Here, E, ρ, $\vec{U}$, u_i, u_j, u_l, μ, τ, $\tau_{i,j}$ are the energy, the density, the velocity vector, and the velocity components in the directions x_i, x_j, x_l ($i, j, l = 1, 2, 3$); the dynamic viscosity, the time, and the stress tensor $\tau_{i,j} = P + \sigma_{i,j}$, $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ for $i \neq j$. P is the pressure of liquid or gas; λ is the thermal conductivity; c_p and c_v are the specific heat at constant pressure and volume, respectively; F is the external force, and $\sigma_{i,j} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \delta_{ij}\left(\xi - \frac{2}{3}\mu\right)\frac{\partial u_l}{\partial x_l}$, $(\tau_{cor}) = \frac{L}{\left(\left(E_{st}\right)_{U,P}/\rho\right)^{1\backslash 2}}$.

Furthermore, $L = L_{U,P} = L_U$ is the scale of turbulence. The subscripts $(_{U,P})$ and $(_U)$ refer to the velocity field and the subscript $(_T)$ refers to the temperature field. L_y on $x_2 = y$, or L_x,

$x_1 = x$. Here, x_1 and x_2 are the coordinates along the wall and normal to it. The subscript "col_{st}" refers to the components, which are actually the deterministic. The subscript "st" refers to the component, which are actually the stochastic. As a result, using the law of the equivalency of measures between the random and deterministic process at the critical point, we obtained the sets of stochastic equations of mass, momentum, and energy for the next space–time areas: (1) the onset of generation (subscript 1, 0, or 1); (2) the generation of turbulence (subscript 1,1); (3) the diffusion (1,1,1) or 1; (1,1), and (4) the dissipation of the turbulent fields.

The resulting set of equations can be described by a correlator, which can be written for each of the four space–time domains (N, M). This correlator also determines the probability of the fractal origin of this interaction instead of only determining the set of equations for the interaction of the random and deterministic fields. Therefore, in accordance with [54–59], this correlator in space–time is

$$\lim_{m_i \to m_c; r_i \to r_c; \Delta\tau_i \to \tau_c} (D_{N,M}(m_i; r_i; \Delta\tau_i)) = 0 \tag{4}$$

$$D_{N,M}(m_c; r_c; \tau_c) = \sum_i \lim_{m_i \to m_c} \lim_{r_i \to r_c} \lim_{\Delta\tau_i \to \tau_c} \left\{ m\left(T^M Z^* \cap T^N Y^*\right) - R_{1_{T^M Z^* T^N Y^*}} m\left(T^M Z^*\right) \right\} \tag{5}$$

The subscript j denotes the parameters m_{cj} (j = 3 means the mass, the momentum, and the energy). For the case of the binary intersections, it was written that $X = Y + Z + W$. Subsets Y, Z, and W are called extended in X as $\{Y^*, Z^*, W^*\}$ if the measures $m(Y)$, $m(Z)$, and $m(W)$ have the properties [54–59]:

$$\begin{gathered}
m(Y) = m(Y^*) = m(T^n Y) + \bigcup_{k=0}^{k=n-1} m(T^k(G_1^{n-k})) \text{and wandering subsets} \bigcup_{k=0}^{k=n-1} m(T^k(G_1^{n-k})) \subset Y; \\
m(Z) = m(Z^*) = m(T^n Z^*) + \bigcup_{k=0}^{k=n-1} m\left(T^k\left(G_2^{n-k}\right)\right) \text{and wandering subsets} \bigcup_{k=0}^{k=n-1} m(T^k(G_2^{n-k})) \subset Z; \\
m(W) = m(W^*) = m(T^n W) + \bigcup_{k=0}^{k=n-1} m\left(T^k\left(G_3^{n-k}\right)\right) \text{and wandering subsets} \bigcup_{k=0}^{k=n-1} m(T^k(G_3^{n-k})) \subset W.
\end{gathered} \tag{6}$$

Here G_1^n is the wandering subset of the expanded subset $Y^* \subset X$, G_2^n is the wandering subset of the expanded subset $Z^* \subset X$, G_3^n is the wandering subset of the expanded subset W^*.

Here subscripts "cr" or "c" refer to the critical point $r(x_{cr}, \tau_{cr})$ or r_c: the space–time point of the onset of the interaction between the deterministic field and the random field, which leads to the turbulence. In addition, subsets Y, Z, W are called extended in X. For the transfer of the substantial quantity Φ (mass (density ρ), momentum ($\rho\boldsymbol{U}$), energy (E)) of the deterministic (laminar) motion into the random (turbulent) one, for domain 1 of the start of turbulence generation, the pair (N, M) = (1, 0) with the equivalence of measures is written $\left(d\Phi_{col_{st}}\right)_{1,0} = -R_{1,0}(\Phi_{st})$ and $\left(\frac{d(\Phi)_{col_{st}}}{d\tau}\right)_{1,0} = -R_{1,0}\left(\frac{\Phi_{st}}{\tau_{cor}}\right)$. Applying the correlation $D_{N,M}(m_c; r_c; \tau_c) = D_{1,1}(m_c; r_c; \tau_c)$ derived in [39–44], the equivalence relation for pair (N,M) = (1,1) was defined as $\left(d\Phi_{col_{st}}\right)_{1,1} = -R_{1,1}(d\Phi_{st})$, $\left(\frac{d(\Phi)_{col_{st}}}{d\tau}\right)_{1,1} = -R_{1,1}\left(\frac{d\Phi_{st}}{d\tau}\right)$, where $R_{1,0}$ and $R_{1,1}$ are the fractal coefficients, $\Phi_{col_{st}}$ is the part of the field of Φ, exactly, its deterministic component (subscript col_{st}) is the stochastic component, the measure of which is zero; Φ_{st} is the part of Φ, exactly, the proper stochastic component (subscript st). It should be noted that the stochastic equations (1)–(3) derived in [54–59] include free terms of gradient and non-gradient structures on their right-hand side.

3. Sets of Stochastic Equations

The flow near a rotating disk, as well as the previous ones, has an important scientific and applied significance as the three-dimensional motion of a fluid on a solid surface, but the forced motion is caused here by the rotational motion of the body. Then, taking into account the previously presented set of stochastic equations of the considered theory of equivalent measures in the case of a continuous isothermal medium, we write set (1)–(3) of equations of mass, momentum, and energy in accordance with [54–59]. For the area (1) —the onset of generation (subscript 1, 0, or 1) referring the pair (N, M) = (1,0) is:

$$\left\{\begin{array}{c} \left(\frac{d(\rho)_{col,st}}{d\tau}\right)_{1,0} = -\frac{\rho_{st}}{\tau_{cor}}; \\ \left\{\begin{array}{c} \left(\frac{d(\rho\vec{U})_{col,st}}{d\tau}\right)_{1,0} = -\left(\frac{(\rho\vec{U})_{st}}{\tau_{cor}}\right); \\ div(\tau_{i,j})_{col,st1} = \frac{(\rho\vec{U})_{st}}{\tau_{cor}} \end{array}\right. \\ \left\{\begin{array}{c} \left(\frac{d(E)_{col,st}}{d\tau}\right)_{1,0} = -\left(\frac{(E)_{st}}{\tau_{cor}}\right)_{1,0}; \\ div(\lambda\frac{\partial T}{\partial x_j} + u_i\tau_{i,j})_{col,st1} = \left(\frac{(E)_{st}}{\tau_{cor}}\right)_{1,0} \end{array}\right. \end{array}\right. \tag{7}$$

Set (1)–(3) of equations for the area (2)—the area of generation of turbulence (subscript 1,1) referring to the pair (N, M) = (1,1) is written as:

$$\left\{\begin{array}{c} \left(\frac{d(\rho)_{col,st}}{d\tau}\right)_{1,1} = -\left(\frac{d\rho_{st}}{d\tau_{cor}}\right); \\ \left\{\begin{array}{c} \left(\frac{d(\rho\vec{U})_{col,st}}{d\tau}\right)_{1,1} = -\left(\frac{d(\rho\vec{U})_{st}}{d\tau}\right); \\ div(\tau_{i,j})_{col,st2} = \frac{d(\rho\vec{U})_{st}}{d\tau} \end{array}\right. \\ \left\{\begin{array}{c} \left(\frac{d(E)_{col,st}}{d\tau}\right)_{1,1} = -\left(\frac{d(E)_{st}}{d\tau}\right)_{1,1}; \\ div(\lambda\frac{\partial T}{\partial x_j} + u_i\tau_{i,j})_{col,st2} = \left(\frac{d(E)_{st}}{d\tau}\right)_{1,1.} \end{array}\right. \end{array}\right. \tag{8}$$

Set (1)–(3) of equations for the area (3) is the diffusion of the turbulence referring to the pair (N = p, M = k, l) = (1,1,0) was written as

$$\begin{array}{c} \frac{d(\rho)_{st}}{d\tau} = -\frac{(\rho)_{st}}{\tau_{cor}}; \frac{d\rho_{st}}{d\tau} = \left(\frac{d\rho_{st}}{d\tau}\right)_{1,0} + \left(\frac{d\rho_{st}}{d\tau}\right)_{1,1}; \\ \frac{d(\rho\vec{U})_{st}}{d\tau} = -\frac{(\rho\vec{U})_{st}}{\tau_{cor}}; \frac{d(\rho\vec{U})_{st}}{d\tau} = \left(\frac{d(\rho\vec{U})_{st}}{d\tau}\right)_{1,0} + \left(\frac{d(\rho\vec{U})_{st}}{d\tau}\right)_{1,1} \\ \left|\frac{(dE_{st})}{d\tau}\right|_{1;(1,0)} = (R_{zTz})_{1;(1,0)}\left|\frac{(E_{st})_j}{\delta\tau}\right|_{1,(1,0)}; \left|\frac{(dE_{st})}{d\tau}\right| = \left|\frac{(dE_{st})_1}{d\tau}\right| + \left|\frac{(dE_{st})_2}{d\tau}\right| \end{array} \tag{9}$$

Therefore, for the area (3) of diffusion, we have two fractal equations. The first equation is written as

$$\frac{d(E_{st})_j}{d\tau} = -(R_{zTz})_{(1,1,1)}\frac{(E_{st})_j}{\tau_{cor1}} \tag{10}$$

Here, (E_{st}) is the field-energy component, which is actually the stochastic one (subscript 'st'), the subscript j = 1 refers to the space–time area of the diffusion of turbulence 3).

4. Equations for Critical Reynolds Number

The solution for the velocity field (u, v, and w are the components of the velocity in the radial "r", circumferential "ϕ", and axial "z" directions) of the deterministic (laminar) motion is presented in [39–41]. According to this solution, the velocity components are expressed as the dependences in the radial, circumferential, and axial directions, respectively: $u = r\omega F(\xi)$, $v_\varphi = r\omega G(\xi)$, $w = \sqrt{\nu\omega}H(\xi)$. The values $F(\xi)$, $G(\xi)$, $H(\xi)$, and their derivatives in the function $\xi = z\sqrt{\omega/\nu}$, are also given in [41]. In this case, the flow-motion mode is represented as a function of the coefficient of the moment of resistance of the disk.

$$C_M = \frac{2M}{0.5{\cdot}\rho{\cdot}\omega^2{\cdot}R^5} \tag{11}$$

Here, $M = -2\pi\int\limits_0^R r^2\tau_{z\varphi}dr$, $Re = \frac{R^2\omega}{\nu}$ is the Reynolds number and ω, R, are the rotational velocity and the radius of the disc. Taking into account that the velocity $v_\varphi = r\omega G(\xi)$, the value of the stress $\tau_{z\varphi}$ is written as

$$\tau_{z\varphi} = \mu\frac{dv_\varphi}{dz} = \rho r\nu^{1/2}\omega^{3/2}G'(\xi) \tag{12}$$

Here $v_\varphi = r\omega G(\xi)$ is the velocity in the circumferential direction, for z = 0, $G(\xi) = 1$, for the current value of z near the critical point $G(\xi) \approx 1 - Kz\left(\frac{L_y}{z}\right)\sqrt{\frac{\omega}{\nu}}$ and δ is the thickness of the boundary layer, in accordance with [41], $\delta^{-1} \approx K\sqrt{\frac{\omega}{\nu}}$. L_x is the scale of the disturbance along the radial direction (along the current radius "r"), and L_y is the scale of the disturbance along the axial direction "z". Then, according to the equivalence of measures of deterministic and random motion, we write that

$$div(u_i \tau_{i,j})_{col,st1} = \left(\frac{(E)_{st}}{\tau_{cor}}\right)_{1,0} \quad (13)$$

In the first approximation, the left-hand side of the equation takes the form:

$$\begin{gathered} div(u_i\tau_{i,j}) \approx div(v_\phi \tau_{z\phi}) \approx G'(\xi)\rho\omega\nu^{1/2}\omega^{3/2}G(\xi)\tfrac{d}{dz}r^2 = 2G'(\xi)]\rho\omega\nu^{1/2}\omega^{3/2}G(\xi)r\tfrac{L_x}{L_y} \approx \\ 2G'(\xi)\rho\omega\nu^{1/2}\omega^{3/2}\tfrac{L_x}{L_y}r\left(1 - Kz\left(\tfrac{L_y}{z}\right)\sqrt{\tfrac{\omega}{\nu}}\right) = 2 * 0.616\rho\omega\nu^{1/2}\omega^{3/2}\tfrac{L_x}{L_y}r\left(1 - Kz\left(\tfrac{L_y}{z}\right)\sqrt{\tfrac{\omega}{\nu}}\right) \\ G'(\xi) = 0.616, \end{gathered} \quad (14)$$

see [41].

Then, we have the expression

$$1.232\,\rho\omega R\nu^{1/2}\omega^{1/2}\frac{L_x}{L_y}\frac{r}{R}\left(1 - Kz\left(\frac{L_y}{z}\right)\sqrt{\frac{\omega}{\nu}}\right) = \frac{E_{st}}{\tau^0_{cor}} \quad (15)$$

From the obtained expression, we can determine the dependence for a dimensionless number at which there is an equivalence of measures between deterministic and random motion called in the hydrodynamics the critical Reynolds number for the flow in the boundary layer near the disk surface corresponding to the values of the correlation times $(\tau^0_{cor})_1$, $(\tau^0_{cor})_2$, $(\tau^0_{cor})_3$, $\tau_{motion} = [\omega]^{-1}$. Thus, for the case of the correlation time $(\tau^0_{cor})_1 = \frac{L}{\sqrt{E_{st}/\rho}}$, we write

$$1.232\left(\tfrac{L_x}{R}\right)\left(\tfrac{r}{R}\right)\left(\tfrac{\omega^2R^2}{E_{st}/\rho}\right)\tfrac{\nu\omega}{E_{st}/\rho}\tfrac{R^2\omega}{R^2\omega}\sqrt{\tfrac{E_{st}/\rho}{\nu\omega}}\left(1 - Kz\left(\tfrac{L_y}{z}\right)\sqrt{\tfrac{\omega}{\nu}}\right) = 1 \quad (16)$$

$$1.232\left(\frac{L_x}{R}\right)\left(\frac{r}{R}\right)\left(\frac{\omega^2R^2}{E_{st}/\rho}\right)\frac{R^2\omega^2}{E_{st}/\rho}\frac{\nu}{R^2\omega}\sqrt{\frac{E_{st}/\rho}{\nu\omega}}\left(1 - Kz\left(\frac{L_y}{z}\right)\sqrt{\frac{\omega}{\nu}}\right) = 1 \quad (17)$$

$$1.232\left(\frac{L_x}{R}\right)\left(\frac{r}{R}\right)\left(\frac{\omega^2R^2}{E_{st}/\rho}\right)^2\frac{\nu}{R^2\omega}\sqrt{\frac{E_{st}/\rho}{\nu\omega}\frac{R^2\omega}{R^2\omega}}\left(1 - Kz\left(\frac{L_y}{z}\right)\sqrt{\frac{\omega}{\nu}}\right) = 1 \quad (18)$$

$$1.232\left(\frac{L_x}{R}\right)\left(\frac{r}{R}\right)\left(\frac{\omega^2R^2}{E_{st}/\rho}\right)^2\frac{\nu}{R^2\omega}\sqrt{\frac{E_{st}/\rho}{R^2\omega^2}\frac{R^2\omega}{\nu}}\left(1 - Kz\left(\frac{L_y}{z}\right)\sqrt{\frac{\omega}{\nu}}\right) = 1 \quad (19)$$

$$\sqrt{\frac{R^2\omega}{\nu}} = 1.232\left(\frac{L_x}{R}\right)\left(\frac{r}{R}\right)\left(\frac{\omega^2R^2}{E_{st}/\rho}\right)^{3/2}\left(1 - Kz\left(\frac{L_y}{z}\right)\sqrt{\frac{\omega}{\nu}}\right) \quad (20)$$

Finally, we obtain

$$\mathrm{Re} \approx 1.5\left(\frac{L_x}{R}\right)^2\left(\frac{r}{R}\right)^2\left(\frac{\omega^2R^2}{E_{st}/\rho}\right)^3\left(1 - Kz\left(\frac{L_y}{z}\right)\sqrt{\frac{\omega}{\nu}}\right)^2, \quad (21)$$

$$\mathrm{Re} \approx 1.5\left(\tfrac{L_x}{R}\right)^2\left(\tfrac{r}{R}\right)^2\left(\tfrac{\omega R}{\sqrt{E_{st}/\rho}}\right)^6\left(1 - Kz\left(\tfrac{L_y}{z}\right)\sqrt{\tfrac{\omega}{\nu}}\right)^2 \quad (22)$$

Correspondently, for the correlation time $(\tau^0_{cor})_2 = \frac{L^2}{\nu}$, we have

$$\mathrm{Re} \approx \left\{1.5\left(\tfrac{L_x}{R}\right)^2\left(\tfrac{r}{R}\right)^2\left(\tfrac{\omega R}{\sqrt{E_{st}/\rho}}\right)^6\left(1 - Kz\left(\tfrac{L_y}{z}\right)\sqrt{\tfrac{\omega}{\nu}}\right)^2\right\}\mathrm{Re}^2{}_{st} \quad (23)$$

or

$$\mathrm{Re} \approx \left\{1.5\left(\tfrac{L_x}{R}\right)^2\left(\tfrac{r}{R}\right)^2\left(\tfrac{\omega R}{\sqrt{E_{st}/\rho}}\right)^6\left(1 - Kz\left(\tfrac{L_y}{z}\right)\sqrt{\tfrac{\omega}{\nu}}\right)^2\right\}\mathrm{Re}^2{}_{st} \quad (24)$$

For the correlation time $(\tau^0_{cor})_3 = \frac{\nu}{E_{st}/\rho}$, we obtain the value

$$\mathrm{Re} \approx \left\{1.5\left(\tfrac{L_x}{R}\right)^2\left(\tfrac{r}{R}\right)^2\left(\tfrac{\omega R}{\sqrt{E_{st}/\rho}}\right)^6\left(1 - Kz\left(\tfrac{L_y}{z}\right)\sqrt{\tfrac{\omega}{\nu}}\right)^2\right\}\frac{1}{\mathrm{Re}^2{}_{st}} \quad (25)$$

or

$$\mathrm{Re} \approx \left\{ 1.5\left(\frac{L_x}{R}\right)^2 \left(\frac{r}{R}\right)^2 \left(\frac{\omega R}{\sqrt{E_{st}/\rho}}\right)^6 \left(1 - Kz\left(\frac{L_y}{z}\right)\sqrt{\frac{\omega}{\nu}}\right)^2 \right\} \frac{1}{\mathrm{Re}^2{}_{st}} \tag{26}$$

5. The Equation for the Critical Point

Now let us determine the position of the critical point. The definition of the critical point is found from the equation as

$$\int_{-\Delta V|2}^{+\Delta V|2} d\left(E_{col_{st}}\right)_{1;0} = \int_X dE_{st} \tag{27}$$

Here E_{st} is the random energy component in the space X with the measure $m(E_{st}) < \infty$

$$E_{st} = E_{st}(\overrightarrow{x_i}, \tau_i, m_i) < \infty \tag{28}$$

In accordance with the ergodic theory [39,40]

$$\int_X dE_{st} = \frac{1}{\Delta V}\int_V E_{st}\delta((\Delta V)_{critic} - \Delta V)dV = \frac{1}{\tau^0_{cor}}\int_\tau E_{st}\delta(\tau^0_{cor} - \tau)d\tau = (E_{st})_{critic} \tag{29}$$

$(E_{st})_{critic}$ is the energy of the stochastic field in the critical point,
or

$$\int_X dE_{st} = \frac{1}{L}\int_L E_{st}\delta((x_i)_{critic} - x_i)dL = \frac{1}{\tau^0_{cor}}\int_\tau E_{st}\delta(\tau^0_{cor} - \tau)d\tau = (E_{st})_{critic} \tag{30}$$

L is the scale of the disturbance.

Then, taking into account the values of functions $F(\xi)$, $G(\xi)$, $H(\xi)$, in the neighborhood of critical point and using the equations for Formulas (11) and (12), we find $\left(E_{col_{st}}\right)_{1;0} = 0.5\rho v_\phi^2$, $v_\phi = r\omega G(\xi)$ is the velocity in the circumferential direction, for $z = 0$, $G(\xi) = 1$, for the current value of z near the critical point $G(\xi) \approx 1 - Kz\left(\frac{L_y}{z}\right)\sqrt{\frac{\omega}{\nu}}$.

Then, we may write that

$$\begin{aligned} \int_{-V|2}^{+V|2} d\left(E_{col_{st}}\right)_{1;0} \approx \int_{-L|2}^{+L|2} d\left(E_{col_{st}}\right)_{1;0} &\approx 0.5\rho\omega^2\left\{\left[(r+L_x/2)^2\left(1-K(z+L_y/2)\right)^2\frac{\omega}{\nu}\right] - \left[(r-L_x/2)^2\left(1-K(z-L_y/2)\right)^2\frac{\omega}{\nu}\right]\right\} \\ &\approx 0.5\rho\omega^2 rL_x\left[1-K^2z^2\left(\frac{L_y}{z}\right)\frac{\omega}{\nu}\right] \end{aligned} \tag{31}$$

So, we obtain

$$\frac{L_x}{R}\frac{r}{R}\left[1 - \left(\frac{L_y}{z}\right)K^2z^2\frac{\omega}{\nu}\right] = \frac{E_{st}/\rho}{(R\omega)^2} \tag{32}$$

Then, for the Equation (32), we can write

$$\left(\frac{L_x}{R}\frac{r}{R}\right)^2 = \left(\frac{E_{st}/\rho}{(R\omega)^2}\right)^2 \frac{1}{\left[1 - K^2z^2\left(\frac{L_y}{z}\right)\frac{\omega}{\nu}\right]^2} \tag{33}$$

6. The Solution for the First Critical Reynolds Number

We substitute Equation (33) in expression (22) for the Reynolds number, then, for the correlation time $\left(\tau^0_{cor}\right)_1 = \frac{L}{\sqrt{E_{st}/\rho}}$, the critical Reynolds number is

$$\mathrm{Re} \approx 1.5\left(\frac{L_x}{R}\right)^2 \left(\frac{r}{R}\right)^2 \left(\frac{\omega R}{\sqrt{E_{st}/\rho}}\right)^6 \left(1 - Kz\left(\frac{L_y}{z}\right)\sqrt{\frac{\omega}{\nu}}\right)^2 \tag{34}$$

Then, we obtain

$$\mathrm{Re} \approx 1.5\left(\frac{\omega^2R^2}{E_{st}/\rho}\right) \frac{\left(1 - Kz\left(\frac{L_y}{z}\right)\sqrt{\frac{\omega}{\nu}}\right)^2}{\left[1 - K^2z^2\left(\frac{L_y}{z}\right)\frac{\omega}{\nu}\right]^2} \tag{35}$$

or

$$\mathrm{Re} \approx 1.5\left(\frac{\omega R}{\sqrt{E_{st}/\rho}}\right)^2 \frac{\left(1 - Kz\left(\frac{L_y}{z}\right)\sqrt{\frac{\omega}{\nu}}\right)^2}{\left[1 - K^2 z^2\left(\frac{L_y}{z}\right)\frac{\omega}{\nu}\right]^2} \tag{36}$$

For the correlation time $\left(\tau^0_{cor}\right)_2 = \frac{L^2}{\nu}$, we have

$$\mathrm{Re} \approx \left(1.5\left(\frac{\omega R}{\sqrt{E_{st}/\rho}}\right)^2 \frac{\left(1 - Kz\left(\frac{L_y}{z}\right)\sqrt{\frac{\omega}{\nu}}\right)^2}{\left[1 - K^2 z^2\left(\frac{L_y}{z}\right)\frac{\omega}{\nu}\right]^2}\right) \mathrm{Re}^2{}_{st} \tag{37}$$

Then, for the correlation time $\left(\tau^0_{cor}\right)_3 = \frac{\nu}{E_{st}/\rho}$, we obtain

$$\mathrm{Re} \approx \left(1.5\left(\frac{\omega R}{\sqrt{E_{st}/\rho}}\right)^2 \frac{\left(1 - Kz\left(\frac{L_y}{z}\right)\sqrt{\frac{\omega}{\nu}}\right)^2}{\left[1 - K^2 z^2\left(\frac{L_y}{z}\right)\frac{\omega}{\nu}\right]^2}\right) \frac{1}{\mathrm{Re}^2{}_{st}} \tag{38}$$

Substituting Equation (33) in expressions (36)—(38) for the Reynolds number and neglecting the terms containing the value K^2, we may write the estimate for the critical Reynolds number in the flow near the rotating disk (K~0.5) as:

$$(\mathrm{Re}_{critic})_1 \approx \left(1.5\left(\frac{\omega R}{\sqrt{E_{st}/\rho}}\right)^2 \left(1 - 2K\left(\frac{L_y}{z}\right)z\sqrt{\frac{\omega}{\nu}}\right)\right)\mathrm{Re}^2{}_{st} \approx 1.5\left(\frac{\omega R}{\sqrt{E_{st}/\rho}}\right)^2 \left(1 - 2\left(\frac{L_y}{\delta}\right)\right)\mathrm{Re}^2{}_{st} \tag{39}$$

For the value of L_y/δ near the wall at the critical point in accordance with [41,49–59], we have

$$\left(\frac{L_y}{\delta}\right) \approx 0.02 \div 0.04 \tag{40}$$

Thus, finally, we have the theoretical solution for the critical Reynolds number for the motion of the flow near the rotating disk

$$(\mathrm{Re}_{critic})_1 \approx 1.3\left(\frac{\omega R}{\sqrt{E_{st}/\rho}}\right)^2 \mathrm{Re}_{st}{}^2 \tag{41}$$

In accordance with [41,77,78], there are the following values for the degree of turbulence observed in the laboratory and the turbulent Reynolds numbers Re_{st} near the wall of the disk: $\left(\omega R/\sqrt{E_{st}/\rho}\right)^{-1} = 0.01 \div 0.02$ and $\mathrm{Re}_{st} = 5 \div 10$. As a result, using Equation (41), we have $\mathrm{Re}_{critic} = 325{,}000$, which agrees with the data [41]. In the case, when the turbulent Reynolds number is $\mathrm{Re}_{st} = 5 \div 15$, we have $3.25 \times 10^5 \leq \mathrm{Re}_{critic} \leq 7.3 \times 10^6$. Therefore, the defined range for the first critical Reynolds number for the motion near a rotating disk is within the experimental values for the transition mode $2.9 \times 10^5 \leq \mathrm{Re}_{critic} < 7 \times 10^5$ [41].

7. Conclusions

Analytical Formulas (39) and (41), for the critical Reynolds number for the motion of the flow near a rotating disk based on the theory of stochastic equations of continuum laws and the equivalence of measures between random and deterministic motion are presented. Also, analytical Formulas (32) and (33) for the critical point in the case of the motion of the flow near a rotating disk are derived. The results of solutions show the satisfactory correspondence between the values obtained with using the analytical dependences for critical Reynolds number (39) and (41), and the experimental data [41]. For the degree of turbulence observed in the laboratory $\left(\omega R/\sqrt{E_{st}/\rho}\right)^{-1} = 0.01 \div 0.02$ and the turbulent Reynolds numbers Re_{st} near the wall of the disk $\mathrm{Re}_{st} = 5 \div 10$, we have $\mathrm{Re}_{critic} = 325{,}000$, which agrees with the data [41]. It seems that the obtained dependences for the critical Reynolds number can be useful for estimating of the flow regime in gas or steam turbines.

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