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Rational Solitons in the Gardner-Like Models

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Abstract: Rational solutions of nonlinear evolution equations are considered in the literature as a mathematical image of rogue waves, which are anomalously large waves that occur for a short time. In this work, bounded rational solutions of Gardner-type equations (the extended Korteweg-de Vries equation), when a nonlinear term can be represented as a sum of several terms with arbitrary powers (not necessarily integer ones), are found. It is shown that such solutions describe first-order algebraic solitons, kinks, and pyramidal and table-top solitons. Analytical solutions are obtained for the Gardner equation with two nonlinear terms, the powers of which differ by a factor of 2. In other cases, the solutions are obtained numerically. Gardner-type equations occur in the description of nonlinear wave dynamics in a fluid layer with continuous or multilayer stratification, as well as in multicomponent plasma, and their solutions are used for the interpretation of rogue waves.

Keywords: soliton; rational solution; Gardner equation

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1. Introduction

Nowadays, there is great interest in rational solutions of nonlinear evolutionary equations, which are associated with mathematical models of rogue waves. Thus, for many years, the well-known Peregrine breather, obtained in the framework of the nonlinear Schrödinger equation [1–4], has been considered as a reference example of a rogue wave on the sea surface, since its amplitude is three times larger than the amplitude of the background waves. Then it was possible to show that within the same equation, there are multiparameter rational solutions of higher orders, in which the maximum wave amplitude can be much higher [5–8]. Such solutions became known as super rogue waves. It should especially be noted that the Peregrine breather, as well as super rogue waves, are observed in laboratory wave tanks and optical fibers [9–13], demonstrating the stability and physical importance of rational solitons/breathers.

Similar rational solutions are also found in the framework of other evolutionary equations solved exactly by the methods of the inverse scattering problem, such as the Benjamin-Ono equation [14,15], the Kadomtsev-Petviashvili equation [16,17], the Hirota equation [18], Ablowitz-Ladik equation [19], and others. Let us focus on the papers [20–24], which consider rational solutions in KdV-like systems, in particular, within the framework of the Korteweg-de Vries equation and its modified version. In some cases, these rational solutions are singular and have no physical meaning [25,26]. In other cases, they are bounded, but exist on a pedestal, which also leads to unlimited mass and energy of such solutions. Meanwhile, when the nonlinearity in KdV-like systems is composite, that is, it includes quadratic and cubic nonlinearity (such an equation is called the Gardner equation), then its rational solution is an algebraic soliton [27,28].

The Gardner equation is one of the main equations in the theory of nonlinear internal waves in oceans stratified by density and shear current; it is actively used in the interpretation of in situ measurement data [29]. If the ocean stratification is complex (multilayer flow or continuous stratification), the quadratic and cubic nonlinearity coefficients can be small, so that higher-order effects have to be included. In particular, with the so-called symmetric stratification in the form of a three-layer flow, the evolution equation includes nonlinearity of the third and fifth orders [30]. In the case of stratification close to exponential (constant buoyancy frequency), a larger number of nonlinear terms may appear in the evolution equation [31,32]. It is important to note that internal waves in the ocean can reach very large amplitudes, leading to the destruction of oil and gas platforms in the sea. They affect the propagation of acoustic signals over long distances, masking the underwater environment [29–32].

Similar equations have been obtained in plasma theory [33–35], Fermi-Pasta-Ulam chains [36], and even compacton theory [37]. High-order Gardner-like equations are not integrable, and there is no efficient way to find a set of rational solutions such as super rogue waves. This was the main stimulus for this study and ensures its novelty. In this paper, we focus on the analysis of rational solutions of the first kind, found both analytically and numerically. In Section 2, a general approach to finding rational bounded solutions (solitons) is discussed. Analytically, such solutions are found in the case of a nonlinearity of the $u^q - u^{2q}$ type for any $q > 1$ (Section 3). Numerically, algebraic solitons are found for a nonlinearity of $u - u^q$ type (Section 4). For certain coefficients of polynomial nonlinearity, rational solutions are obtained in the form of a kink, i.e., a jump between two constant values (Section 5). If the nonlinearity is described by a high-order polynomial, then a rational soliton can be a table-top or pyramidal soliton (Section 6). The results obtained are summarized in the Conclusions.

2. General Approach to Obtaining Rational Solutions of the First Order

Let us consider KdV-type equations with arbitrary nonlinearity:

$$\frac{\partial u}{\partial t} + f(u)\frac{\partial u}{\partial x} + \delta\frac{\partial^3 u}{\partial x^3} = 0. \tag{1}$$

It is convenient to set δ equal to 1 in this equation and rewrite (1) as

$$\frac{\partial u}{\partial t} + \frac{\partial F(u)}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad f(u) = dF/du, \tag{2}$$

assuming that $F(0) = 0$. We will look for bounded rational solutions of Equation (2) with tails falling to infinity on the entire x axis. It was already noted in [38] that in the framework of KdV-like models, rational (algebraic) solutions are obtained at zero wave propagation velocity (in the framework of the original equations, this is the propagation velocity of long waves), so that rational solutions are found from the ordinary differential equation

$$\frac{d^2u}{dx^2} + F(u) = 0, \tag{3}$$

where the constant of integration is chosen to be zero, ensuring the damping of the solutions at infinity.

If in the lowest order a nonlinear function is a power function $F(u) = au^b$ with constant coefficients a and b, then the tails of a possible rational soliton have an algebraic asymptotic behavior:

$$u(x \to \pm\infty) \approx \left[-\frac{2(1+b)}{a(b-1)^2}\right]^{\frac{1}{b-1}} |x|^{-\frac{2}{b-1}}. \tag{4}$$

From here, it is clear that $b > 1$. Already from this simple estimate, it follows that if the nonlinearity in the KdV-like equation is simple, i.e., Equation (1) contains only one nonlinear term, then all rational solutions are singular and have no physical meaning. Thus, for the existence of bounded rational solutions, a composite nonlinearity is needed, where the subsequent term is a higher-order nonlinearity that will make a large contribution at the top of the soliton.

The general approach to finding rational solutions follows from (3) after a single integration:

$$\frac{1}{2}\left(\frac{du}{dx}\right)^2 + \Pi(u) = 0, \quad F(u) = d\Pi/du. \tag{5}$$

Moreover, the integration constant is again chosen to be zero, ensuring the decreasing of the solution at infinity. If a rational solution exists, then its amplitude is uniquely found as the minimum nonzero root of the algebraic equation

$$\Pi(A) = 0, \tag{6}$$

and the solution itself is expressed through the integral

$$x = \pm \int_u^A \frac{dv}{\sqrt{-2\Pi(v)}}. \tag{7}$$

Since there are no free parameters in Equation (7), the rational solution is uniquely determined.

Although formally the technology for finding rational solutions is quite simple, it describes a wide class of algebraic solitons that are of great practical importance.

3. Analytical Rational Solitons

It was already mentioned that the algebraic soliton in the framework of the classical Gardner equation (with the nonlinearity $u - u^2$) has been known for a long time [27]. Here we present a class of non-integrable Gardner-like equations that have solutions in the form of rational functions. For this we use the following ansatz,

$$u(x) = \frac{A}{(1 + B^2x^2)^p} \tag{8}$$

with non-vanishing arbitrary parameters A, B and p ($p > 0$), and substitute it into (5). After trivial calculations, we obtain the following expression for the function $\Pi(x)$:

$$\Pi(u) = -2p^2A^2B^2\left[\left(\frac{u}{A}\right)^{2+1/p} - \left(\frac{u}{A}\right)^{2+2/p}\right]. \tag{9}$$

Differentiating this function twice, we find the nonlinear function in (1):

$$f(u) = -2p^2B^2\left[\frac{(2+1/p)(1+1/p)}{A^{1/p}}u^{1/p} - \frac{(2+2/p)(1+2/p)}{A^{2/p}}u^{2/p}\right]. \tag{10}$$

Denoting the coefficients as

$$\alpha = 2p^2B^2\frac{(2+1/p)(1+1/p)}{A^{1/p}}, \tag{11}$$

$$\beta = 2p^2B^2\frac{(2+2/p)(1+2/p)}{A^{2/p}}, \tag{12}$$

We reduce Equation (1) to a Gardner-type equation with positive coefficients:

$$\frac{\partial u}{\partial t} - \left(\alpha u^{1/p} - \beta u^{2/p}\right)\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{13}$$

The parameters of a rational soliton are uniquely found from (11) and (12) by the given coefficients of Equation (13):

$$A = \left[\frac{\alpha}{\beta}\frac{(2p+4)}{(2p+1)}\right]^p, \tag{14}$$

$$B = \sqrt{\frac{\alpha^2}{\beta}\frac{(2+p)}{(p+1)(2p+1)^2}}. \tag{15}$$

We present here a family of Gardner-type equations that admit an analytical description of rational solitons.

The Equation (13) turns out to be the classic Gardner equation when $p = 1$:

$$\frac{\partial u}{\partial t} - \left(\alpha u - \beta u^2\right)\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \tag{16}$$

$$u(x) = \frac{2\alpha/\beta}{1 + \frac{\alpha^2}{6\beta}x^2}. \tag{17}$$

This solution, as noted in the Introduction, was obtained in [27].

In case of $p = 1/2$, the (2 + 4) Korteweg-de Vries equation was obtained in [30]:

$$\frac{\partial u}{\partial t} - \left(\alpha u^2 - \beta u^4\right)\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \tag{18}$$

$$u(x) = \pm\frac{\sqrt{5\alpha/2\beta}}{\sqrt{1 + \frac{5\alpha^2}{12\beta}x^2}}. \tag{19}$$

Finally, the higher-order Gardner equation takes the following form ($p = 1/3$):

$$\frac{\partial u}{\partial t} - \left(\alpha u^3 - \beta u^6\right)\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \tag{20}$$

$$u(x) = \left(\frac{14\alpha/5\beta}{1 + \frac{63\alpha^2}{100\beta}x^2}\right)^{1/3}. \tag{21}$$

Modular nonlinearities arise in a number of plasma problems; see for example the Schamel equation [39,40]. In this case, the Gardner generalization of the Schamel equation has the following form ($p = 2$):

$$\frac{\partial u}{\partial t} - \left(\alpha |u|^{1/2} - \beta u\right)\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \tag{22}$$

and its rational solution is written as

$$u(x) = \left(\frac{120\alpha}{75\beta + 4\alpha^2 x^2}\right)^2. \tag{23}$$

A comparison of the shapes of various rational solitons in dimensionless coordinates (u/A, Bx) is shown in Figure 1. As we can see, with an increase in the degree of nonlinear terms in the Gardner equation ($1/p$), the rational algebraic soliton broadens, and its tails decrease more slowly at infinity.

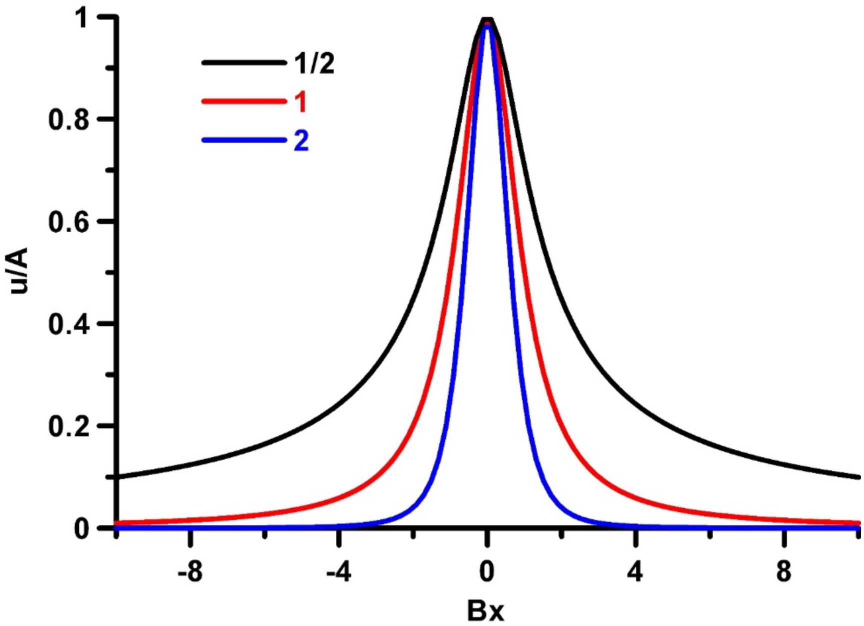

Figure 1. The shape of rational solitons (8) for various values of p.

4. Rational Solitons in the Korteweg-de Vries Equation with Added High-Order Nonlinearity—Numerical Results

Analytical rational solutions, as shown in Section 3, are found within the framework of the Gardner Equation (13) when the ratio between the powers of the nonlinear terms is equal to 2. Here we consider another form of the Gardner equation,

$$\frac{\partial u}{\partial t} - \left(u - \frac{(q+1)(q+2)}{6} u^q \right) \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \tag{24}$$

in which there is only one free parameter $q > 1$. The coefficient in front of the highest derivative is chosen so that the amplitude of the rational soliton is $A = 1$. In this case, the function $\Pi(u)$ in Formula (5) is equal to

$$\Pi(u) = -\frac{u^3 - u^{q+2}}{6}. \tag{25}$$

From the analysis that is given in Section 2, it follows that the power asymptotics of a rational soliton at infinity do not depend on q and are determined only by a quadratic nonlinearity. However, the waveform depends on q, since the function $\Pi(u)$ depends on this parameter. Qualitatively, the graphs of the function $\Pi(u)$ for different q are similar to each other (Figure 2); therefore, rational solitons are also similar to each other. They are found numerically by direct integration of Equation (7) with function (25), and their shape is shown in Figure 3.

As can be seen from Figure 3, with an increase in the degree q, the shape of the solitons becomes almost the same. The latter is obvious, since at large values of q the leading nonlinear term in Equation (24) becomes small at $u < 1$, and the slopes of the wave tend to the universal algebraic function that follows from (4) at $b = 2$,

$$u(x) \approx \frac{12}{x^2}, \tag{26}$$

and the influence of the nonlinear term with q appears only at the top of the soliton.

Qualitatively, the same solutions can be obtained in the case when the first nonlinear term in Equation (24) has any degree, not necessarily quadratic and integer.

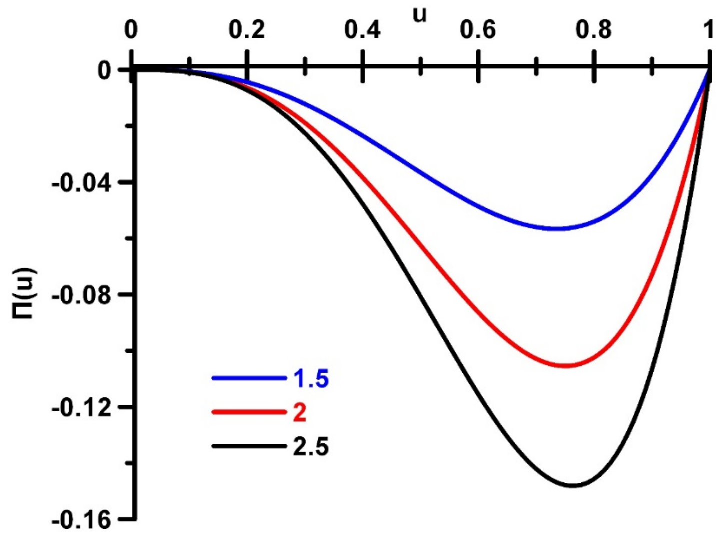

Figure 2. Graph of the function $\Pi(u)$ for various values of the parameter q.

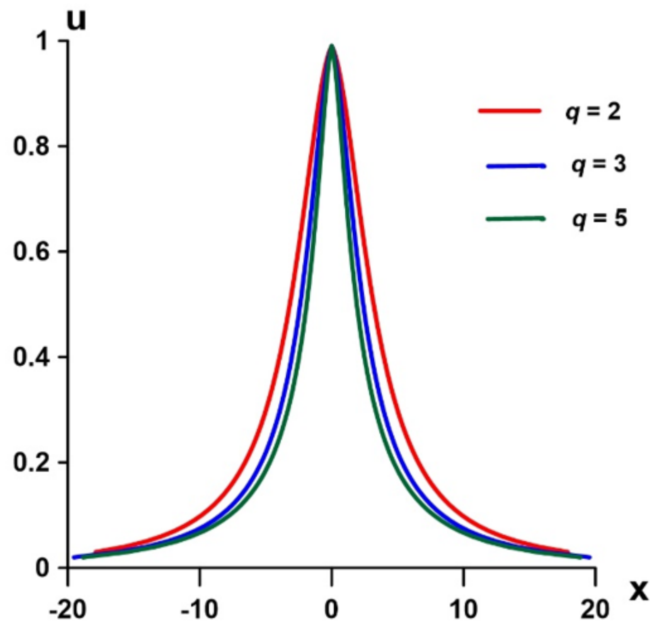

Figure 3. Shapes of rational solutions of Equation (24) for different powers q.

5. Kinks

In all the cases considered above, the form of a rational soliton was a one-parameter bell-shaped function, which is essentially due to the "one-humpness" of the function $\Pi(u)$, as in Figure 2. In all these cases, the curve $\Pi(u)$ approaches the point $A = 1$ (soliton amplitude) with a nonzero derivative. A different situation will be realized when the tangency at the point $A = 1$ will occur with a zero derivative. For analysis, it suffices to modify the function $\Pi(u)$ as follows:

$$\Pi(u) = -Wu^d(1-u)^r, \tag{27}$$

where coefficients $d \geq 3$ (in the lowest order, the nonlinearity is quadratic or higher order), $r > 1$, and W is a positive constant. In this case, $\frac{d\Pi}{du}(u=1) = 0$ (see Figure 4), and the function $u(x)$ will slowly change in the vicinity of the soliton top, as can be seen from Equation (7).

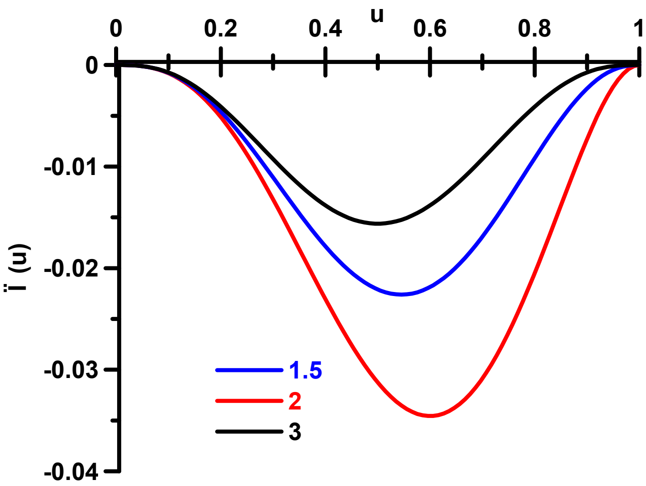

Figure 4. Graph of the function $\Pi(u)$ for various values of the parameter r (d = 3).

An analytical example arises in the case when

$$\Pi(u) = -\frac{1}{2}u^4(1-u)^2; \tag{28}$$

then the solution is written explicitly (Figure 5) as

$$\pm x = \ln\frac{u}{1-u} - \frac{1}{u}. \tag{29}$$

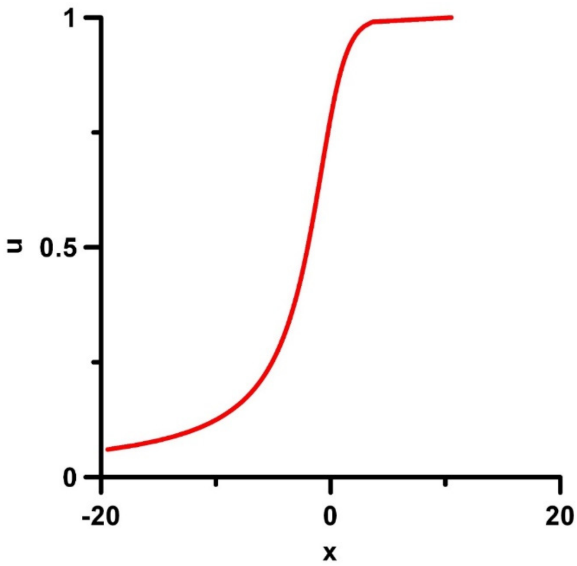

Figure 5. Rational kink (28).

It defines a kink—the jump between two constant values (with a plus sign, this jump is from zero to one). At the left end, the asymptotics of the kink is algebraic $u \sim x^{-1}$, and at the right end, it is exponential. We give here an explicit expression for the modified Gardner equation with a solution in the form of a kink (29):

$$\frac{\partial u}{\partial t} - 6u^2\left[1 - \frac{10}{3}u + \frac{5}{2}u^2\right]\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{30}$$

Thus, for the existence of a kink, there must be several nonlinear terms (at least three). Naturally, kinks qualitatively similar to (29) also arise in the case of a more complex nonlinearity, but only when the condition $\frac{d\Pi}{du}(u = 1) = 0$ is fulfilled at one of the ends.

6. Pyramidal Rational Solitons

A more complex form of a rational soliton occurs when the function $\Pi(u)$ has several (at least three) extremes within the interval, as shown in Figure 6. For analysis, we choose this function in the form

$$\Pi = Wu^3\left[(u-u_1)^2+\varepsilon^2\right](u-A), \tag{31}$$

assuming that $u_1 < A$ is an "intermediate" root, where the curve approaches the u axis in Figure 6 (here we chose $W = 1$, $A = 1$, and $u_1 = 0.5$). A small value of epsilons allows controlling the shape of a rational soliton. In this case, the solution in the region (0–0.5) will resemble the superposition of the kinks described above, and above it—an "ordinary" soliton, so that in general we get a pyramidal rational soliton. It is solved numerically and is shown in Figure 7 for different values of the small epsilon parameter.

If, moreover, u_1 tends to A, then the soliton top becomes flat, and its becomes a "table-top" soliton, and its width tends to be a constant value (Figure 8).

We present here the generalized Gardner equation, the solution of which are pyramidal and table-top solitons ($W = 1$):

$$\frac{\partial u}{\partial t}-\left[6(u_1^2+\varepsilon^2)u-12(u_1^2+\varepsilon^2+2Au_1)u^2+20(2u_1+A)u^3-30u^4\right]\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}=0. \tag{32}$$

That is, the nonlinearity here is described by a polynomial of the fifth degree. At higher degrees, more complex pyramidal structures may occur. If the polynomial has a lower degree, then there are no rational solutions in the form of pyramidal and table-top solitons.

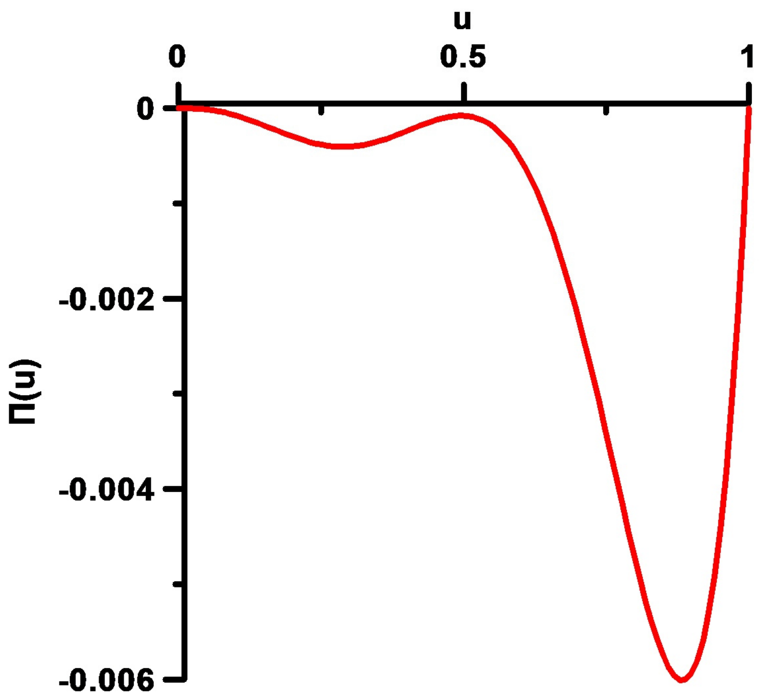

Figure 6. Graph of the function $\Pi(u)$ for $W = -1$, $A = 1$, $u_1 = 0.5$.

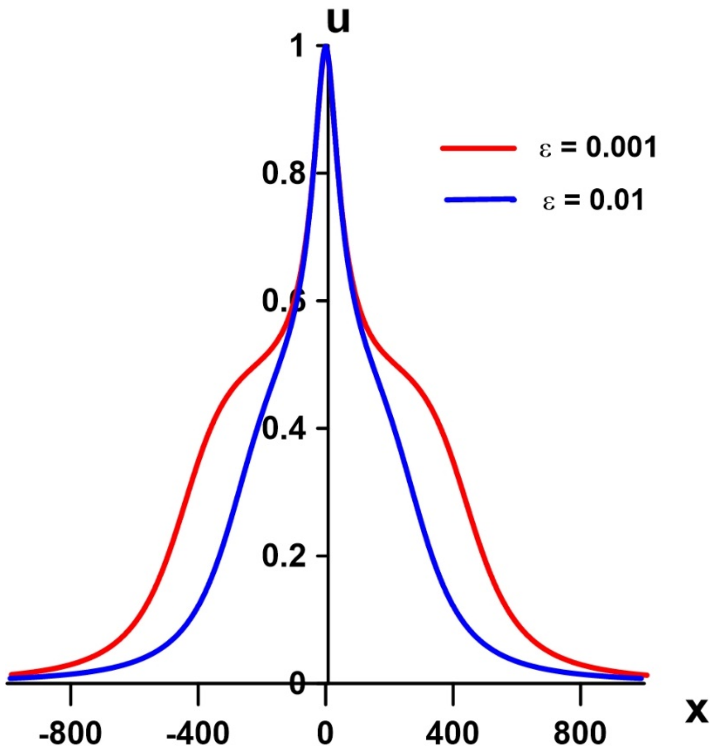

Figure 7. Shape of a pyramidal soliton for various epsilon values.

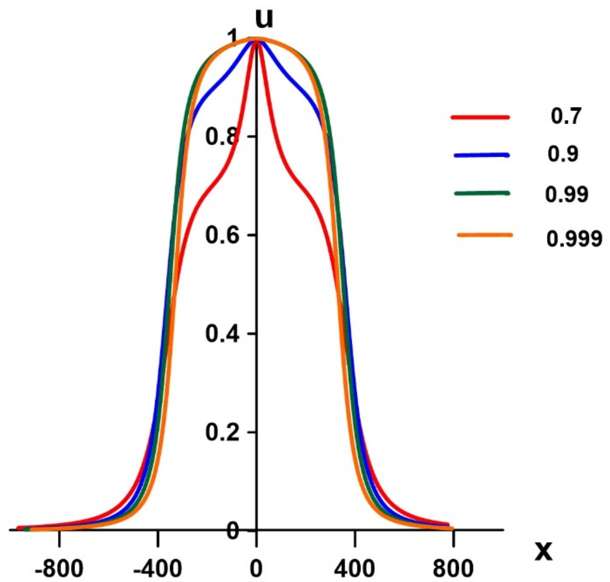

Figure 8. The shape of a pyramidal and table-top soliton for different values of parameter u_1/A.

7. Conclusions

In this paper, we study rational bounded solutions of the generalized Gardner equation that describe the so-called algebraic solitons, since their tails are described by power functions. Equations of this type are encountered in the theory of wave motions of stratified flows. An analytical family of rational solitons for the Gardner equation, in which there are two nonlinear terms, where the degrees differ by a factor of 2, is found. For an arbitrary ratio of the powers of these terms, rational solutions are obtained numerically. If the number of nonlinear terms is more than 2, then new types of rational solutions arise, which are kinks, pyramidal, or table-top solitons. Some solutions here are obtained analytically, and the others numerically.

Being a mathematical representation of large-amplitude waves, rational solutions are known to play an important role in the problem of rogue waves. They are studied mainly in the framework of integrable evolution equations. It is shown here that they can exist in the framework of non-integrable systems. Their stability with respect to external disturbances and noise fields is yet to be investigated.

The practical significance of the study is related to the problem of rogue waves in a fluid stratified by density and shear flow. In the ocean, internal waves can reach very large amplitudes up to 100 m, and they appear for a short time and unexpectedly. With a complex structure of stratification in the evolution equations, complex combinations of nonlinear terms arise, which can be modeled by the Gardner-type equation considered in this paper. The rational solutions found in our article are the prototype of rogue waves in a stratified fluid.

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