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Oscillatory Bifurcations in Porous Layers with Stratified Porosity, Driven by Each Coefficient of the Spectrum Equation

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Abstract: The onset of oscillatory bifurcations in a porous horizontal layer L, uniformly rotating about a vertical axis, with vertically stratified porosity, heated from below and salted from above and below, is investigated. Denoting by P_i, $(i = 1, 2)$, the Prandtl numbers of the salt S_i salting L from below $(i = 1)$ and above $(i = 2)$ respectively, it is shown that: (i) in L the oscillatory bifurcations can occur only if one of the structural conditions $P_1 > 1,\ P_2 < 1$ or $P_1 = 1,\ P_2 < 1$ or $P_1 > 1,\ P_2 = 1$ is verified; (ii) exists a bound $\bar{R}_2$ for the Rayleigh number R_2 of S_2 such that $R_2 < \bar{R}_2$ guarantees the absence of cold convection; (iii) via a new approach based on the instability power of each coefficient of the spectrum equation, criteria of existence, location and frequency of oscillatory (Hopf) bifurcations are furnished for any porosity stratification law. These criteria, as far as we know are, for the case at stake, the first criteria of Hopf bifurcations appearing in literature. We are confident that, via experimental results, will be validated.

Keywords: convection; Hopf bifurcation; rotation; vertically stratified viscosity

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1. Introduction

The onset of thermal convection in porous layers with vertically stratified permeability and/or viscosity, for its importance in geophysical phenomena and in the construction of artificial porous materials, has attracted—in the past as nowadays—the attention of scientists [1–14]. The increase in viscosity with depth in the earth's mantle has been studied in [1,2]; the changes in permeability due to mineral diagenesis in fractured crust has been analyzed in [3]; in references [4,5] the porosity changes due to the subterranean movement and the increase in permeability and porosity near solid wall, are considered; the influence of porosity stratification on the onset of thermal convection in the construction of artificial porous materials is studied in [2].

In the present paper a porous horizontal layer with depth-dependent permeability and viscosity—heated from below, rotating uniformly about a vertical axis and salted from above and from below—is considered. The scope is to analyze the effects of such stratifications on the onset of Hopf bifurcations. The paper is organized as follows. In Sections 2 and 3 same basic preliminaries concerning the model equations (Section 2) and the linear stability of the thermal solution (Section 3) are given. Section 4 is devoted to the spectrum equation of the problem at stake, while in the subsequent Section 5 the instability basic property of the coefficients of the spectrum equation is recalled. In Section 6 it is shown the existence of hidden symmetries and structural conditions on the salts necessary for the existence of oscillatory bifurcations are found.The condition for avoiding the onset of instability for each value of the thermal Rayleigh number (*cold convection*) is found in Section 7. The criteria for the onset of oscillatory bifurcations are obtained in the subsequent Section 8. Section 9 is devoted to the exponentially increasing porosity. The paper ends with some final remarks (Section 10).

2. Preliminaries

Let two different chemical components ("salts") S_α $(\alpha = 1, 2)$, be dissolved in the fluid porous layer L and let the equation of state be

$$\rho = \rho_0[1 - \alpha_*(T - T_0) + \sum_{\alpha=1}^{2} A_\alpha(C_\alpha - \tilde{C}_\alpha)] \tag{1}$$

where ρ, T, C_α are the density, temperature and salts concentrations with ρ_0, T_0, $\tilde{C}_\alpha$ reference values and α_*, A_α thermal and solute expansion coefficients. We denote by $Oxyz$ an orthogonal frame of reference with fundamental unit vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ ($\mathbf{k}$ pointing vertically upwards). Let $d = \text{const.} > 0$ and $L = z \in [0, d]$. The isochoric motions in L—rotating uniformly around the z axis with velocity $\bar{\omega}$ – are governed, in the Boussinesq approximation [6], by

$$\begin{cases} \nabla P = -\dfrac{\mu}{\tilde{k}}\mathbf{v} - \rho\mathbf{g} - 2\rho_0\bar{\omega}\mathbf{k} \times \mathbf{v} \\ T_{,t} + \mathbf{v} \cdot \nabla T = k\Delta T, \quad \nabla \cdot \mathbf{v} = 0 \\ C_{\alpha,t} + \mathbf{v} \cdot \nabla C_\alpha = k_\alpha \Delta C_\alpha \end{cases} \tag{2}$$

with the list of symbols given by

$$\begin{cases} P \text{ pressure field}, \ T \text{ temperature field}, \\ \mathbf{v} \text{ seepage velocity}, \ \mu = \tilde{\mu} f_1(z) \text{fluid viscosity}, \\ \tilde{k} = \bar{k} f_2(z) \text{ permeability}, \ k \text{ thermal diffusivity}, \\ k_\alpha \text{ diffusivity of } S_\alpha, \ \bar{k} \text{ reference permeability}, \\ \tilde{\mu} \text{ reference viscosity} \end{cases}$$

defined in [6]. Passing to the boundary conditions, since in $(2)_1$ there are not derivatives in the velocity, one needs only to prescribe the normal component of $\mathbf{v}$: we require that this component is null. As concerns the temperature and the salts, we assume that their values are fixed. Therefore to (2) we append precisely the boundary conditions

$$\begin{cases} T(0) = T_1, \ T(d) = T_2, \ \mathbf{v} \cdot \mathbf{k} = 0, \quad \text{at } z = 0, d \\ C_\alpha(0) = C_{\alpha l}, \ C_\alpha(d) = C_{\alpha u} \qquad \alpha = 1, 2, \ \delta C_1 > 0, \ \delta C_2 < 0, \end{cases} \tag{3}$$

with $T_1, T_2, C_{\alpha l}, C_{\alpha u}$ $(\alpha = 1, 2)$ positive constants and $C_{\alpha l} - C_{\alpha u} = \delta C_\alpha$ $(\alpha = 1, 2)$, $T_1 > T_2$. The boundary value problem (2) and (3) admits the conduction solution $m_0 = (\tilde{\mathbf{v}}, \tilde{p}, \tilde{T}, \tilde{C_\alpha})$ given by

$$\begin{cases} \tilde{\mathbf{v}} = \mathbf{0}, \ \tilde{T} = -\beta z + T_1, \ \beta = \dfrac{T_1 - T_2}{d}, \ \tilde{C_\alpha} = C_{\alpha l} - \dfrac{z(\delta C_\alpha)}{d} \\ \tilde{P} = p_0 + \rho_0 g z^2 \left[-\dfrac{\alpha_* \beta}{2} + A_1 \dfrac{(\delta C_1)}{2d} + A_2 \dfrac{(\delta C_2)}{2d}\right] + \\ \qquad -\rho_0 g z [1 - \alpha_*(T_1 - T_0) + A_1(C_{1l} - \tilde{C}_1) + A_2(C_{2l} - \tilde{C}_2)] \end{cases} \tag{4}$$

where p_0 is a constant. Setting

$$\mathbf{v} = \tilde{\mathbf{v}} + \mathbf{u}, \quad p = \tilde{P} + \Pi, \quad T = \tilde{T} + \theta, \quad C_\alpha = \tilde{C_\alpha} + \Phi_\alpha \tag{5}$$

and introducing the scalings

$$\begin{cases} t = t^* \dfrac{d^2}{k}\,,\ \mathbf{u} = \mathbf{u}^* \dfrac{k}{d}\,,\ \Pi = \Pi^* \dfrac{\tilde{\mu} k}{\bar{k}}\,,\ \mathbf{x} = \mathbf{x}^* d\,,\ \theta = \theta^* T^\sharp\,, \\ \Phi_\alpha = (\Phi_\alpha)^* \Phi^\sharp_\alpha\,,\ T^\sharp = \left(\dfrac{\tilde{\mu} k |\delta T|}{\alpha_* \rho_0 g \bar{k} d}\right)^{1/2}\,,\ \Phi^\sharp_\alpha = \left(\dfrac{\tilde{\mu} k P_\alpha |\delta C_\alpha|}{A_\alpha \rho_0 g \bar{k} d}\right)^{1/2}\,, \\ R = \left(\dfrac{\alpha_* \rho_0 g \bar{k} d |\delta T|}{\tilde{\mu} k}\right)^{1/2}\,,\ R_\alpha = \left(\dfrac{A_\alpha \rho_0 g \bar{k} d P_\alpha |\delta C_\alpha|}{\tilde{\mu} k}\right)^{1/2}\,,\ \mathcal{T} = \dfrac{2\rho_0 \bar{\omega} k}{\tilde{\mu}}\,, \\ \delta T = T_1 - T_2\,,\ H = \text{sign}(\delta T)\,,\ H_\alpha = \text{sign}(\delta C_\alpha)\,,\ P_\alpha = \dfrac{k}{k_\alpha}\,, \end{cases} \tag{6}$$

since in the case at stake the layer is heated from below and salted from below by S_1 and from above by S_2, it follows hat $H = H_1 = 1$, $H_2 = -1$ and the equations governing the dimensionless perturbations $\{\mathbf{u}^*, \Pi^*, \theta^*, (\Phi_\alpha)^*\}$, omitting the stars, and setting $f = \dfrac{f_1}{f_2}$, are $(\alpha = 1, 2)$

$$\begin{cases} \nabla \Pi = -f(z)\mathbf{u} + \left(R\theta - \displaystyle\sum_{\alpha=1}^{2} R_\alpha \Phi_\alpha\right)\mathbf{k} + \mathcal{T}\mathbf{u} \times \mathbf{k} \\ \nabla \cdot \mathbf{u} = 0,\ \theta_t + \mathbf{u} \cdot \nabla\theta = R\mathbf{u} \cdot \mathbf{k} + \Delta\theta \\ P_\alpha \left(\dfrac{\partial \Phi_\alpha}{\partial t} + \mathbf{u} \cdot \nabla \Phi_\alpha\right) = H_\alpha R_\alpha \mathbf{u} \cdot \mathbf{k} + \Delta\Phi_\alpha\,, \end{cases} \tag{7}$$

under the boundary conditions

$$\mathbf{u} \cdot \mathbf{k} = \theta = \Phi_\alpha = 0 \quad \text{on} \quad z = 0, 1 \tag{8}$$

In (6) and (7) R and R_α are the thermal and salt Rayleigh numbers respectively while P_α are the salt Prandtl numbers and $\mathcal{T}$ is the Taylor-Darcy number. We set $\mathbf{u} = (u, v, w)$ and assume, as usually done, that:

(i) *the perturbations* $(u, v, w, \theta, \Phi_1, \Phi_2)$ *are periodic in the x and y directions, respectively of periods* $2\pi/a_x$, $2\pi/a_y$;
(ii) $\Omega = [0, 2\pi/a_x] \times [0, 2\pi/a_y] \times [0, 1]$ *is the periodicity cell;*
(iii) $\mathbf{u}, \Phi_1, \Phi_2, \theta$ *belong to* $W^{2,2}(\Omega)$ *and are such that all their first derivatives and second spatial derivatives can be expanded in Fourier series uniformly convergent in* Ω

and denote by $L_2^*(\Omega)$ the set of the functions Φ such that

(1) $\Phi : (\mathbf{x}, t) \in \Omega \times \mathbb{R}^+ \to \Phi(\mathbf{x}, t) \in \mathbb{R}$, $\Phi \in W^{2,2}(\Omega), \forall\, t \in \mathbb{R}^+, \Phi$ *is periodic in the x and y directions of period* $2\pi/a_x$, $2\pi/a_y$ *respectively and* $|\Phi|_{z=0} = |\Phi|_{z=1} = 0$;
(2) Φ *together with all the first derivatives and second spatial derivatives can be expanded in a Fourier series absolutely uniformly convergent in* Ω, $\forall\, t \in \mathbb{R}^+$.

3. Preliminaries to Linear Instability

Since $(7)_1$ is linear, the linear stability of m_0 is governed by

$$\begin{cases} \nabla \Pi = -f(z)\mathbf{u} + \left(R\theta - \sum_{\alpha=1}^{2} R_\alpha \phi_\alpha\right)\mathbf{k} + \mathcal{T}\mathbf{u} \times \mathbf{k}, \\ \nabla \cdot \mathbf{u} = 0, \\ \theta_t = Rw + \Delta\theta, \\ P_\alpha \phi_{\alpha t} = H_\alpha R_\alpha w + \Delta\phi_\alpha, \end{cases} \tag{9}$$

under the boundary conditions

$$w = \theta = \phi_\alpha = 0 \quad \text{on } z = 0, 1. \tag{10}$$

Let $f \in C^1[0,1]$ a.e. and set

$$\zeta = (\nabla \times \mathbf{u}) \cdot \mathbf{k} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \tag{11}$$

In view of

$$\begin{cases} \mathbf{k} \cdot [\nabla \times f\mathbf{u}] = f\zeta, \\ \mathbf{k} \cdot \{\nabla \times [(R\theta - \sum_{\alpha=1}^{2} R_\alpha \phi_\alpha)\mathbf{k}]\} = 0, \\ \mathbf{k} \cdot [\nabla \times (\mathbf{u} \times \mathbf{k})] = \mathbf{k} \cdot [\nabla(v\mathbf{i} - u\mathbf{j})] = w_z, \end{cases} \tag{12}$$

the third component of the curl of (9)$_1$ gives

$$\zeta = \mathcal{T} f^{-1} w_z. \tag{13}$$

Furthermore, in view of

$$\begin{cases} \mathbf{k} \cdot \nabla \times [\nabla \times (-f\mathbf{u})] = f' w_z + f\Delta w, \\ \mathbf{k} \cdot \nabla \times [\nabla \times (\mathbf{u} \times \mathbf{k})] = \zeta_z, \\ \mathbf{k} \cdot \nabla \times [\nabla \times (R\theta - \sum_{\alpha=1}^{2} R_\alpha \phi_\alpha)\mathbf{k}] = -\Delta_1 \Big(R\theta - \sum_{\alpha=1}^{2} R_\alpha \phi_\alpha\Big), \end{cases} \tag{14}$$

where $\Delta_1 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$, one has that the third component of the double curl of (9) is

$$\mathcal{T}\zeta_z + f' w_z + f\Delta w = \Delta_1 \left(R\theta - \sum_{\alpha=1}^{2} R_\alpha \phi_\alpha \right) \tag{15}$$

and (13) implies

$$F = \Big[\mathcal{T}^2 (f^{-1})' + f'\Big] w_z + \mathcal{T}^2 (f^{-1}) w_{zz} + f\Delta w - \Delta_1 \left(R\theta - \sum_{\alpha=1}^{2} R_\alpha \phi_\alpha \right) = 0. \tag{16}$$

Since the set $\{\sin n\pi z\}_{n \in \mathbb{N}}$ is a complete orthogonal basis of $L_2^*(0,1)$, one has

$$\phi \in \{w, \theta, \phi_1, \phi_2\} \Rightarrow \phi = \sum_{n=1}^{\infty} \phi_n = \sum_{n=1}^{\infty} \tilde{\phi}_n(x, y, t) \sin n\pi z \tag{17}$$

and the periodicity in the x, y directions implies

$$\begin{cases} \Delta_1 \phi_n = -a^2 \phi_n, \quad \Delta \phi_n = -\xi_n \phi_n \\ a^2 = a_x^2 + a_y^2, \quad \xi_n = a^2 + n^2 \pi^2 \end{cases} \tag{18}$$

The following property holds.

Property 1. *Setting*

$$\begin{cases} A_n = \displaystyle\int_0^1 \left[\frac{a^2}{\xi_n} \sin^2(n\pi z) + \Big(1 - \frac{a^2}{\xi_n}\Big) \cos^2(n\pi z) \right] f(z)\, dz \\ B_n = \Big(1 - \dfrac{a^2}{\xi_n}\Big) \mathcal{T}^2 \displaystyle\int_0^1 (f^{-1}) \cos^2(n\pi z)\, dz \end{cases} \tag{19}$$

one has

$$\tilde{w}_n = \eta_n \left(R\tilde{\theta}_n - \sum_{\alpha=1}^{2} R_\alpha \phi_{\alpha n} \right) \tag{20}$$

where

$$\eta_n = \frac{a^2}{2\xi_n(A_n + B_n)} > 0 \tag{21}$$

Proof. One easily obtains that

$$\begin{aligned} F_n =& \tilde{w}_n(n\pi f' \cos n\pi z - \xi_n f \sin n\pi z) + \\ & \tilde{w}_n \mathcal{T}^2[(f^{-1})' n\pi \cos n\pi z - n^2\pi^2 f^{-1} \sin n\pi z] + \\ & a^2(R\tilde{\theta}_n - \sum_{\alpha=1}^{2} R_\alpha \phi_\alpha) \sin n\pi z = 0 \end{aligned} \tag{22}$$

which implies

$$\int_0^1 F_n \sin n\pi z \, dz = 0 \tag{23}$$

i.e.,

$$\begin{cases} \tilde{w}_n \displaystyle\int_0^1 (n\pi f' \sin n\pi z \cos n\pi z - \xi_n f \sin^2 n\pi z) \, dz + \\ \tilde{w}_n \mathcal{T}^2 \displaystyle\int_0^1 [(f^{-1})' n\pi \sin n\pi z \cos n\pi z - n^2\pi^2 f^{-1} \sin^2 n\pi z] \, dz + \\ a^2 \left(R\tilde{\theta}_n - \displaystyle\sum_{\alpha=1}^{2} R_\alpha \phi_\alpha \right) \displaystyle\int_0^1 \sin^2 n\pi z \, dz = 0 \end{cases} \tag{24}$$

Then in view of

$$\begin{cases} \displaystyle\int_0^1 \sin^2(n\pi z) \, dz = \frac{1}{2}, \quad \int_0^1 f' \sin(2n\pi z) \, dz = -2n\pi \int_0^1 f \cos(2n\pi z) \, dz \\ \displaystyle\int_0^1 [-n^2\pi^2 f^{-1} \sin^2(n\pi z) + \frac{n\pi}{2} (f^{-1})' \sin(2n\pi z)] \, dz = -n^2\pi^2 \int_0^1 f^{-1} \cos^2(n\pi z) \, dz, \end{cases} \tag{25}$$

one has that (23) gives (21). □

Remark 1. *We remark that the previous proof of property 1 follows, step by step, the which one given in [2], but some missprints concerning A_n and B_n (appearing in [2]) have been eliminated.*

4. Spectrum Equation

Let $H_1 = 1, H_2 = -1$, i.e., let L be salted from below by S_1 and from above by S_2. Then (9) implies

$$\begin{cases} \dfrac{\partial \tilde{\theta}_n}{\partial t} = R\tilde{w}_n + \Delta\tilde{\theta}_n \\ P_1 \dfrac{\partial \tilde{\phi}_{1n}}{\partial t} = R_1 \tilde{w}_n + \Delta\tilde{\phi}_{1n} \\ P_2 \dfrac{\partial \tilde{\phi}_{2n}}{\partial t} = -R_2 \tilde{w}_n + \Delta\tilde{\phi}_{2n} \end{cases} \tag{26}$$

where $\tilde{w}_n$ is given by (20). It follows that

$$\frac{\partial}{\partial t} \begin{pmatrix} \tilde{\theta}_n \\ \tilde{\phi}_{1n} \\ \tilde{\phi}_{2n} \end{pmatrix} = \mathscr{L}_n \begin{pmatrix} \tilde{\theta}_n \\ \tilde{\phi}_{1n} \\ \tilde{\phi}_{2n} \end{pmatrix} \tag{27}$$

where

$$\mathscr{L}_n = \begin{pmatrix} a_{1n} & a_{2n} & a_{3n} \\ b_{1n} & b_{2n} & b_{3n} \\ c_{1n} & c_{2n} & c_{3n} \end{pmatrix} \tag{28}$$

and

$$\begin{cases} a_{1n} = R^2\eta_n - \xi_n, \quad a_{2n} = -RR_1\eta_n, \quad a_{3n} = -RR_2\eta_n \\ b_{1n} = \dfrac{RR_1}{P_1}\eta_n, \quad b_{2n} = -\dfrac{(R_1^2\eta_n + \xi_n)}{P_1}, \quad b_{3n} = -\dfrac{R_1R_2}{P_1}\eta_n \\ c_{1n} = -\dfrac{RR_2}{P_2}\eta_n, \quad c_{2n} = \dfrac{R_1R_2}{P_2}\eta_n, \quad c_{3n} = \dfrac{(R_2^2\eta_n - \xi_n)}{P_2} \end{cases} \tag{29}$$

The spectrum equation is given by

$$\mathscr{P}_n(\lambda) = \lambda_n^3 + \bar{A}_{1n}\lambda_n^2 + \bar{A}_{2n}\lambda_n + \bar{A}_{3n} = 0 \tag{30}$$

where

$$\begin{cases} \bar{A}_{1n} = -I_{1n},\ \bar{A}_{2n} = -I_{2n},\ \bar{A}_{3n} = -I_{3n} \\ I_{1n} = a_{1n} + b_{2n} + c_{3n} = \sum_{\alpha=1}^{3} \lambda_{\alpha n},\ I_{3n} = \det \mathscr{L}_n = \lambda_{1n}\lambda_{2n}\lambda_{3n}, \\ I_{2n} = \begin{vmatrix} a_{1n} & a_{2n} \\ b_{1n} & b_{2n} \end{vmatrix} + \begin{vmatrix} a_{1n} & a_{3n} \\ c_{1n} & c_{3n} \end{vmatrix} + \begin{vmatrix} b_{2n} & b_{3n} \\ c_{2n} & c_{3n} \end{vmatrix} = \lambda_{1n}(\lambda_{2n} + \lambda_{3n}) + \lambda_{2n}\lambda_{3n}. \end{cases} \tag{31}$$

One easily obtains that

$$\begin{cases} I_{1n} = \Big\{R^2 + \dfrac{R_2^2}{P_2} - \Big[\dfrac{R_1^2}{P_1} + \dfrac{\xi_n}{\eta_n}\Big(1 + \dfrac{1}{P_1} + \dfrac{1}{P_2}\Big)\Big]\Big\}\eta_n \\ I_{2n} = \dfrac{P_1 + P_2}{P_1P_2}\Big[\Big(\dfrac{1 + P_1 + P_2}{P_1 + P_2}\Big)\dfrac{\xi_n}{\eta_n} + \dfrac{1 + P_2}{P_1 + P_2}R_1^2 - \Big(\dfrac{1 + P_1}{P_1 + P_2}R_2^2 + R^2\Big)\Big]\eta_n\xi_n \\ I_{3n} = \dfrac{1}{P_1P_2}\Big[(R^2 + R_2^2) - \Big(R_1^2 + \dfrac{\xi_n}{\eta_n}\Big)\Big]\xi_n^2\eta_n \end{cases} \tag{32}$$

Remark 2. *In the sequel we will set* $(r = 1, 2, 3)$

$$I_{r1} = I_r,\ \bar{A}_{r1} = A_r,\ \eta_1 = \eta,\ \xi_1 = \xi \tag{33}$$

and, since it is sufficient for the instability, we consider only $n = 1$*. Then it follows that the spectrum equation is*

$$\mathscr{P}(\lambda) = \lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0 \tag{34}$$

with

$$\begin{cases} I_1 = \Big\{R^2 + \dfrac{R_2^2}{P_2} - \Big[\dfrac{R_1^2}{P_1} + \dfrac{\xi}{\eta}\Big(1 + \dfrac{1}{P_1} + \dfrac{1}{P_2}\Big)\Big]\Big\}\eta \\ I_2 = \dfrac{P_1 + P_2}{P_1P_2}\Big[\Big(\dfrac{1 + P_1 + P_2}{P_1 + P_2}\Big)\dfrac{\xi}{\eta} + \dfrac{1 + P_2}{P_1 + P_2}R_1^2 - \Big(\dfrac{1 + P_1}{P_1 + P_2}R_2^2 + R^2\Big)\Big]\ \underline{\eta\xi} \\ I_3 = \dfrac{1}{P_1P_2}\Big[(R^2 + R_2^2) - \Big(R_1^2 + \dfrac{\xi_n}{\eta_n}\Big)\Big]\xi^2\eta \end{cases} \tag{35}$$

$$\begin{cases} \xi^2 = a^2 + \pi^2, \quad \tilde{w}_1 = \tilde{w}\Big(R\tilde{\theta} - \sum_{\alpha=1}^{2} R_\alpha\phi_\alpha\Big) \\ \eta = \eta_1 = \dfrac{a^2}{2\xi(A + B)}, \quad A = A_1, \ \ B = B_1 \\ \tilde{\theta}_1 = \tilde{\theta},\ \ \tilde{\phi}_{s1} = \tilde{\phi}_s,\ \ s = 1, 2, \\ a_{r1} = a_r,\ \ b_{r1} = b_r,\ \ c_{r1} = c_r,\ \ r = 1, 2, 3. \end{cases} \tag{36}$$

5. Power Property of the Spectrum Equation Coefficients

In the present section the existence and location property of the Hopf bifurcations in dynamical system, via the spectrum equation instability coefficients power, is recalled.

Let $\|a_{ij}\|$ be a $n \times n$ real matrix. As it is well known, the set $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ of its eigenvalues is called the *spectrum* of $\|a_{ij}\|$ and the algebraic equation $(r = 1, 2, \ldots, n)$

$$\mathscr{P}(\lambda) = \det \|a_{ij} - \lambda\delta_{ij}\| = \lambda^n + \sum_{r=1}^{n} A_r \lambda^{n-r} = 0, \tag{37}$$

with δ_{ij} Kronecker coefficients, is the *spectrum equation*, since the eigenvalues are its roots.

Further

$$A_r = (-1)^r I_r; \quad r \in \{1, 2, \ldots, n\} \tag{38}$$

with I_r characteristic values of $\|a_{ij}\|$ given by

$$I_1 = \sum_{r=1}^{n} \lambda_r,\ I_2 = \sum_{i \neq j}^{n-r} \lambda_i \lambda_j,\ I_n = \lambda_1 \lambda_2 \cdots \lambda_n. \tag{39}$$

In terms of the entries a_{ij}, I_r is obtained by adding the determinants of the principal diagonal minors of order r of $\|a_{ij}\|$ [7].

The *spectrum* σ and $\|a_{ij}\|$ are said to be

- unstable if at least one eigenvalue has positive real part,
- bifurcating if and only if contains—at least—a zero or pure imaginary eigenvalue.

The following properties hold.

1. Each condition

$$A_r > 0, \quad \forall\, r \in \{1, 2, \ldots, n\} \tag{40}$$

is necessary for the stability of σ and each condition

$$A_r \leq 0, \quad \forall\, r \in \{1, 2, \ldots, n\} \tag{41}$$

is sufficient for the instability (*coefficient instability power*).

2. If

$$A_n = 0,\ \ A_r > 0, \quad r \in \{1, 2, \ldots, n-1\} \tag{42}$$

then the instability is implied by the existence of a zero eigenvalue and one has steady instability.

3. If exists a positive number φ such that the pure imaginary number φi belongs to σ:

$$\mathscr{P}(i\varphi) = 0 \tag{43}$$

and one has oscillatory or Hopf bifurcation.

4. If the entries depend on a positive parameter $\mathscr{R}$, denoting by $\mathscr{R}_S$ the lowest value of $\mathscr{R}$ at which $A_n = 0$ and by $\mathscr{R}_H$ the lowest value of $\mathscr{R}$ at which $\mathscr{P}(i\varphi) = 0$ for at least a real φ, one has

$$\begin{cases} \mathscr{R}_S < \mathscr{R}_H \iff \text{steady bifurcation} \\ \mathscr{R}_S > \mathscr{R}_H \iff \text{oscillatory bifurcation} \\ \mathscr{R}_S = \mathscr{R}_H \iff \text{steady+Hopf bifurcation} \end{cases} \tag{44}$$

A direct proof of (1) is given in the appendix of [7]; (2) and (3) are obvious; (4) depends on the fact that at the growing of $\mathscr{R}$, (44) implies the occurrence of instability respectively via: a zero eigenvalue, a pure imaginary eigenvalue or via the presence of both such eigenvalues [7]. In [7] Rionero has put in evidence that the *coefficients* of the *spectrum equation* have the property of driving not only the onset of instability via the condition

$$A_r = 0. \tag{45}$$

In fact he has shown that, via (45), the following property guaranteeing the existence of Hopf bifurcations holds.

Property 2. *Let $\|a_{ij}\|$ be stable at $\mathscr{R} = 0$ and let $\bar{\mathscr{R}}$ be the lowest positive value of $\mathscr{R}$ at which a coefficient of the spectrum equation is zero. Then*

$$\bar{r} < n \tag{46}$$

implies that exists a

$$\mathscr{R}^* \in]0, \bar{\mathscr{R}}[\tag{47}$$

at which an oscillatory bifurcation occurs.

Proof. Since the instability occurs only via a steady state $\{\mathscr{R}_S = 0, \mathscr{R}_H > 0\}$ or via a rotatory bifurcation $\{\mathscr{R}_S > 0, \mathscr{R}_H = 0\}$ eventually coupled to a steady state $\{\mathscr{R}_S = \mathscr{R}_H\}$ – at the growing of $\mathscr{R}$ from the stability state at $\mathscr{R} = 0$ to the instability state at $\{\mathscr{R} = \bar{\mathscr{R}}\}$ – (46) implies the existence of an $\mathscr{R}^* \in]0, \bar{\mathscr{R}}[$ at which an oscillatory bifurcation occurs. □

6. Salts Structural Conditions, Necessary for the Onset of Oscillatory Bifurcations

Property 3. *In a porous horizontal layer with stratified porosity, rotating uniformly about a vertical axis, heated from below and salted from below by S_1, and from above by S_2, the oscillatory bifurcations can occur only if the salts satisfy one of the structural conditions*

$$\begin{cases} P_1 > 1 \\ P_2 < 1 \end{cases} \qquad \begin{cases} P_1 = 1 \\ P_2 < 1 \end{cases} \qquad \begin{cases} P_1 > 1 \\ P_2 = 1 \end{cases} \tag{48}$$

Proof. Let us consider the one to one transformation between ϕ_α and ψ_α given by

$$\psi_1 = R_1\theta - P_1 R\phi_1, \qquad \psi_2 = R_2\theta + P_2 R\phi_2 \tag{49}$$

with

$$\begin{vmatrix} R_1 & -P_1 R \\ R_2 & P_2 R \end{vmatrix} = R\,(R_1 P_2 + R_2 P_1) \neq 0 \tag{50}$$

Setting

$$R^* = R^2 - \frac{R_1^2}{P_1} + \frac{R_2^2}{P_2} \tag{51}$$

(9) becomes

$$\begin{cases} \nabla\Pi = -f(z)\mathbf{u} + \dfrac{1}{R}\left(R^*\theta + \dfrac{R_1}{P_1}\psi_1 - \dfrac{R_2}{P_2}\psi_2\right)\mathbf{k} + \mathcal{T}\mathbf{u}\times\mathbf{k} \\ \nabla\cdot\mathbf{u} = 0 \\ \theta_t = Rw + \Delta\theta \\ P_\alpha\psi_{\alpha t} = \Delta\psi_\alpha + R_\alpha(P_\alpha - 1)\Delta\theta \end{cases} \tag{52}$$

under the boundary conditions

$$w = \theta = \psi_\alpha = 0 \qquad \text{on } z = 0, 1. \tag{53}$$

In view of (20) and (49), the linear system governing the evolution of the first component of $(\tilde{\theta}, \tilde{\psi}_1, \tilde{\psi}_2)$ is

$$\frac{d}{dt}\begin{pmatrix}\tilde{\theta}\\ \tilde{\psi}_1\\ \tilde{\psi}_2\end{pmatrix} = \mathscr{L}^*\begin{pmatrix}\tilde{\theta}\\ \tilde{\psi}_1\\ \tilde{\psi}_2\end{pmatrix} \tag{54}$$

where

$$\mathscr{L}^* = \begin{pmatrix} R^*\eta - \xi & \frac{R_1}{P_1}\eta & -\frac{R_2}{P_2}\eta \\ (1-P_1)\xi\frac{R_1}{P_1} & -\frac{\xi}{P_1} & 0 \\ (1-P_2)\xi\frac{R_2}{P_2} & 0 & -\frac{\xi}{P_2} \end{pmatrix}. \tag{55}$$

Setting

$$\tilde{\psi}_1 = \bar{a}X \qquad \tilde{\psi}_2 = \bar{b}Y \tag{56}$$

with $\bar{a}, \bar{b}$ real constant to be determined, one has

$$\frac{d}{dt}\begin{pmatrix}\tilde{\theta}\\ X\\ Y\end{pmatrix} = \mathscr{L}^{**}\begin{pmatrix}\tilde{\theta}\\ X\\ Y\end{pmatrix} \tag{57}$$

with

$$\mathscr{L}^{**} = \begin{pmatrix} R^*\eta - \xi & \bar{a}\frac{R_1}{P_1}\eta & -\bar{b}\frac{R_2}{P_2}\eta \\ \frac{\xi}{\bar{a}}(1-P_1)\frac{R_1}{P_1} & -\frac{\xi}{P_1} & 0 \\ \frac{\xi}{\bar{b}}(1-P_2)\frac{R_2}{P_2} & 0 & -\frac{\xi}{P_2} \end{pmatrix} \tag{58}$$

and requiring

$$\bar{a}\frac{R_1}{P_1}\eta = \frac{\xi}{\bar{a}}(1-P_1)\frac{R_1}{P_1}, \qquad \bar{b}\frac{R_2}{P_2}\eta = \frac{\xi}{\bar{b}}(P_2-1)\frac{R_2}{P_2} \tag{59}$$

one has

$$\bar{a}^2 = \frac{(1-P_1)\xi}{\eta}, \qquad \bar{b}^2 = \frac{(P_2-1)\xi}{\eta} \tag{60}$$

which are admissible only for

$$P_1 \leq 1, \qquad P_2 \geq 1. \tag{61}$$

The problem at stake has therefore, when (61) holds, *hidden symmetries*. Since the eigenvalues of real symmetric matrices are all real numbers, it follows that oscillatory bifurcations cannot occur when the structural condition (61) holds. □

7. Cold Convection Influence

Let us choose, as requested by the physics of the phenomenon at stake, the thermal Rayleigh number R as *bifurcation parameter*.

Then in order to apply property 2, one has to require linear stability at $R = 0$. On the other hand, the existence of the salt S_2 salting L from above, implies the existence of the *cold convection* which implies instability at $R = 0$. We called *cold convection* the onset of instability for any value of R, $R = 0$ included [8]. In view of (35) one has that the instability $\forall R \geq 0$ is implied by each one of the following conditions

$$\begin{cases} R_1^2 \geq \mathscr{R}_1^2 = \frac{P_2}{P_1}R_1^2 + \left(1 + P_2 + \frac{P_2}{P_1}\right)\frac{\xi}{\eta} \Rightarrow A_1 \leq 0 \\ R_2^2 \geq \mathscr{R}_2^2 = \frac{1+P_2}{1+P_1}R_1^2 + \left(1 + \frac{P_2}{1+P_1}\right)\frac{\xi}{\eta} \Rightarrow A_2 \leq 0 \\ R_2^2 \geq \mathscr{R}_3^2 = R_1^2 + \frac{\xi}{\eta} \Rightarrow A_3 \leq 0. \end{cases} \tag{62}$$

Therefore one has

Property 4. *The cold convection is avoided and one has a pure thermal convection only if*

$$R_2^2 < \bar{R}_2^2 = \min(\mathscr{R}_1^2, \mathscr{R}_2^2, \mathscr{R}_3^2). \tag{63}$$

Setting

$$\bar{f} = \inf_{[0,1]} f, \qquad \bar{\bar{f}} = \max_{[0,1]} f \tag{64}$$

in view of

$$1 - \frac{a^2}{\xi} = \frac{\pi^2}{a^2+\pi^2}, \qquad \int_0^1 \sin^2 \pi z \, dz = \int_0^1 \cos^2 \pi z \, dz = \frac{1}{2} \tag{65}$$

one has

$$\begin{cases} \displaystyle\int_0^1 f^{-1} \sin^2 \pi z \, dz \geq \frac{1}{2} \bar{\bar{f}}^{-1} \\ \displaystyle\frac{\bar{f}}{2} \leq \min\left(\int_0^1 f \sin^2 \pi z \, dz, \int_0^1 f \cos^2 \pi z \, dz \right) \end{cases} \tag{66}$$

and it follows that

$$\frac{\xi}{\eta} = 2\frac{\xi}{a^2}\left[\pi^2\mathcal{T}^2 \int_0^1 f^{-1} \cos^2 \pi z \, dz + \int_0^1 (a^2 \sin^2 \pi z + \pi^2 \cos^2 \pi z) \, f \, dz\right] > \pi^2[\mathcal{T}^2 \bar{\bar{f}}^{-1} + \bar{f}]. \tag{67}$$

Therefore setting

$$\begin{cases} H(a^2) = \dfrac{\xi}{\eta} = 2\pi^2 \dfrac{a^2+\pi^2}{a^2} g_1 + 2(a^2+\pi^2) g_2 \\ g_1 = \displaystyle\int_0^1 (\mathcal{T}^2 f^{-1} \sin^2 \pi z + f \cos^2 \pi z) \, dz \\ g_2 = \displaystyle\int_0^1 f \sin^2 \pi z \, dz \end{cases} \tag{68}$$

in view of

$$\begin{cases} \lim\limits_{a^2 \to 0} H(a^2) = \lim\limits_{a^2 \to \infty} H(a^2) = \infty \\ \dfrac{d}{da^2} H(a^2) = \left(g_2 - \dfrac{\pi^4}{a^4} g_1 \right) = 0 \Leftrightarrow a^4 = \dfrac{\pi^4 g_1}{g_2} \end{cases} \tag{69}$$

it follows that

$$H^* = \min_{a^2 \in \mathbb{R}^+} H = H(a_c^2), \qquad a_c^2 = \pi^2 \sqrt{\frac{g_1}{g_2}} \tag{70}$$

i.e.,

$$H^* = 2\pi^2 \left(\int_0^1 f \, dz + 2(g_1 g_2)^{\frac{1}{2}} + \mathcal{T}^2 \int_0^1 f^{-1} \sin^2 \pi z \, dz \right) \tag{71}$$

and hence setting

$$\begin{cases} \bar{\mathscr{R}}_1^2 = \dfrac{P_2}{P_1} R_1^2 + \left(1 + P_2 + \dfrac{P_2}{P_1} \right) H^* \\ \bar{\mathscr{R}}_2^2 = \dfrac{1+P_2}{1+P_1} R_1^2 + \left(1 + \dfrac{P_2}{1+P_1} \right) H^* \\ \bar{\mathscr{R}}_3^2 = R_1^2 + H^* \end{cases} \tag{72}$$

one has that the cold convection is avoided by requiring (63).

8. Oscillatory Bifurcations via the Spectrum Equation Coefficients Power Approach

Let (48) and (63) hold in view of (33)–(36) the spectrum equation is

$$\lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0 \tag{73}$$

with

$$A_1 = -I_1 \qquad A_2 = I_2 \qquad A_3 = -I_3 \tag{74}$$

and I_1, I_2, I_3 given by (35). Setting

$$\begin{cases} R^2_{C_1} = \dfrac{R_1^2}{P_1} - \dfrac{R_2^2}{P_2} + \left(1 + \dfrac{1}{P_1} + \dfrac{1}{P_2}\right)H^* \\ R^2_{C_2} = (1+P_2)R_1^2 - (1+P_1)R_2^2 + (1+P_1+P_2)H^* \\ R^2_{C_3} = R_1^2 - R_2^2 + H^* \end{cases} \tag{75}$$

one has that

$$\begin{cases} R^2 = R^2_{C_k} \Leftrightarrow A_k = 0 \qquad (k = 1,2,3) \\ R^2_{C_k} > 0. \end{cases} \tag{76}$$

The "instability coefficient power" $(ICP)_k$ of A_k can be defined by setting

$$(ICP)_k = \frac{1}{R^2_{C_k}} \tag{77}$$

and the following property holds

Property 5. *Let $A_{\bar{k}}$ be the spectrum equation coefficient with the biggest (ICP) and let the thermal conduction m_0 be linearly asymptotically stable at $R = R_2 = 0$. Then, at the growing of R and R_2, from $R = R_2 = 0$, the instability occurs at $R^2 = R^2_{C_{\bar{k}}}$ and one has a steady bifurcation if $\bar{k} = 3$, while an oscillatory bifurcation occurs at an $R^2 \in]0, R^2_{C_{\bar{k}}}[$ if $k < 3$.*

Proof. Let us begin by recalling that the (Routh-Hurwitz) stability conditions in the case at stake are

$$A_1 > 0, \qquad A_2 > 0, \qquad A_1A_2 - A_3 > 0 \tag{78}$$

and that

$$\mathscr{P}(i\varphi) = 0 \Leftrightarrow A_1A_2 - A_3 = 0. \tag{79}$$

At $R = R_2 = 0$, in view of (29) $\mathscr{L} = \mathscr{L}_1$ reduces to

$$\begin{Vmatrix} -\xi & 0 & 0 \\ 0 & -\dfrac{R^2\eta + \xi}{P_1} & 0 \\ 0 & 0 & -\dfrac{\xi}{P_2} \end{Vmatrix} \tag{80}$$

with ξ and η bigger than zero. The eigenvalues are

$$\lambda_1 = -\xi, \qquad \lambda_2 = -\frac{R^2\eta + \xi}{P_1}, \qquad \lambda_3 = -\frac{\xi}{P_2} \tag{81}$$

i.e., m_0 is asymptotically linearly stable and one has

$$A_1 > 0, \qquad A_2 > 0, \qquad A_1A_2 - A_3 > 0 \qquad \text{at } R = R_2 = 0 \tag{82}$$

(1) Let $\bar{k} = 1$. Then at $R^2 = R^2_{C_1}$ one has

$$A_1 = 0, \qquad A_3 > 0, \qquad A_1A_2 - A_3 = -A_3 < 0. \tag{83}$$

Therefore, in view of the continuity of $A_1A_2 - A_3$, exists a $\bar{R} \in]0, R_{C_1}[$ in which

$$A_1A_2 = A_3 \qquad R = \bar{R} \tag{84}$$

which is the lowest root of (84)$_1$ in $]0, R_{C_1}[$ and a *simple oscillatory bifurcation* (SOB) occurs at $\bar{R}$ having the frequency $\varphi/2\pi$ such that

$$\mathscr{P}(i\varphi, \bar{R}) = -i\varphi^3 - A_1(\bar{R})\varphi^2 + iA_2(\bar{R})\bar{\varphi} + A_3(\bar{R}) = 0 \tag{85}$$

with

$$\varphi^2 = \frac{A_3(\bar{R})}{A_1(\bar{R})} = A_2(\bar{R}) \tag{86}$$

(2) if $R_{C_3} = R_{C_1} < R_{C_2}$, then at $R = R_{C_1}$ the spectrum equation reduces to

$$\lambda(\lambda^2 + A_2) = 0 \tag{87}$$

and at $R = R_{C_1}$ one has a steady+oscillatory bifurcation of frequency $\varphi/2\pi$ with

$$\varphi = [A_2(R_{C_1})]^{\frac{1}{2}} \tag{88}$$

(3) if $R_{C_1} = R_{C_2} < R_{C_3}$, the spectrum equation at $R = R_{C_2} = R_{C_1}$ reduces to

$$\begin{cases} \lambda^3 + A_3 = (\lambda + \gamma)(\lambda^2 - \gamma\lambda + \gamma^2) = 0 \\ \gamma = A_3^{\frac{1}{3}}, \qquad A_3(R_{C_1}) > 0. \end{cases} \tag{89}$$

Therefore

$$\lambda_1 = -\gamma, \qquad \lambda_{2,3} = \frac{\gamma}{2}(1 \pm \sqrt{3}) \tag{90}$$

and a SOB occurs at an $\bar{R} \in]0, R_{C_1} = R_{C_2}[$.

(4) if $R_{C_2} < R_{C_1} < R_{C_3}$, a SOB occurs at a $\bar{R} \in]0, R_{C_2}[$ with frequency φ given by (86) with $\bar{R}$ lowest root of $A_1A_2 - A_3 = 0$.

□

In view of (1)–(4) and (75), criteria guaranteeing the existence of oscillatory bifurcations are easily obtained. We confine ourselves to the following.

Property 6. *Let $R_2 < \bar{R}_2$ and let one of (48) holds. Then*

$$P_2(1 - P_1)R_1^2 - P_1(1 - P_2)R_2^2 + (P_1 + P_2 - P_1P_2)H^* < 0 \tag{91}$$

guarantees the existence of a $\bar{R} \in]0, R_{C_1}[$ at which an oscillatory bifurcation occurs.

Proof. In fact (85) implies

$$R_{C_1}^2 - R_{C_3}^2 = \left(\frac{1}{P_1} - 1\right)R_1^2 - \left(\frac{1}{P_2} - 1\right)R_2^2 + \left(\frac{1}{P_1} + \frac{1}{P_2} - 1\right)H^* \tag{92}$$

and hence (97) implies $R_{C_1} < R_{C_3}$. □

Property 7. *Let $R_2 < \bar{R}_2$ and let one of (48) holds. Then*

$$P_2R_1^2 - P_1R_2^2 + (P_1 + P_2)H^* < 0 \tag{93}$$

guarantees the existence of a $\bar{R} \in]0, R_{C_2}[$ at which an oscillatory bifurcation occurs.

Proof. In fact (75) implies

$$R_{C_2}^2 - R_{C_3}^2 = P_2R_1^2 - P_1R_2^2 + (P_1 + P_2)H^* \tag{94}$$

and $R_{C_2} < R_{C_3}$ is guaranteed by (93). □

Property 8. *Let $R_2 < \bar{R}_2$ and let one of (48) holds. Then*

$$\begin{cases} P_2(1-P_1)R_1^2 - P_1(1-P_2)R_2^2 + (P_1+P_2-P_1P_2)H^* = 0 \\ \left(1+P_2-\dfrac{1}{P_1}\right)R_1^2 - \left(1+P_1-\dfrac{1}{P_2}\right)R_2^2 + \left(P_1+P_2-\dfrac{1}{P_1}-\dfrac{1}{P_2}\right)H^* < 0 \end{cases} \tag{95}$$

guarantees that at $R = R_{C_1}$ a steady-oscillatory bifurcation occurs.

Proof. In fact: $(95)_1 \Rightarrow R_{C_1} = R_{C_3}$, $(95)_2 \Rightarrow R_{C_1} < R_{C_2}$. □

Property 9. *Let $R_2 < \bar{R}_2$ and let one of (48) holds. Then*

$$\begin{cases} P_2R_1^2 - P_1R_2^2 + (P_1+P_2)H^* = 0 \\ \left(1+P_2-\dfrac{1}{P_1}\right)R_1^2 - \left(1+P_1-\dfrac{1}{P_2}\right)R_2^2 + \left(P_1+P_2-\dfrac{1}{P_1}-\dfrac{1}{P_2}\right)H^* > 0 \end{cases} \tag{96}$$

guarantees that at $R = R_{C_2}$ a steady-oscillatory bifurcation occurs.

Proof. In fact: $(96)_1 \Rightarrow R_{C_2} = R_{C_3}$, $(96)_2 \Rightarrow R_{C_2} < R_{C_3}$. □

We end by remarking that:

(1) the values of H^* have to be evaluated via (71) with g_1, g_2 given by (68);
(2) the values of P_1, P_2 have to be taken into account;
(3) in the case $\{P_1P_2 > 1, P_2 \le \frac{1}{1+P_1}\}$ the following criterion holds.

Property 10. *Let*

$$\begin{cases} P_1P_2 < 1, \qquad P_2 \le \dfrac{1}{1+P_1}, \qquad R_2 < \bar{R}_2 \\ \left(1-\dfrac{1}{P_1}\right)R_1^2 + \left(\dfrac{1}{P_2}-1\right)R_2^2 > \left(\dfrac{1}{P_1}+\dfrac{1}{P_2}\right)H^* \end{cases} \tag{97}$$

then an oscillatory bifurcation occurs at a $\bar{R} \in]0, R_{C_1}[$.

Proof. In fact one has $\{P_2 < 1,\ P_1 > 1\}$ and $R_{C_1} < \min(R_{C_2}, R_{C_3})$. □

9. Applications

The knowledge of the function $H^*(\mathcal{T})$ given by (71), is necessary for the applications of Hopf bifurcation criteria. One has to remark that—accounting for $(68)_2$ and $(68)_3$ and the presence of $(g_1g_2)^{\frac{1}{2}}$ in (71), does not simplify $H^*(\mathcal{T})$.

We here, for the sake of simplicity and concreteness, confine ourselves to the case

$$f(z) = e^z. \tag{98}$$

One has, $\forall p \in \mathbb{R}$,

$$\begin{cases} \displaystyle\int_0^1 e^{pz}\cos\pi z\,dz = \left[e^{pz}\frac{p\cos\pi z + \pi\sin\pi z}{p^2+\pi^2}\right]_0^1 = \frac{p(e^p-1)}{p^2+\pi^2}, \\ \displaystyle\int_0^1 e^{pz}(\cos^2\pi z + \sin^2\pi z)\,dz = \frac{e^p-1}{p}, \\ \displaystyle\int_0^1 e^{pz}(\cos^2\pi z - \sin^2\pi z)\,dz = \int_0^1 e^{pz}\cos\pi z\,dz = \frac{p(e^p-1)}{p^2+4\pi^2}, \end{cases} \tag{99}$$

which imply

$$\begin{cases} \int_0^1 e^{pz}\sin^2 \pi z\,dz = \dfrac{2\pi^2(e^p-1)}{p(p^2+4\pi^2)}, \\ \int_0^1 e^{pz}\cos^2 \pi z\,dz = \dfrac{(e^p-1)(p^2+2\pi^2)}{p(p^2+4\pi^2)} \end{cases} \tag{100}$$

$$\begin{cases} \int_0^1 e^{z}\sin^2 \pi z\,dz = \dfrac{2\pi^2(e-1)}{1+4\pi^2}, \ \int_0^1 e^{-z}\sin^2 \pi z\,dz = \dfrac{2\pi^2(e-1)}{e(1+4\pi^2)}, \\ \int_0^1 e^{z}\cos^2 \pi z\,dz = \dfrac{(1+2\pi^2)(e-1)}{1+4\pi^2}, \ \int_0^1 e^{-z}\cos^2 \pi z\,dz = \dfrac{(1+2\pi^2)(e-1)}{e(1+4\pi^2)} \end{cases} \tag{101}$$

$$\begin{cases} g_1 = \dfrac{2\pi^2(\mathcal{T}^2+(1+2\pi^2)/2\pi^2)(e-1)}{e(1+4\pi^2)}, \ g_2 = \dfrac{2\pi^2(e-1)}{1+4\pi^2}, \\ (g_1g_2)^{\frac{1}{2}} = \dfrac{2\pi^2(e-1)}{1+4\pi^2}\sqrt{e^{-1}\left(\mathcal{T}^2+\dfrac{1+2\pi^2}{2\pi^2}\right)} \end{cases} \tag{102}$$

$$H^*(\mathcal{T}) = 4\pi^4(e-1)\left[e^{-1}\mathcal{T}^2 + 2\sqrt{e^{-1}\left(\mathcal{T}^2+\frac{1+2\pi^2}{2\pi^2}\right)}+1\right] \tag{103}$$

plotted in Figure 1. We end by remarking that:

(1) the contruction of $H^*(\mathcal{T})$ in the cases of stratification laws of type $f = e^{cz}$, with $c \in \mathbb{R}$, is obtained following, step by step, the previous procedure. In particular, one can consider the law $e^{c(1/2-z)}$, $c =$const.> 0 proposed in [1] for the increase of viscosity in the earth's mantle;

(2) in [2], upper and lower bounds of $H^*(\mathcal{T})$ are furnished for any stratification law.

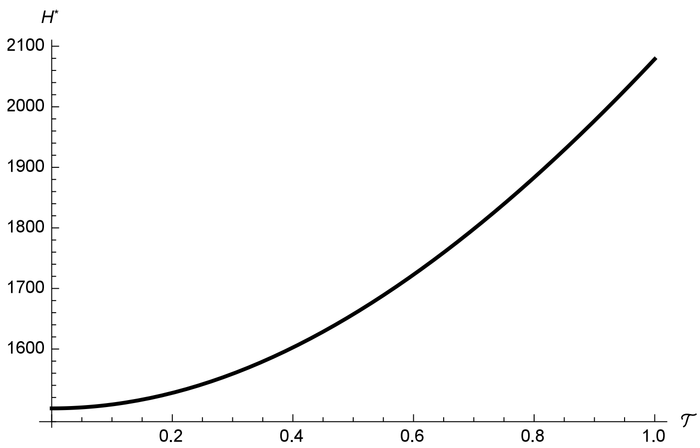

Figure 1. $f = e^z, H^* = H^*(\mathcal{T})$.

10. Final Remarks

1. The results obtained can be applied for any stratification law of porosity f and the oscillatory bifurcations depend on f via H^* given in (71).
2. Property 5 guarantees the existence of oscillatory bifurcations (giving also an estimate of their locations).
3. The condition $R_{C_k} < R_{C_3}$ for at least a $k < 1$ is simpler than the looking for the roots of $A_1A_2 - A_3 = 0$.

Compliance with ethical standards.

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