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Chaotic Dynamics of the Interface between Dielectric Liquids at the Regime of Stabilized Kelvin-Helmholtz Instability by a Tangential Electric Field

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Abstract: The nonlinear dynamics of the interface between two immiscible dielectric liquids at the regime of suppressed Kelvin-Helmholtz instability by external horizontal electric field is studied theoretically. The initial equations of the fluids motion are reduced to a single weakly nonlinear integro-differential equation that describes the interaction of solitary waves (rational solitons) propagating along the interface. The dynamics of two interacting solitons is regular and integrable; they can combine into a stable wave packet (breather). It is shown that the interaction of three solitons becomes complex and, for a wide rang of initial conditions, chaotic. The numerically obtained Poincaré sections demonstrate the destruction of toroidal trajectories in the phase space during the transition of the system to a chaotic regime of fluid motion. Such a behaviour is consistent with the Kolmogorov-Arnold-Moser theory describing quasi-periodic chaotic motion in Hamiltonian systems. At the developed chaotic state, the system fast loses the information on its initial state; the corresponding estimate for Lyapunov exponent is obtained. From the physical point of view, the chaotic behavior of the system is related with structural instability of the soliton triplet. The triplet can decay into a solitary wave and stable breather consisting of two interacting solitons.

Keywords: Kelvin-Helmholtz instability; rational solitons; electric field; liquid interface; nonlinear waves; liquid dielectrics

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1. Introduction

One of the most widespread types of hydrodynamic instabilities is the Kelvin-Helmholtz (KH) instability occurring at the interface between two fluids moving with different velocities [1,2]. It also known that an external electric or magnetic field directed tangentially to the unperturbed interface has a stabilizing effect [3,4]. Thus, the interest to study the interfacial dynamics under the action of external electric field is caused by the possibility of suppressing and controlling hydrodynamic instabilities [5–13]. In this work, we investigate the dynamics of interface between dielectric liquids at the regime of stabilized KH instability by tangential electric field. In our previous work [5], a family of exact solutions describing the weakly nonlinear dynamics of the interface between non-conducting liquids in the regime of complete stabilization of the KH instability was obtained. The solutions describe propagation and interaction of nonlinear interfacial localized waves (the so-called rational solitons). In the current work, we will focus on the study of complicated and chaotic interaction between solitons propagating along the interface.

Note that the search for soliton solutions in various nonlinear equations of mathematical physics is an important and challenging problem, see the reviews [14,15] and references therein. The nonlinear Schrödinger (NLS) and Korteweg de Vries (KdV) equations are the classic examples of equations that admits solitary wave solutions [16,17]. The NLS equation originally arises in describing the propagation of narrow spectral wave packets on the fluid boundary in a gravitational field [16] (the soliton dynamics is studied in the recent works [18–20]). The KdV equation describes nonlinear waves in the long-wave region of the spectrum, that is, where the length of surface perturbations is significantly greater than the depth of the fluid [17]. Various modifications of the NLS and KdV equations were used to describe the surface electrohydrodynamic waves, for example, see the works [21–26].

An advantage of both models is the possibility of reducing the original complex problem of describing the motion of a fluid to solving relatively simple nonlinear partial differential equations. An important feature of the solutions described in [5] is that they are obtained without restrictions on the spectrum width and depth of the liquid layer. In other words, the solutions [5] are obtained on the basis of equations of motion that include non-local integro-differential operators. The basic idea to find exact analytical solutions is to consider a special regime of fluid motion where their flow occurs along the electric field lines [27]. A similar approach was used in [28–31] to analytically describe nonlinear dynamics of liquid boundaries in the framework of non-local integro-differential equations.

Let us specify the physical conditions under which the nonlinear dynamics of the fluid boundary can be described analytically. The dispersion relation for linear waves at the interface between dielectric liquids in the presence of tangential electric field and the boundary velocity jump has the form [26]

$$(\omega + V_c k_x)^2 = (c_e^2 - c^2)k_x^2, \tag{1}$$

where ω is the frequency, k_x is the x-component of the wave vector, $V_c = (\rho_1 V_1 + \rho_2 V_2)/(\rho_1 + \rho_2)$ is the velocity of the center of mass of the system, $\rho_{1,2}$ and $V_{1,2}$ are the fluids densities and velocities, respectively. The indexes "1" and "2" correspond to the lower and upper liquid, respectively. Here we do not take into account the effects of gravity and capillarity which dominate in the long-wave and short-wave limits, respectively. We consider the intermediate wavelength range, where electrostatic forces play a major role. The quantities c and c_e are determined by the expressions

$$c^2 = \frac{\rho_1 \rho_2 (V_1 - V_2)^2}{(\rho_1 + \rho_2)^2}, \quad c_e^2 = \frac{\varepsilon_0 (\varepsilon_1 - \varepsilon_2)^2 E^2}{(\rho_1 + \rho_2)(\varepsilon_1 + \varepsilon_2)},$$

where ε_0 is the electric constant, $\varepsilon_{1,2}$ are the dielectric constants of liquids, and E is the absolute value of the external horizontal electric field.

The stability of the interface between the fluids depends on the sign of the right-hand side of the dispersion relation (1): at $c_e^2 < c^2$ the KH instability is developing, whereas at $c_e^2 > c^2$ the interface is stable and waves propagate along it in the direction of applied electric field. The situation where the right-hand side of Equation (1) vanishes ($c_e^2 = c^2$) corresponds to the neutral equilibrium state—the destabilizing effect of the tangential discontinuity of the velocity is compensated by the stabilizing action of the horizontal electric field. Such regime of fluid motion is realised for the following value of the external electric field:

$$E_c^2 = \frac{\rho_1 \rho_2 (\varepsilon_1 + \varepsilon_2)(V_1 - V_2)^2}{(\rho_1 + \rho_2)\varepsilon_0 (\varepsilon_1 - \varepsilon_2)^2}.$$

The exact soliton solutions [5] were found precisely for the neutral stability regime, that is, for $E = E_c$.

In this paper, we investigate the complex interaction of interfacial solitons. It will be shown that the fluid boundary can demonstrate complex and chaotic behavior. In our opinion, this fact is especially important for the problem of describing the wave turbulence of a liquid surface in an external electric (magnetic) field [32–36].

2. The Model Equation

In this section, we briefly describe the derivation of the key nonlinear integro-differential equation describing the dynamics of solitons at the interface between liquids. We consider the dynamics of the interface between two immiscible dielectric fluids with the densities $\rho_{1,2}$

and relative permittivities $\varepsilon_{1,2}$ in the external horizontal electric field. The liquid interface in the unperturbed state is the $z = 0$ (the x axis coincides with the direction of the external field $\mathbf{E}$ and the z axis is perpendicular to it). The fluids flow along the x axis at different velocities $V_{1,2}$; that is, the difference $\Delta V \equiv V_1 - V_2$ is nonzero. The shape of the interface in the perturbed state is specified by the equation $z = \eta(x, y, t)$. We assume that the horizontal length of the problem is infinitely long.

The potentials of the velocity $\Phi_{1,2}$ and the electric field $\varphi_{1,2}$ satisfy the Laplace equations

$$\nabla^2 \Phi_{1,2} = 0, \qquad \nabla^2 \varphi_{1,2} = 0.$$

The normal components of the velocities of the upper and lower fluids at the interface should be equal to each other $\partial_n \Phi_1 - \partial_n \Phi_2 = 0$, where ∂_n is the derivative in the direction of the normal to the surface. The potentials of the electric field satisfy the boundary conditions $\varphi_1 - \varphi_2 = 0$ and $\varepsilon_1 \partial_n \varphi_1 - \varepsilon_2 \partial_n \varphi_2 = 0$. These conditions mean the requirements of the continuity of the tangential component of the electric field strength and the normal component of the electric displacement field at the interface (we assume that interfacial free charges are absent). The flow of the fluids at infinite distances from the interface ($z \to \mp\infty$) becomes steady and the electric field becomes uniform: $\Phi_{1,2} \to V_{1,2}x$ and $\varphi_{1,2} \to -Ex$. The Hamiltonian H of the system under study has the form [37]

$$H = \int_{z \le \eta} \left[\frac{\rho_1}{2} (\nabla \Phi_1)^2 - \frac{\varepsilon_0 \varepsilon_1}{2} (\nabla \varphi_1)^2 \right] d^3 r +$$

$$+ \int_{z \ge \eta} \left[\frac{\rho_2}{2} (\nabla \Phi_2)^2 - \frac{\varepsilon_0 \varepsilon_2}{2} (\nabla \varphi_2)^2 \right] d^3 r.$$

Here, capillary and gravitational forces are not taken into account. The interface motion is described by the Hamiltonian equations:

$$\frac{\partial \eta}{\partial t} = \frac{\delta H}{\delta \psi}, \qquad \frac{\partial \psi}{\partial t} = -\frac{\delta H}{\delta \eta}. \tag{2}$$

The quantity ψ in (2) is defined as $\psi(x, y, t) = \rho_1 \Phi_1 - \rho_2 \Phi_2$, at $z = \eta(x, y, t)$.

The next step in deducing the equations of the boundary motion in explicit form is to apply the small slope approximation $|\nabla_\perp \eta| \ll 1$. Let us also consider only plane waves propagating along the x axis (any dependence on the y variable is absent). For definiteness, let $\Delta V > 0$ and $c > 0$ (these inequalities specify the direction of the discontinuity of the velocity). In order to obtain the equations of the interface motion, the Hamiltonian of the system H should be expressed in terms of the canonical variables η and ψ. This Hamiltonian is represented as a surface integral by the Ostrogradsky–Gauss theorem. Further, the integrand in the Hamiltonian is expanded in a power series η and ψ (the method was described in detail in [4,29,30,38]). It is convenient to use the quantities introduced as

$$\psi \to \psi c \lambda (\rho_1 + \rho_2), \qquad \eta \to \eta \lambda, \qquad t \to t\lambda / c,$$

$$x \to x\lambda, \qquad E \to eE_c, \qquad V_c \to v_c c,$$

where λ is the characteristic wavelength. As a result, the Hamiltonian with allowance of terms up to cubic in the integrand has the form

$$H = \frac{1}{2} \int \psi \hat{k} \psi \, dx + \frac{1}{2} \int \eta \Big[(e^2 - 1) \hat{k} \eta - 2 v_c \psi_x + A \left(\psi_x^2 - (\hat{k}\psi)^2 \right)$$

$$+ (A_e e^2 + A) \left(\eta_x^2 - (\hat{k}\eta)^2 \right) - 2\sqrt{1 - A^2} \left(\eta_x \hat{k} \psi + \psi_x \hat{k} \eta \right) \Big] dx,$$

where $A = (\rho_1 - \rho_2)/(\rho_1 + \rho_2)$ is the Atwood number, $A_e = (\varepsilon_1 - \varepsilon_2)/(\varepsilon_1 + \varepsilon_2)$ is its analog for dielectric constants, and $\hat{k} = -\hat{H}\partial_x$, where $\hat{H}$ is the Hilbert operator. Equation (2) with this Hamiltonian have the form:

$$\eta_t - \hat{k}\psi - v_c\eta_x = -\sqrt{1-A^2}\Big(\hat{k}(\eta\eta_x) - (\eta\hat{k}\eta)_x\Big) - A\Big((\eta\psi_x)_x + \hat{k}(\eta\hat{\psi})\Big), \tag{3}$$

$$\psi_t + (e^2-1)\hat{k}\eta - v_c\psi_x = \frac{A}{2}\Big((\hat{k}\psi)^2 - \psi_x^2\Big) - \sqrt{1-A^2}\Big(\eta\hat{k}\psi_x - \psi_x\hat{k}\eta - \hat{k}(\eta\psi_x)\Big) + \frac{A_e e^2 + A}{2}\Big((\hat{k}\eta)^2 - \eta_x^2 + 2(\eta\eta_x)_x + 2\hat{k}(\eta\hat{k}\eta)\Big). \tag{4}$$

Thus, the evolution of the interface is described by a quite complicated system of nonlinear integro-differential equations. In the absence of electric field, the Equations (3) and (4) were analyzed in [38].

The realization of the neutral stability regime discussed above corresponds to the condition, $e^2 = 1$. The Equations (3) and (4) can be greater simplified under the additional condition: $A = -A_e$ or $\varepsilon_1\rho_1 = \varepsilon_2\rho_2$ (the examples of liquids with such properties are presented in [30]). When the both conditions are satisfied, the Equation (4) becomes compatible with the expression: $\psi = 0$. The equality, $\psi = 0$, physically means that the fluids flow along the electric field lines (see [5]). Finally, the system (3) and (4) can be reduced to the only nonlinear equation:

$$\eta_t - v_c\eta_x = \sqrt{1-A^2}\big[\hat{H}(\eta\eta_x)_x - (\eta\hat{H}\eta_x)_x\big]. \tag{5}$$

Our further attention will be paid to the analysis of this key equation.

3. Soliton Dynamics

It turns out that in complex variables, the Equation (5) can be reduced to an even more compact form. Let represent the surface elevation as $\eta = \eta^+ + \eta^-$, where $\eta^\pm = \hat{P}^\pm\eta$ are the analytical continuations of the function η in the upper and lower half-planes of the complex variable x, respectively, and $\hat{P}^\pm = (1 \mp i\hat{H})/2$ are the projectors. Let us also pass to the system of the center of mass of liquids: $x \to x + v_c t$. Such substitution eliminates the second term on the left-hand side of the equation. The resulting compact complex equation has the form

$$\tau\eta_t^+ = 2i\hat{P}^+(\eta^+\eta_x^-)_x, \tag{6}$$

where $\tau = (1-A^2)^{-1/2}$ is the characteristic nonlinear time in the problem.

The specific feature of the Equation (6) is that it allows reduction to the system of ordinary differential equations describing the interaction of an arbitrary number of structurally stable solitary waves. We seek solution (6) in the form

$$\eta^+(x,t) = \sum_{n=1}^{N} \frac{iS_n/2}{x - p_n(t)}, \tag{7}$$

where the complex functions $p_n(t)$ determine the positions of singular points (poles) in the lower half-plane of the complex variable x ($\mathrm{Im}p_n < 0$) and S_n are real constants. Each pole corresponds to an individual localized perturbation of the interface:

$$\eta = \sum_{n=1}^{N} \eta_n, \qquad \eta_n = \frac{S_n|\mathrm{Im}p_n(t)|}{(x - \mathrm{Re}p_n(t))^2 + (\mathrm{Im}p_n(t))^2}.$$

It can be seen that quantities S_n specify the areas of solitary perturbations ($\pi S_n = \int \eta_n dx$), quantities $\mathrm{Re}p_n$ determine the positions of waves on the x axis, $|\mathrm{Im}p_n|$ are the characteristic widths of perturbations, and $S_n/|\mathrm{Im}p_n|$ are their amplitudes. The substitution of

Equation (7) into Equation (6) gives the following system of equations for the motion of poles

$$\tau \frac{dp_n}{dt} = \sum_{j=1}^{N} \frac{S_j}{(p_n - \bar{p}_j)^2}, \qquad n = 1, 2, \ldots N. \tag{8}$$

We will show below that this system can describe both regular and chaotic regimes of the soliton motion.

3.1. Regular Dynamics

For the simplest case of one soliton $N = 1$, the solution of the system (8) gives the trajectory of the pole

$$p_1(t) = v(t - t_0) - iS_1/a, \qquad v = -a^2/(4\tau S_1),$$

where a is the amplitude of the wave and t_0 is a constant. Thus, the solitary wave propagates without distortions at a constant velocity v, which is proportional to the square of its amplitude. The direction of wave motion depends on the sign of S_1. Positive and negative perturbations propagate against and along the x axis, respectively (the direction of motion changes to opposite at a change in the sign of the jump of the velocity ΔV, which was previously chosen positive).

In the case $N > 1$, we define the position of the center of the system of poles as

$$p_0 = \frac{1}{S} \sum_{n=1}^{N} S_n p_n, \qquad S = \sum_{n=1}^{N} S_n.$$

The quantity $I = \mathrm{Im} p_0$ is an integral of motion of the system (8), so that the center p_0 moves in parallel to the real axis. Let $|p_n - p_0| \ll |I|$ for any n, that is, the poles are close to each other on the complex plane. In this situation, the center of the system of poles will move at a constant velocity

$$p_0 = v(t - t_0) + iI, \qquad v = -S/(4\tau I^2),$$

which is similar to the behavior of one pole with $S_1 = S$ for the above case $N = 1$. In this case, the poles rotate about the common center at the frequency $\Omega = v/I$. Indeed, the motion of the poles with respect to the center p_0 is described by N independent equations

$$\frac{d(p_n - p_0)}{dt} = i\Omega(p_n - p_0), \qquad n = 1, 2, \ldots N,$$

obtained from Equation (8) by expansion in the small parameters $p_n - p_0$. Thus, in the case of strong interaction, solitary waves are joined in a wave packet (breather). The regular dynamics of breathers evolution is described in details in [5]. Further, we will focus on complex irregular dynamics of soliton interaction with $N \geq 3$.

3.2. Chaotic Dynamics

In this section, we study the dynamics of soliton interaction based on the numerical solution of the nonlinear differential Equation (8). For simplicity, we consider the dynamics of interaction of three solitons, that is, $N = 3$. Without loss of generality, set the system parameters as follows: $\tau = 1$, and $S_1 = S_2 = S_3 = 1$. The phase space of the system is characterized by six variables—the pole coordinates depending on time. Due to the presence of the integral of motion I, the system (8) has five independent degrees of freedom. Let us show that in this case, the dynamics of three interacting solitons can demonstrate the chaotic behaviour.

To numerically simulate the soliton dynamics we use the explicit fourth order Runge-Kutta method with the time step $dt = 2.5 \times 10^{-3}$ for the solution of the system (8). At the initial moment of time, we combine the maxima of three solitons at the origin of the coordinate system, that is, $\mathrm{Re}(p_1) = \mathrm{Re}(p_2) = \mathrm{Re}(p_3) = 0$ at $t = 0$. We present first four calculation series demonstrating that the interaction of three solitons has quite complicated character. The initial conditions are shown in the table below.

To characterize the system dynamics we use Poincaré sections shown in Figure 1 for the initial conditions presented in Table 1. Figure 1 shows the distances between poles with the indexes "1" and "3" at the moment when the amplitudes of second and third solitons are equal to each other, that is, $\mathrm{Im}(p_2 - p_3) = 0$. Figure 1a corresponds to the initially close position between poles. In this case, the dynamics of solitons is regular; their positions oscillate around a common center of mass. When the initial position of the first pole is shifted to the real numerical axis, the Poincaré section is deformed as shown in Figure 1b. With a further shift of the pole to the real axis, the toroidal trajectories in phase space begin to break down, see Figure 1c. This behavior is consistent with the Kolmogorov-Arnold-Moser theory developed for description of chaotic motion in Hamiltonian systems. Finally, Figure 1d shows the developed chaotic state of the system under study for the initial condition presented in fourth line of Table 1.

Table 1. The initial poles positions for the numerical solution of the system (8).

#	p_1	p_2	p_3
1	$-0.400i$	$-0.6i$	$-1.15i$
2	$-0.200i$	$-0.6i$	$-1.15i$
3	$-0.175i$	$-0.6i$	$-1.15i$
4	$-0.100i$	$-0.6i$	$-1.15i$

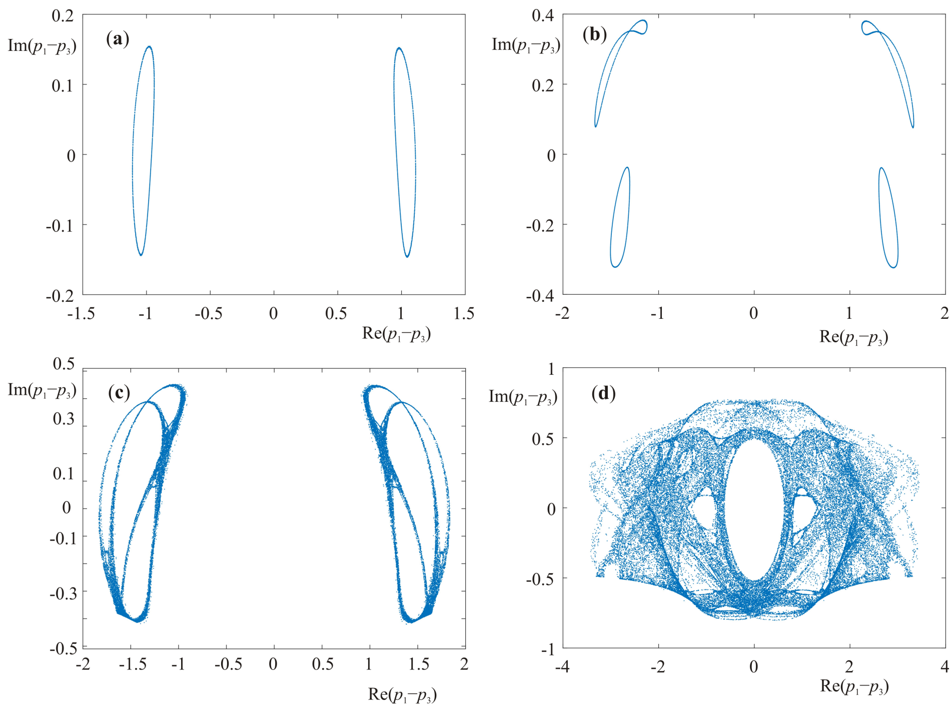

Figure 1. Poincaré sections are shown for the different initial conditions presented in Table 1: **(a–d)** correspond to 1, 2, 3, and 4 lines in Table 1, respectively.

The evolution of interacting solitons is shown in Figure 2. Figure 2a corresponds to the regular behavior which phase portrait is shown in Figure 1b. We can see that dynamics of the soliton interaction is indeed regular and periodic. Figure 2b shows completely different situation; the soliton motions are chaotic and non-periodic. To plot Figure 2b we used the initial conditions from fourth line in Table 1, that is, the Poincaré section for the soliton dynamics is shown in Figure 1d.

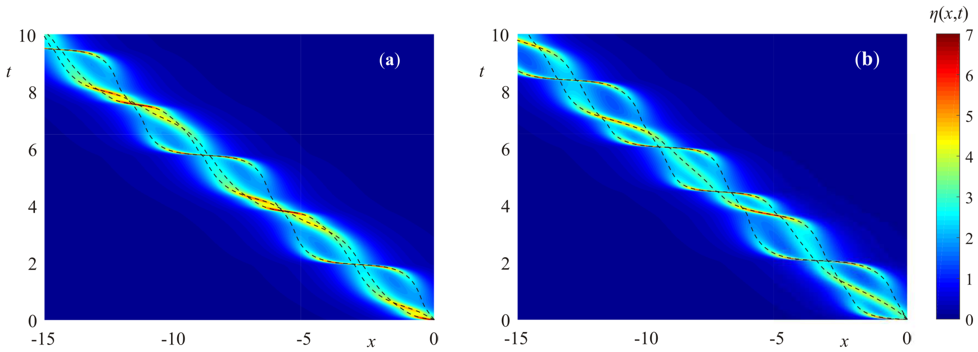

Figure 2. The evolution of the interface is shown for the different initial conditions: (**a**,**b**) correspond to Figure 1b,d, respectively. Black dashed lines correspond to the pole positions at real x-axis.

A distinctive feature of chaotic behavior is the impossibility of accurate prediction of the state of the system at long times. To demonstrate the chaotic nature of the soliton dynamics, let us consider the evolution of a liquid surface for two initially close conditions. For the first realization, we will keep the initial conditions given by the fourth line in Table 1. For the second implementation, we will slightly change the initial condition for the third soliton $p_3 = -1.15001i$ at $t = 0$. Figure 3 shows the time dependence of the quantity characterizing the difference between two realizations:

$$\Delta p_3(t) = |\mathrm{Im}(p_3(t) - p_3'(t))|,$$

where $p_3(t)$ and $p_3'(t)$ are the third pole coordinates for two different initial conditions. Figure 3 shows that initially close trajectories diverge exponentially with time with the value of Lyapunov exponent about 0.1. Thus, in a fairly short time of the order of 100 dimensionless units, the system completely loses memory of its initial data. Note that Figure 1d was obtained from the simulation of long-term dynamics of the system: the total computational interval was equal to 10^5 of non-dimensional units of time.

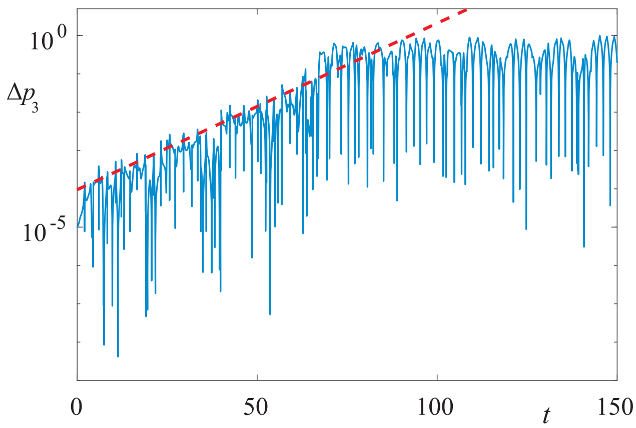

Figure 3. The difference between two initially close positions of third pole is shown versus time, the red dashed line corresponds to the exponential fit: $10^{-4}\exp(0.1t)$.

In general, the obtained results demonstrate a tendency to transition from regular to chaotic regime of the fluid motion. The key parameter in the simulations presented is the amplitude of the first soliton. As the soliton amplitude increases, its velocity also increases. For a some critical value of the initial pole position, the group motion of soliton triplet can be unstable. Figure 4 shows that with a further slight increase in the amplitude of the first soliton, the joint motion of the waves becomes impossible. It can be seen that the first soliton leaves the region of joint motion. The triplet of solitons becomes unstable: it decays into a solitary wave and a stable breather consisting of two solitons. Thus, the chaotic behavior of the system is observed near the instability threshold, when the interaction between solitons becomes weak, but the triplet does not decay yet.

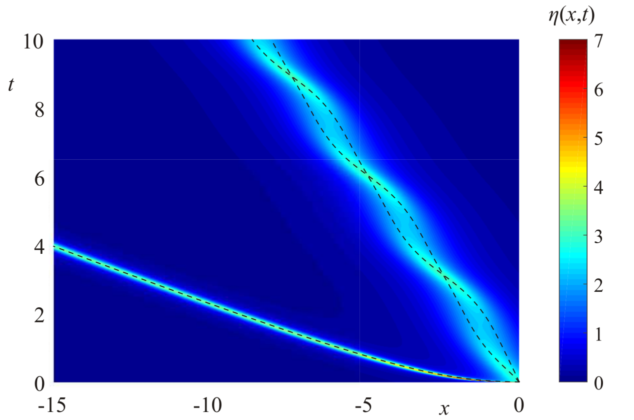

Figure 4. The evolution of the interface is shown for the initial conditions: $p_1 = -0.075i$, $p_2 = -0.6i$, $p_3 = -1.15i$. Black dashed lines correspond to the pole positions at real x-axis.

4. Conclusions

In the present work, the nonlinear dynamics of the interface between two immiscible dielectric fluids is investigated in the regime of stabilization of the Kelvin-Helmholtz instability by a horizontal electric field (capillary and gravity forces are not take into account). In a situation where the instability is completely stabilized by the external field (neutral equilibrium), the description of the weakly nonlinear evolution of the liquid surface can be reduced to the analysis of ordinary differential equations describing the propagation and interaction of structurally stable localized waves (rational solitons). The dynamics of two interacting nonlinear waves is regular and integrable [5]. In this work, we show that the motion of soliton triplet can be complex and chaotic. The Poincaré sections obtained on the basis of numerical analysis demonstrate the destruction of toroidal trajectories in the phase space during the transition of the system to a chaotic regime. Such a behaviour is consistent with Kolmogorov-Arnold-Moser theory describing quasi-periodic chaotic motion in Hamiltonian systems. In the work, we have estimated the Lyapunov exponent in the regime of developed chaotic state. It is shown that the system fast loses the information on its initial state in the chaotic regime. From the physical point of view, the chaotic motion of soliton triplet is related with its structural instability. With a slight increase in the velocity of one soliton, the triplet decays into a solitary wave and stable breather consisting of two solitons. Thus, within the framework of one model Equation (5), one can observe a rich picture of nonlinear types of motion including chaos, solitons and collapses [39]. We hope that our work will motivate researchers to investigate, both experimentally and theoretically, a new type of fluid motion, where two immiscible fluids separated by an interface flow along the electric (or magnetic) field lines.

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