

Article

# Comparison between XY Spin Chains with Spin 1/2 or 1 Interacting with Quantized Electromagnetic Field by One and Two Photon Jaynes-Cummings Model

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**Abstract:** This paper describes two cases of interaction between a quantized electromagnetic field and two different XY spin molecules; one with spins  $\frac{1}{2}$ , and the other with spins 1. Both interact with a quantized electromagnetic field, with one of the spins in the chain interacting with the electromagnetic field. The interaction between the field mode and the spin chain with spins 1 is described by the one- and two-photon Jaynes-Cummings model (JC model). On the other hand, the interaction between the spins  $\frac{1}{2}$  and the electromagnetic field is described only by the one-photon Jaynes-Cummings model. Analytical and numerical calculations were made for the case of a different number of photons in the field mode, a different number of spins, and a different position of spin, interacting with the electromagnetic field. The invariant and block structures of such a chain are shown with a comparison made between the evolution of the magnetic moment and the number of photons in both cases.

**Keywords:** spin chains; spin 1; Jaynes-Cummings model; energy spectrum; evolution



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## 1. Introduction

The main task in quantum optics is the interaction between a quantized electromagnetic field and the energy levels of the atom or ion. Such an interaction is enshrined in the Jaynes-Cummings quantum model [1–3]. The Jaynes-Cummings (JC) model describes the dipole transition between two levels in a quantum system. The model describes the transition between electronic levels of an atom due to absorption or emission of an electromagnetic quantum. This model is used to describe the transition between states of electron in an ion situated in an ion trap [4–6]; the manipulation of the state of a quantum dot [7,8]. Other applications of this model are studying the squeezed states [9,10], the theoretical study of single-photon lasers [11], in single-photon photodetectors [12], etc. Areas of fundamental importance, where these studies play a major role, are atomic clocks [13,14], quantum gates [15,16] for quantum computers, laser theory [17,18], and many others.

The two-photon JC model with three-level atom has also been extensively studied [19]. The transitions between levels, from third to second and from second to first, are made by the one-photon JC model, with one photon emission. The two-photon JC model also includes two photon absorption/emission, and has been used for transferring the population between the first and third levels. The two-photon JC model was an important step in developing advanced methods for better population transfer, such as STIRAP (stimulated-Raman adiabatic passage) [20]. Nowadays, STIRAP has applications in many fields [21], such as the formation of ultracold molecules [21], the creation of gates for quantum information in different schemes [21], the control of nitrogen vacancy centers and semiconductor quantum dots [21], and the quantum optical analogues that are used in waveguide optics [21] and frequency conversion [21]. The two photon JC model has also been used to study the effects of the spin orbital coupling [22,23] of ultracold atoms located in the ring optical cavity. Such atoms play the role of pseudo-spins. Spin orbital coupling

in such a system can change the properties of the JC model, and also in the Dicke model. For example it can inhibit transition to the super radiant phase in some cases [22].

There are some models that describe the interaction between spins in a chain. Examples of such models are the Ising [24], Heisenberg [25], XY [26], etc. Each of these models is used to study different processes, in different types of system. In the XY model, the spins interact with two of their components. Examples of areas where the XY model has been used are spin glasses [27] and their phase transitions, entanglement [26], neural networks [28], and many others. Many publications in this field have only described interactions between closest neighbors in the spin chain.

A system of spin-one chains is of theoretical interest. Such a system is used to study phase transitions [29], entanglement [30], critical behaviors [31], and spin chain dynamics [32,33]. A frustration-free system has a strong entanglement in its ground state [34]. A frustrated system, however [35], or a system with anisotropy [31], can have critical behavior and entanglement that has been intensively studied.

There are models describing the interaction between a spin chain and an electromagnetic field, for instance, the Dicke model [36–38]. It describes the interaction of the spin chain as a whole with a quantized electromagnetic field in an ion trap. Dicke's chain-to light interaction model plays a major role in the study of super-radiation phase transition [39–41], the squeezing of spin states [36], and quantum entanglement [36,40].

Another model describing the interaction between a spin chain and a quantum electromagnetic field, which is being worked on by Ref. [42] and also at the "Institute of Solid-State Physics" part of "Bulgarian Academy of Science", is shown in Ref. [43]. This model has been used to study the interaction between coupled resonators, each of which has one spin [44]. The present work is a continuation of the work of the author in this field.

The study of XY spin chains in which each spin has spin 1 interacting with a quantized electromagnetic field is not only interesting from a fundamental point of view [45], but can also be used to study super radiance lasers [45,46], and has applications in some quantum computer realizations [45]. The XY model is already used to build W-entangled state [47] and quantum gates [48]. Spin 1 particles can be used as qutrits [49] (quantum analogue of classical trit). Using qutrits will allow more information to be stored and processed by a smaller number of particles, and qutrits also allow better performance of some quantum algorithms, such as the Grover search algorithm [50] and the Quantum counting algorithm [51]. We hope this study will help develop an ion trap [52], a quantum dot [53], and cavity [53] based quantum computers.

This paper compares a spin 1/2 chain with another consisting of spin 1. The XY model gives the spin–spin interaction in both chains. The photons are absorbed/emitted by a single spin in the chain. The spin photon interaction is described by the one- and two-photon Jaynes–Cummings model. This paper is organized as follows: The first subsection of the next chapter, Section 2.1.1, briefly discusses the spin 1/2 cases of a spin chain interacting with a quantized electromagnetic field. The field interacts with one of the spins in the chain, which is described by the one photon Jaynes–Cummings model. The Hamiltonian, the invariant (operator related to the number of excited states in the system), the block-matrix structure of the Hamiltonian matrix, the size of the blocks, and their matrix elements are considered. In the second subpart, Section 2.1.2, numerical simulations of the evolution of the number of photons and the magnetic moment of the spin chain are shown. The first two subparts of Part Three, Sections 2.2.1 and 2.2.2, show the same characteristics (Hamiltonian, its block structure, matrix elements, and invariant) for a spin 1 chain interacting with an electromagnetic field. A single spin in a chain can absorb one or two photons through the one- and two-photon model of Jaynes–Cummings. The next two chapters, Sections 2.2.3 and 2.2.4, show the numerical simulations of the energy spectrum, and the evolution of the number of photons and the magnetic moment of the spin chain. The last section, Section 3, is the conclusion, comparing the two spin chain cases.

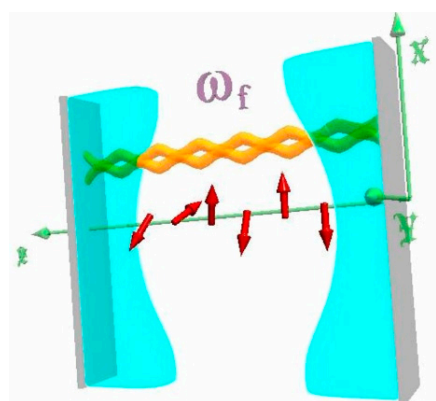
## 2. Results

### 2.1. Spin Chain with Spin 1/2 Interacting with the Electromagnetic Field

The system under consideration here is a linear spin chain consisting of  $N$  spins, located in the cavity. A laser field is applied to the cavity. The laser field is focused on the  $k$ -th spin. The cavity leads to quantization of the field, so let there be one mode with frequency  $c_2$ . The electromagnetic field interacts with the  $z$ -th component of the spin. The difference between the two levels of the  $k$ -th spin is  $c_1$ . An external magnetic field is applied to the cavity coinciding with the  $Z$ -detection of the spins. The magnitude of the external magnetic field is  $c_0$ . All spins have spin  $\frac{1}{2}$ , and interact with their nearest neighbors through the  $X$  and  $Y$  components of their spin (described by the  $XY$  model).

#### 2.1.1. Hamiltonian, Invariant, and Block Structure of the Hamiltonian

Let us first consider the case of a linear chain consisting of spins  $1/2$  interacting with a single mode of quantized electromagnetic field. The spin chain is open, and there is only first neighbor interaction. The interaction between the spins is in the  $XY$  model. This case has been studied in detail in the articles [42,43], and the system is shown in Figure 1.



**Figure 1.** Particle with spin  $\frac{1}{2}$  in  $XY$  spin chain interacting with a quantized electromagnetic field in a cavity.

The Hamiltonian of this system is:

$$\hat{H} = -G_1(\hat{a}^\dagger \hat{S}_k^- - \hat{a} \hat{S}_k^+) - J \sum_{i=1}^{N-1} (\hat{S}_i^+ \hat{S}_{i+1}^- - \hat{S}_i^- \hat{S}_{i+1}^+) + c_2 \hat{a}^\dagger \hat{a} + c_1 \hat{S}_k^z + c_0 \sum_{i=1, i \neq k}^N \hat{S}_i^z. \quad (1)$$

With  $N$  denoting the number of spins in the chain.  $G_1$  is the interaction constant between spin and field mode. Exchange coupling constant of spin–spin interaction between the nearest neighboring spins (at position  $i$  and  $i + 1$ ) is  $J$ . Both constants  $G_1$  and  $J$  can be positive or negative. The Hamiltonian can be block diagonalized using the invariant, as shown in [42,43].

The operators that change the number of photons in the mod of the field are creation operators of the mode  $\hat{a}^\dagger$  and annihilation operators of the mode  $\hat{a}$ :

$$\hat{a}|n, s_1 \dots s_N\rangle = \sqrt{n}|n-1, s_1 \dots s_N\rangle \quad \hat{a}^\dagger|n, s_1 \dots s_N\rangle = \sqrt{n+1}|n+1, s_1 \dots s_N\rangle. \quad (2)$$

In wave function  $|n, s_1 \dots s_N\rangle$ , first is written the number of photons in the field (denoted with  $n$ ), second are written the states of spins in the chain,  $s_i$  (can be  $\uparrow$  for spin up and  $\downarrow$  for spin down).

The state of the  $k$ -th spin (denoted by  $|n, s_1 \dots s_k \dots s_N \equiv |s\rangle_k$ ) is changed by the following operators:

$$\hat{S}_k^- |\uparrow\rangle_k = |\downarrow\rangle_k, \quad \hat{S}_k^+ |\downarrow\rangle_k = |\uparrow\rangle_k, \quad (3)$$

$$\hat{S}_k^\pm = (\hat{S}_k^x \mp i\hat{S}_k^y). \quad (4)$$

For each position in the chain  $i \in [1, N - 1]$ , the operators  $\hat{S}_i^+ \hat{S}_{i+1}^-$  and  $\hat{S}_i^- \hat{S}_{i+1}^+$  turn down the state of one spin, and up the state of the other spin. The total number of spins oriented upwards in the system remains constant. The operators  $\hat{a}^\dagger \hat{S}_k^-$  and  $\hat{a} \hat{S}_k^+$  turn the spin state down, and one further photon is emitted, or the number of photons is reduced when one spin is turned up.  $I$  is an imaginary unit.

The parts of the Hamiltonian:

$$c_2 \hat{a}^\dagger \hat{a} + c_1 \hat{S}_k^z + c_0 \sum_{i=1, i \neq k}^N \hat{S}_i^z \tag{5}$$

do not change the state of the spin or the number of photons.

The invariant is defined by  $[\hat{H}, I\hat{nv}] = 0$ . Then the invariant in case of spin 1/2 of the system is  $I\hat{nv}_{1/2}$  [54]:

$$I\hat{nv}_{1/2} = \frac{1}{2} \sum_{i=1}^N \hat{S}_i^z + \hat{n}, \tag{6}$$

where  $\hat{S}_i^z$  is the Pauli matrix (of the  $i$ -th spin) in the Z direction, and  $\hat{n} = \hat{a}^\dagger \hat{a}$  is the operator for number of photons in the field's mode.

When the operator,  $I\hat{nv}_{1/2}$ , is applied to one of the Hamiltonian eigenstates,  $|\psi(t_i)\rangle$ , the corresponding eigenvalue,  $\mu_k$ , is whole- or half-integer. The eigenvalue,  $\mu_k$ , makes sense of the total number of excitations of the state [42].

$$\mu_i = \langle \psi(t_i) | I\hat{nv}_{1/2} | \psi(t_i) \rangle, \tag{7}$$

where  $|\psi(t_i)\rangle$  are the states of the system at time  $t_i$ .

The application of Hamiltonian to each state results in another state with the same invariant value,  $I\hat{nv}_{1/2}$ . By reordering the basis vectors of the matrix, a block diagonal matrix can be obtained with each block corresponding to a different value of the invariant

$$\hat{H} = \begin{pmatrix} \hat{H}(\mu_1) & 0 & 0 & 0 & \dots \\ 0 & \hat{H}(\mu_2) & 0 & 0 & \dots \\ 0 & 0 & \hat{H}(\mu_3) & 0 & \dots \\ 0 & 0 & 0 & \hat{H}(\mu_4) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{8}$$

where  $\hat{H}(\mu_w)$  are block matrices and each block corresponds to a different value of the invariant  $\mu_w$ . Index  $w$  is the integer and corresponds to the block number. The block size for different invariant values is:

$$Dim(\hat{H}(\mu_w)) = \sum_{k=0}^{\mu_w} \frac{N!}{(N-k)!k!}. \tag{9}$$

The maximum value of  $Dim(\hat{H}(\mu_w))$  is obtained when  $k = N$ . Then, as the value of  $w$  increases, the block size remains constant. The matrix elements of these block matrices can be calculated as follows:

$$\langle n_1, s_1 s_2 \dots s_i \dots s_N | \hat{H} | n_2, z_1 z_2 \dots z_j \dots z_N \rangle = -G\sqrt{n_2} \delta_{n_1, n_2+1} \delta_{z_k, \uparrow} - G\sqrt{n_2+1} \delta_{n_1, n_2-1} \delta_{z_k, \downarrow} + \left( \sum_{i=1}^N (-1)^{\delta_{s_i, \downarrow}} c_0 + (-1)^{\delta_{s_k, \downarrow}} c_1 + n_2 c_2 \right) \delta_{n_1, n_2} - 2 \sum_{i=1}^N \sum_{j=1}^N J_{i,j} \left( \delta_{s_i, \uparrow} \delta_{z_j, \downarrow} + \delta_{s_i, \downarrow} \delta_{z_i, \uparrow} \right), \tag{10}$$

$$\langle n_1, s_1 s_2 \dots s_i \dots s_N | \hat{H}(\mu_w) | n_2, z_1 z_2 \dots z_j \dots z_N \rangle = \langle n_1, s_1 s_2 \dots s_i \dots s_N | \hat{H} | n_2, z_1 z_2 \dots z_j \dots z_N \rangle \delta_{\mu_w, N_{\uparrow, w} + m_w}, \tag{11}$$

where  $|n_1, s_1 s_2 \dots s_i \dots s_N\rangle$  and  $|n_2, z_1 z_2 \dots z_j \dots z_N\rangle$  are two states of the system. The states have  $n_1$  and  $n_2$  photons, respectively. For vectors  $s_i$  and  $z_j$ , the states of the  $i$ -th spin of the system can be  $\uparrow$  or  $\downarrow$ .

### 2.1.2. Evolution of Magnetic Moment and Number of Photons

One way to calculate the evolution is by using a full Hamiltonian to obtain an evolution operator,  $\hat{E}$ . The task can be simplified by taking into account the fact that during the state evolution, the total number of excitations remains constant. Therefore, the evolution of each state can be found using the block matrix with the same number of excitations as the initial state:

$$\hat{E} = \exp(-i\hat{H}(t_1 - t_0)/\hbar), \quad (12)$$

$$\hat{E}(\mu_x) = \exp(-i\hat{H}(\mu_x)(t_1 - t_0)/\hbar). \quad (13)$$

Diagonalizing the Hamiltonian makes it easy to find the operator of evolution. The diagonal elements of the Hamiltonian correspond to the energy spectrum. Diagonalization also helps to find the eigenstates of the system, and shows degeneracy.

Using the evolution operator, the state of the system can be calculated from the formula:

$$|\psi(t_1)\rangle = \hat{E}(\mu_x)|\psi(t_0)\rangle, \quad (14)$$

where  $|\psi(t_i)\rangle$  is the current state of the system in the time  $t_i$ .

$$\hat{M}^z = \sum_{i=1}^N \hat{S}_i^z, \quad (15)$$

$$Mz(t_i) = \langle \psi(t_i) | \hat{M}^z | \psi(t_i) \rangle, \quad (16)$$

$$\hat{n} = \hat{a}^\dagger \hat{a}, \quad (17)$$

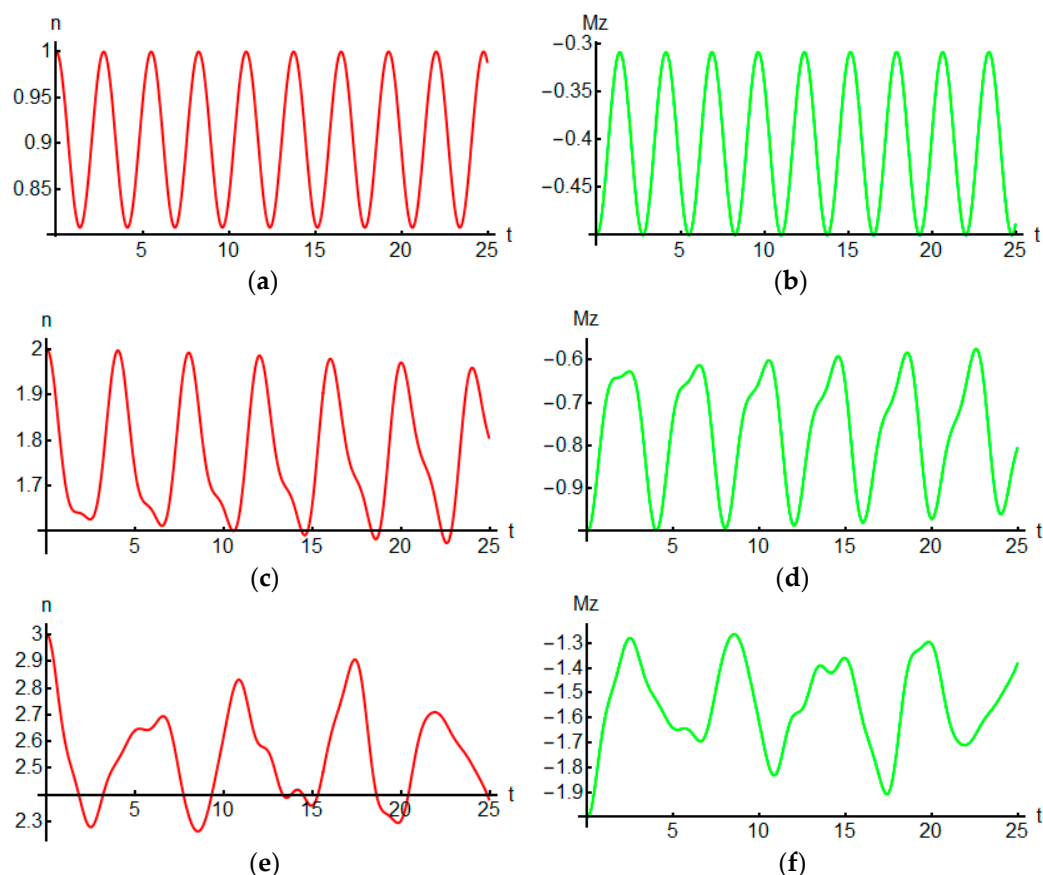
$$n(t_i) = \langle \psi(t_i) | \hat{n} | \psi(t_i) \rangle, \quad (18)$$

where  $\hat{M}_z$  is the magnetic moment operator in the Z direction.  $Mz(t_i)$  is the magnetic moment of the chain in the Z direction, when the system is in state  $|\psi(t_i)\rangle$ . The number of photons in the mode of the field, in the state of the system  $|\psi(t_i)\rangle$  is  $n(t_i)$ .

Figure 2 shows a numerical simulation of the evolution of the magnetic moment and the number of photons of the spin chain interacting with a quantized electromagnetic field. The spin chain consists of 1 spin in Figure 2a,b, 2 spins in Figure 2c,d and 4 spins in Figure 2e,f. In the case of four spins, the second spin in the chain interacts with the field. The values for the number of excitations in the case of the above figures for one, two, and four spins are equal to one, two, and three. The initial states of spins and mode in the Figure 2 are the following: All excitations are in the mode of the field, so all spins are in state  $|\downarrow\rangle$ . The parameters used for this numerical simulation are:

$$\begin{aligned} c_0 &= 1.957 \times 10^{-2}, c_1 = 2.674 \times 10^{-2}, \\ c_2 &= 2, J = 8.66 \times 10^{-1}, \\ G_1 &= 0.5. \end{aligned} \quad (19)$$

The Hamiltonian connects all states that have the same number of excitations. In the case of more than one spin, there is an interaction between spins. The spin flipping part of the Hamiltonian  $J \sum_{i=1}^{N-1} (\hat{S}_i^+ \hat{S}_{i+1}^- - \hat{S}_i^- \hat{S}_{i+1}^+)$  by itself does not change the number of photons in the mode (or magnetization), but when the spin chain consists of more than one spin, the excitations can pass between spins in the chain. It can be seen that increasing the number of spins increases the number of evolution-connected states. The interferences that occur lead to a difference in the evolution of the number of photons in the mode for more spins than the evolution of the number of photons in the mode in the case of only one spin. During evolution, the system can be in superposition of the states. The magnitude of the spin-spin interaction determines how much each spin in the chain contributes to magnetization, depending on the distance of the spin interacting with the field. These are the reasons why magnetization has continuous spectrum.



**Figure 2.** Evolution of magnetic moment amplitude (a,c) and the number of photons (b,d) for the spin chain interacting with the quantized electromagnetic field. The spin chain consists of 1 spin and the initial state is  $|1, \downarrow\rangle$  in figures (a,b), of 2 spins and the initial state is  $|1, \downarrow\downarrow\rangle$  in figures (c,d), and 4 spins and the initial state is  $|3, \downarrow\downarrow\downarrow\rangle$  (e,f).

In the case of one spin, the Hamiltonian connects the following states,  $|n, \uparrow\rangle$  and  $|n + 1, \downarrow\rangle$ . There is only one excitation that can be exchanged between the spin “chain” and the mode. The invariant value is 0.5. This is the typical Rabi oscillation, well known from quantum optics.

Figure 2c,d shows two spins and two excitations. In this case the invariant value is 1. The eigenvectors connected during evolution are  $|2, \downarrow\downarrow\rangle$ ,  $|1, \downarrow\uparrow\rangle$ ,  $|1, \uparrow\downarrow\rangle$  and  $|0, \uparrow\uparrow\rangle$ . The states with the same number of photons  $|1, \downarrow\uparrow\rangle$  and  $|1, \uparrow\downarrow\rangle$  have the same value of  $Mz$  equal to zero. The states  $|0, \uparrow\uparrow\rangle$  and  $|2, \downarrow\downarrow\rangle$  have magnetization values 1 and  $-1$ , respectively.

Figure 2e,f shows the evolution in the case of four spins, the chain has three excitations. In this case the invariant value is equal to 1. Examples of such states are  $|3, \downarrow\downarrow\downarrow\rangle$ ,  $|2, \downarrow\downarrow\downarrow\uparrow\rangle$ ,  $|2, \downarrow\downarrow\uparrow\downarrow\rangle$ ,  $|2, \downarrow\uparrow\downarrow\downarrow\rangle$ ,  $|1, \uparrow\downarrow\uparrow\downarrow\rangle$ ,  $|0, \uparrow\downarrow\uparrow\uparrow\rangle$ , and nine others. The states with an equal number of photons have the same value of  $Mz$ . For example  $|2, \uparrow\downarrow\downarrow\rangle$ ,  $|2, \downarrow\uparrow\downarrow\rangle$ ,  $|2, \downarrow\downarrow\uparrow\downarrow\rangle$ ,  $|2, \downarrow\downarrow\downarrow\uparrow\rangle$  have the same value of  $Mz$ , equal to  $-1$ . Other states with same value of invariant have different  $Mz$ , for example  $|1, \uparrow\downarrow\uparrow\downarrow\rangle$ ,  $|1, \downarrow\uparrow\uparrow\downarrow\rangle$ , and  $|1, \downarrow\downarrow\uparrow\uparrow\rangle$  have a value of  $Mz$  equal to zero.

## 2.2. Spin Chain with Spin 1 Interacting with the Electromagnetic Field

The system of our study is a linear spin chain consisting of  $N$  spins, all of them are with spin 1. All spins are located in a cavity, and the laser field is focused on the  $k$ -th spin, when the field interacts with the  $z$ -th component of the spin. The electromagnetic field is quantized in the cavity, and has a frequency of  $c_2$ . An external magnetic field is applied to the cavity, coinciding with the  $Z$  detection of the spins. The magnitude of the external magnetic field is  $c_0$ . The  $k$ -th spin between neighboring levels has the same transition



frequency,  $c_1$ . Spins interact with their closest neighbors through the X and Y components of their spin.

### 2.2.1. Hamiltonian

A spin chain consisting of spin 1 particles interacts with one or two photons, and one of the spins interacts with an electromagnetic field mode. The Hamiltonian of such a system is:

$$\hat{H} = \hat{H}_{JC2} + \hat{H}_{XY} + \hat{H}_{EF}, \quad (20)$$

where  $\hat{H}_{XY}$  is the Hamiltonian of spin–spin interaction through the XY model. The spin chain is open, and each spin only interacts with its first neighbors.

The Hamiltonian of the two-photon Jaynes–Cummings model,  $\hat{H}_{JC2}$ , in the system Hamiltonian,  $\hat{H}$ , describes the interaction between one spin in the chain and two photons in one mode from a quantized electromagnetic field:

$$\hat{H}_{JC2} = \hat{H}_{1P} + \hat{H}_{2P} + c_2 \hat{a}^\dagger \hat{a} + c_1 \hat{Q}_k^z, \quad (21)$$

$$\hat{H}_{1P} = -G_1 \left( \hat{a}^\dagger \hat{\zeta}_k^- - \hat{a} \hat{\zeta}_k^+ \right), \quad (22)$$

$$\hat{H}_{2P} = -G_2 \left( \hat{a}^\dagger \hat{a}^\dagger \hat{\zeta}_k^- \hat{\zeta}_k^- - \hat{a} \hat{a} \hat{\zeta}_k^+ \hat{\zeta}_k^+ \right). \quad (23)$$

Here,  $\hat{H}_{1P}$  and  $\hat{H}_{2P}$  correspond to interaction between the  $k$ -th spin and the field mode, while  $\hat{H}_{1P}$  corresponds to the one-photon absorption, and  $\hat{H}_{2P}$  corresponds to the two-photon absorption effect. In this case, the spin has to absorb two photons from the mode to change its state to two levels higher, and accordingly to emit the corresponding number of photons to go to a lower level.  $G_1$  and  $G_2$  are the interaction constants between the  $k$ -th spin and field mode. Both constants,  $G_1$  and  $G_2$ , can be positive or negative. The constant  $c_1$  is the energy difference between the two states of the  $k$ -th spin, and  $c_2$  is the energy carried from the photons in the mode of the field.  $\hat{a}^\dagger$  and  $\hat{a}$  are creation and annihilation operators of one photon in the mode, correspondingly, and the creation and annihilation operators for  $k$ -th spin are  $\hat{\zeta}_k^+$  and  $\hat{\zeta}_k^-$ . The Pauli matrix analogue,  $\hat{\zeta}_k^z$ , corresponds with the spin Z component. Concerning the Z axis, the other two axes, X and Y, are equivalent. From a mathematical point of view, operators  $\hat{\zeta}_k^+$  and  $\hat{\zeta}_k^-$  are a linear combination of the Pauli matrix analogues:

$$\hat{\zeta}_k^\pm = \left( \hat{\zeta}_k^x \mp i \hat{\zeta}_k^y \right). \quad (24)$$

A Hamiltonian part describing interaction between spins, according to the XY model, is  $\hat{H}_{XY}$ . Another Hamiltonian part describes an external magnetic field,  $\hat{H}_{EF}$ .

$$\hat{H}_{XY} = -J \sum_{i=1}^{N-1} \left( \hat{\zeta}_i^+ \hat{\zeta}_{i+1}^- - \hat{\zeta}_i^- \hat{\zeta}_{i+1}^+ \right), \quad (25)$$

$$\hat{H}_{EF} = c_0 \sum_{i=1; i \neq k}^N \hat{\zeta}_i^z, \quad (26)$$

where  $k$  is the position of the spin interacting with the quantized electromagnetic field; constant  $c_0$  corresponds to the external magnetic field, and  $\hat{\zeta}_i^z$  is the Pauli matrix analogue that corresponds to the third component of the spin. Here,  $J$  is the exchange coupling constant of the spin-spin interaction between the  $i$ -th and  $(i + 1)$ -th spin. Constant  $J$  can be positive or negative. The spin chain consists of  $N$  spins. Illustrative example for such a system is shown in Figure 1.

### 2.2.2. Invariant and Block Structure

An important feature of the model describing the chain of particles with spin 1 interacting with the quantized electromagnetic field is the invariant. By definition, the invariant is:

$$\left[ \hat{H}, \hat{I} \hat{n} v_1 \right] = 0 \quad (27)$$

and:

$$\hat{I} \hat{n} v_1 |\psi(t_i)\rangle = \mu_i |\psi(t_i)\rangle. \quad (28)$$

The Hamiltonian (Equation (21)) retains the total number of excitations during evolution, and has the following invariant:

$$\hat{I} \hat{n} v_1 = - \sum_{i=1}^N \hat{\zeta}_i^Z + \hat{n} \quad (29)$$

where  $|\psi(t_i)\rangle$  are the Hamiltonian eigenvectors,  $\hat{M}^z = - \sum_{i=1}^N \hat{\zeta}_i^Z$  is the magnetic moment in the Z direction, and  $\hat{n} = \hat{a}^\dagger \hat{a}$  is number of photons in the mode of the field. When  $\hat{I} \hat{n} v_1$  is applied to one of the Hamiltonian eigenstates, the corresponding eigenvalue,  $\mu_i$ , is an integer. The operators,  $\hat{\zeta}_i^Z$ , are the Z-th Pauli matrix analogue for spin 1.

In this form the invariant does not match the number of excitations in the system. The number of excitations can be found by applying the following operator to the state:

$$\hat{E} x_1 = - \sum_{i=1}^N (\hat{\zeta}_i^Z + \hat{\zeta}_i^0) + \hat{n}, \quad (30)$$

$\hat{E} x_1$  gives the number of excitations of the system, and is also an invariant of the system.

The analogs of the Pauli matrix for spin 1 are:

$$\begin{aligned} \hat{\zeta}_i^0 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \hat{\zeta}_i^x &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \\ \hat{\zeta}_i^y &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & \hat{\zeta}_i^z &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \\ \hat{\zeta}_i^- &= \frac{1}{\sqrt{2}} (\hat{\zeta}_i^x + i \hat{\zeta}_i^y) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \\ \hat{\zeta}_i^+ &= \frac{1}{\sqrt{2}} (\hat{\zeta}_i^x - i \hat{\zeta}_i^y) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \end{aligned} \quad (31)$$

Here, the spin projection in direction Z is considered. The eigenvalues of the Pauli matrix for spin 1 are  $-1$ ,  $0$ , and  $1$ . The states will be denoted as  $\downarrow$ ,  $\rightarrow$ , and  $\uparrow$ , respectively.

By using the invariant, the Hamiltonian can be block diagonalized by analogy with the 1/2 spin systems. The maximal size of the block is  $N^2$ . After reaching this maximum amount, the block matrix size remains constant when increasing the number of photons in the mode. The formula for the block matrix size of a spin chain, with spin 1 interacting with electromagnetic field, can be obtained analytically. The expression for the block matrix size is the following:

$$\begin{aligned} \text{Dim}(\hat{H}(\mu_w)) &= \sum_{j_0=0}^{\mu_w} \sum_{j_2=0}^{\lfloor (\mu_w - j_0)/2 \rfloor} h_{j_0 j_2 \mu_w N}, \\ h_{j_0 j_2 \mu_w N} &= \binom{N}{j_2} \binom{|N - j_2|}{|\mu_w - j_0 - 2j_2|} & \mu_w - j_0 - j_2 > N \\ &= 0 & \mu_w - j_0 - j_2 > N \end{aligned} \quad (32)$$



In this formula  $n$  is the number of the spins,  $\mu_w$  is the number of the excitations, and  $m_1 = |\mu_w - j_0 - 2j_2|$  is the number of the spins in the state zero (one excitation). The number of spins with zero excitations,  $m_0$  (spin state is  $-1$ ), and the number of spins with two excitations is  $m_2$  (spin state is  $1$ ). The brackets  $\lfloor$  and  $\rfloor$  round the number in them down. The condition for the number of spins is obtained by  $m_1 + 2m_2 \leq N$ .

The matrix elements of the Hamiltonian can be found as follows:

$$\begin{aligned} \langle n_1, s_1 s_2 \dots s_i \dots s_N | \hat{H} | n_2, z_1 z_2 \dots z_i \dots z_N \rangle = & \left( c_0 \sum_{i=1; i \neq k}^N (\delta_{z_i, 2} - \delta_{z_i, 0}) + c_1 (\delta_{z_k, 2} - \delta_{z_k, 0}) + c_2 \eta \right) \delta_{n_1, \eta} \prod_{i=1}^N \delta_{s_i, z_i} + \\ & J_{ij} \delta_{n_1, \eta} \left( (\delta_{z_i, 0} + \delta_{z_i, 1}) (\delta_{z_j, -1} + \delta_{z_j, 0}) \delta_{s_i, z_{i-1}} \delta_{s_j, z_{j+1}} + (\delta_{z_j, -1} + \delta_{z_j, 0}) (\delta_{z_i, 0} + \delta_{z_i, 1}) \delta_{s_i, z_{i+1}} \delta_{s_j, z_{j-1}} \right) \prod_{t=1; t \neq i, j}^N \delta_{s_t, z_t} + \\ & G_1 \left( (\delta_{z_k, 0} + \delta_{z_k, 1}) \sqrt{\eta + 1} \delta_{n_1, \eta+1} \delta_{s_k, z_{k-1}} + (\delta_{z_k, -1} + \delta_{z_k, 0}) \sqrt{\eta} \delta_{n_1, \eta-1} \delta_{s_k, z_{k+1}} \right) \prod_{t=1; t \neq k}^N \delta_{s_t, z_t} + \\ & G_2 (\delta_{z_k, -1} \delta_{z_k, 1} \sqrt{\eta} \sqrt{\eta - 1} \delta_{n_1, \eta-2} + \delta_{z_k, 1} \delta_{s_k, -1} \sqrt{\eta + 1} \sqrt{\eta + 2} \delta_{n_1, \eta+1}) \prod_{t=1; t \neq k}^N \delta_{s_t, z_t}. \end{aligned} \tag{33}$$

Similarly, to the spin  $\frac{1}{2}$  case, block matrices have the same size and the corresponding positions of the matrix elements, but the elements can have different values, no matter which spin interacts with the photon.

The Equations (35)–(37) describe examples of block matrices of the open two-spin chain with spin 1, where one of the spins interacts with the quantized electromagnetic field. The interaction is only between the closest neighbors  $J_{i, i+1} = J$  and  $J_{i, j \neq i+1} = 0$ . The number of excitations is 1, 2, 3, and 4, with blocks sizes 1, 3, and 6, respectively:

$$\hat{H}_{2S}(\mu = 1) = c_0 + c_1, \tag{34}$$

$$\hat{H}_{2S}(\mu = 2) = \begin{pmatrix} c_1 & J & 0 \\ J & c_0 & G_1 \\ 0 & G_1 & c_0 + c_1 + c_2 \end{pmatrix}, \tag{35}$$

$$\hat{H}_{2S}(\mu = 3) = \begin{pmatrix} c_1 & J & 0 & 0 & 0 & 0 \\ J & 0 & J & G_1 & 0 & 0 \\ 0 & J & c_0 - c_1 & 0 & G_1 & \sqrt{2}G_2 \\ 0 & G_1 & 0 & c_1 + c_2 & J & 0 \\ 0 & 0 & G_1 & J & c_0 + c_2 & \sqrt{2}G_1 \\ 0 & 0 & \sqrt{2}G_2 & 0 & \sqrt{2}G_1 & c_0 + c_1 + 2c_2 \end{pmatrix}. \tag{36}$$

The block matrices reach their maximum size when the invariant value is equal to  $2N$ . The maximum size can be calculated using the number of spin states (in this case equal to three) and the number of spins in the chain,  $3^N$ .

In the same way, as is shown in Section 2.1.2., evolution can be calculated using the full Hamiltonian or using only its block corresponding to the relevant number of excitations and equal to that of the initial state of the system:

$$\hat{E} = \exp(-\hat{H}(t_1 - t_0)/\hbar), \tag{37}$$

$$\hat{E} = \exp(-\hat{H}(t_1 - t_0)/\hbar). \tag{38}$$

System states can be calculated using the operator of an evolution formula:

$$|\psi(t_1)\rangle = \hat{E}(\mu_x) |\psi(t_0)\rangle, \tag{39}$$

where  $|\psi(t_i)\rangle$  are the system states at time  $t_i$ ;  $\hat{M}_z(t_i)$  is the magnetic moment the operator in the Z direction, and the same moment in time  $t_i$  and  $\hat{n}$  is the number of photons in the field mode.

Each state of the system has its magnetic moment in the Z direction,  $Mz(t_i)$ , and number of photons,  $n(t_i)$ . They can be computed using equations:

$$\hat{M}_z = \sum_{i=1}^N \hat{q}_i^Z, \tag{40}$$

$$Mz(t_i) = \langle \psi(t_i) | \hat{M}_z | \psi(t_i) \rangle, \quad (41)$$

$$\hat{n} = \hat{a}^\dagger \hat{a}, \quad (42)$$

$$n(t_i) = \langle \psi(t_i) | \hat{n} | \psi(t_i) \rangle. \quad (43)$$

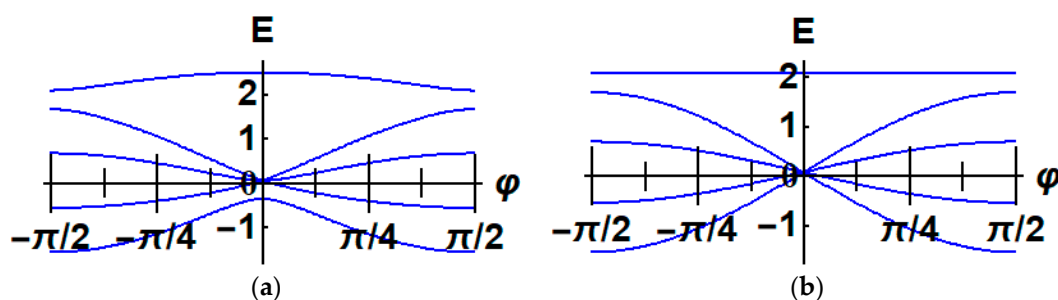
### 2.2.3. Numerical Simulation for Energy Spectrum

The eigenvalues of the Hamiltonian can be used for facilitated computation of the evolution operator. They also show connections between eigenvalues with changing parameters. The following parameterizations,  $G_1 = \cos(\varphi)$ ,  $G_2 = \cos^2(\varphi)$ , and  $J_{i,i+1} = \sin(\varphi)$ , can be used. This description of these constants is general, since arbitrary values of  $G$  and  $J$  can be written as  $G = A \cos(\varphi)$  and  $J = A \sin(\varphi)$ . Here the amplitude is  $A = \sqrt{G^2 + J^2}$ .

The Hamiltonian in the case of single-photon absorption can be obtained from Equations (19)–(26), with  $G_1 \neq 0$  and  $G_2 = 0$ . Similarly, in the case of two-photon absorption, they can be obtained from the same equations, when  $G_1 = 0$  and  $G_2 \neq 0$ . In both cases the invariant is the same and is equal to the one, where  $G_1 \neq 0$  and  $G_2 \neq 0$ .

Important limiting cases are when there is only one spin or when there is no spin–spin interaction between the spins. This case can be reduced to a Jaynes–Cummings model of a two or three-level system, similar to a two or three-level atom. In the case of single-photon absorption, the energy for the transition between levels  $1 \rightarrow 2$  and  $2 \rightarrow 3$  is equal. The two-photon simultaneous absorption case without spin–spin interaction can be effectively reduced to one-photon absorption for spin  $\frac{1}{2}$ .

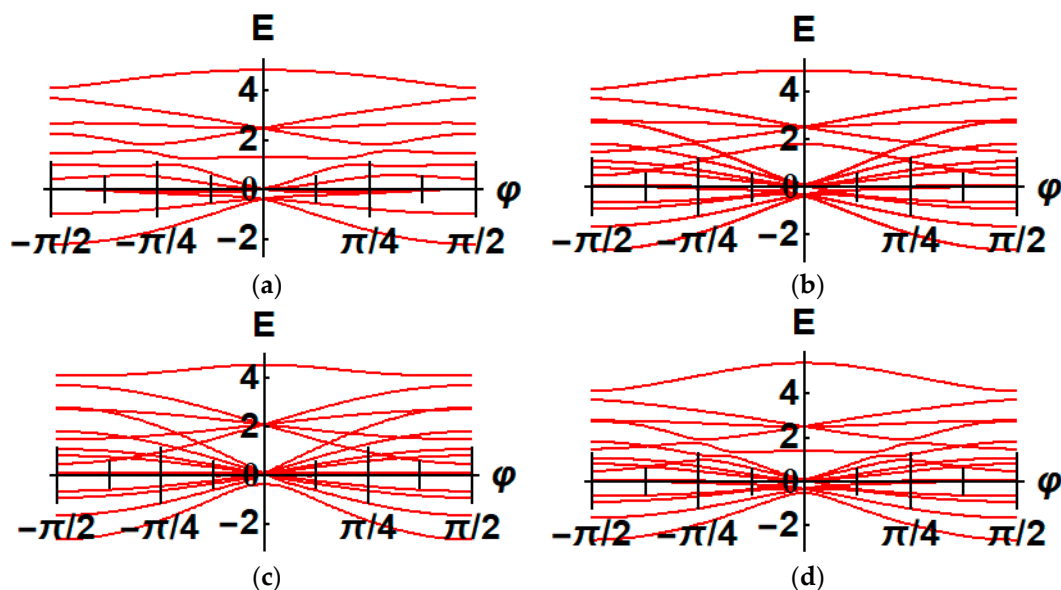
Figures 3–6 show the energy spectrum for spin chains consisting of 4 spins and a second spin interacting with a quantized electromagnetic field. Each figure corresponds to a different number of excitations: Figure 3 for one excitation, Figure 4 for two, Figure 5 for three, and Figure 6 for four. In the figures, Figures 3a, 4a, 5a and 6a corresponds to a spin chain consisting of spins  $\frac{1}{2}$ . Figures 3a, 4b, 5b and 6b correspond to a spin chain consisting of spins 1 interacting when only one photon emission/absorption is present. Figures 3b, 4c, 5c and 6c corresponds to the spin chain consisting of spin 1, in the case where there are only two photon parts of spin–light interaction of the Hamiltonian. The last, Figures 3a, 4d, 5d and 6d of the figures shows the case when both a one-photon and two-photon part of the JC Hamiltonian are present. The parameter  $\varphi$  has initial value of  $-\pi/2$  and goes to  $+\pi/2$ . The values of the other parameters are  $J = \sin(\varphi)$ ,  $c_0 = 0.01957$ ,  $c_1 = 0.026743$ , and  $c_2 = 2$ .



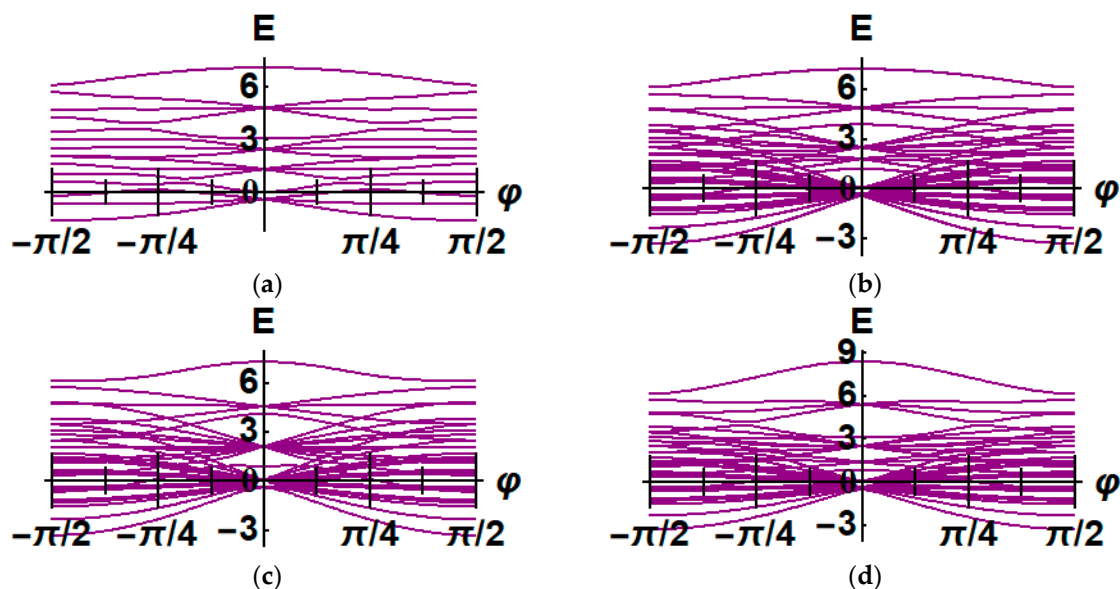
**Figure 3.** Energy spectrum of four spins in a chain and a second spin interacting with the quantized electromagnetic field. The number of excitations is equal to one. (a) Results for the spectrum are the same for the case of spin chain consist of spins  $1/2$ ; the case of spin chain consists of spin 1,  $G_1 = \cos(\varphi)$ , and  $G_2 = 0$ ; the case of spin chain consists of spin 1,  $G_1 = \cos(\varphi)$  and  $G_2 = \cos^2(\varphi)$  (b) spin chain consists of spin 1,  $G_1 = 0$  and  $G_2 = \cos^2(\varphi)$ .

In the case of one excitation, the energy spectrum of both cases of spin  $\frac{1}{2}$  and spin 1 (with only one photon and with one and two photon interaction) is the same, because there is not enough excitation for spin to go to state 1. When there is only one spin interacting with the electromagnetic field, and there are no other spins interacting with the first one, the system only has two states,  $|1, \downarrow\rangle$  and  $|0, \rightarrow\rangle$ .

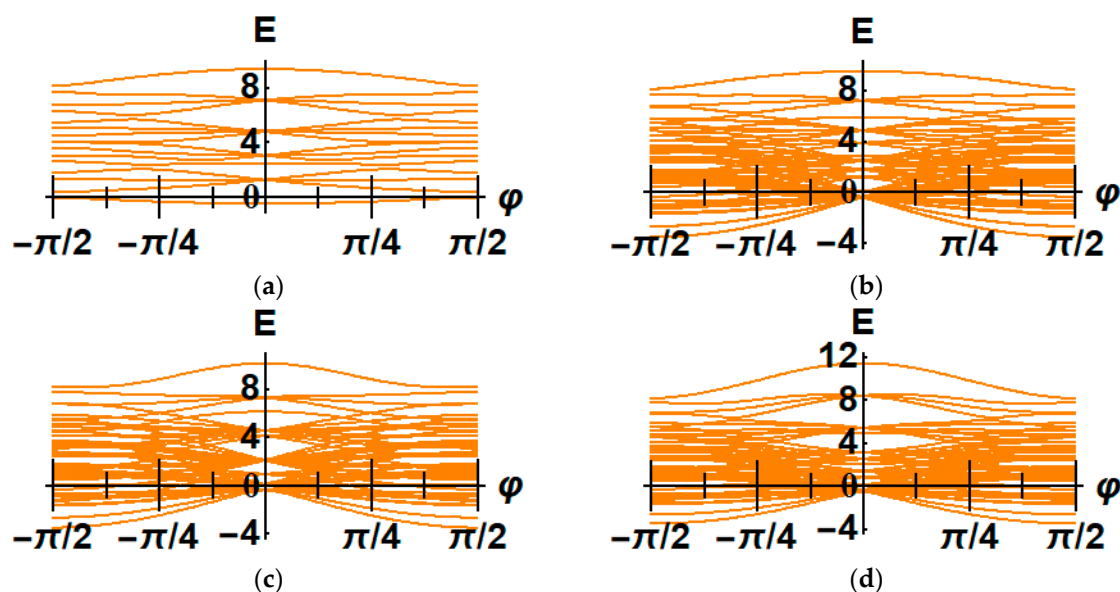
In the case of more excitations, when  $J = G_1 = 0$  and  $G_2 \neq 0$  the eigenvalues are close to those with spin  $\frac{1}{2}$  chain, with two times more excitations and changed values of parameters of spin  $\frac{1}{2}$  Hamiltonian— $c_1, c_2$ , and  $G_1$ . When  $J \neq 0$  and  $G_1 = G_2 = 0$  the energy spectrum is the same as that of the spin  $\frac{1}{2}$  case, with the same number of excitations. With a change of  $\varphi$  between  $\pm\pi/2$  and 0, these states “connect” with each other. Differences in energy spectrum occur due to different numbers of excitations transported by different parts of the Hamiltonian. The XY interaction between spins in the chain transports only one excitation between the spins at a time. However, the spin–light interaction transports “quanta” of two excitations at once.



**Figure 4.** Energy spectrum of four spins in a chain and a second spin interacting with the quantized electromagnetic field. The number of excitations is equal to two. (a) Spin chain consist of spins  $1/2$ ; (b) Spin chain consists of spin 1,  $G_1 = \cos(\varphi)$  and  $G_2 = 0$ ; (c) Spin chain consists of spin 1,  $G_1 = 0$  and  $G_2 = \cos^2(\varphi)$ ; (d) Spin chain consists of spin 1,  $G_1 = \cos(\varphi)$  and  $G_2 = \cos^2(\varphi)$ .



**Figure 5.** Energy spectrum of four spins in a chain and a second spin interacting with the quantized electromagnetic field. The number of excitations is equal to three. (a) Spin chain consist of spins  $1/2$ ; (b) Spin chain consists of spin 1,  $G_1 = \cos(\varphi)$  and  $G_2 = 0$ ; (c) Spin chain consists of spin 1,  $G_1 = 0$  and  $G_2 = \cos^2(\varphi)$ ; (d) Spin chain consists of spin 1,  $G_1 = \cos(\varphi)$  and  $G_2 = \cos^2(\varphi)$ .



**Figure 6.** Energy spectrum of four spins in a chain and a second spin interacting with the quantized electromagnetic field. The number of excitations is equal to four. (a) Spin chain consist of spins  $\frac{1}{2}$ ; (b) Spin chain consists of spin 1,  $G_1 = \cos(\varphi)$  and  $G_2 = 0$ ; (c) Spin chain consists of spin 1,  $G_1 = 0$  and  $G_2 = \cos^2(\varphi)$ ; (d) Spin chain consists of spin 1,  $G_1 = \cos(\varphi)$  and  $G_2 = \cos^2(\varphi)$ .

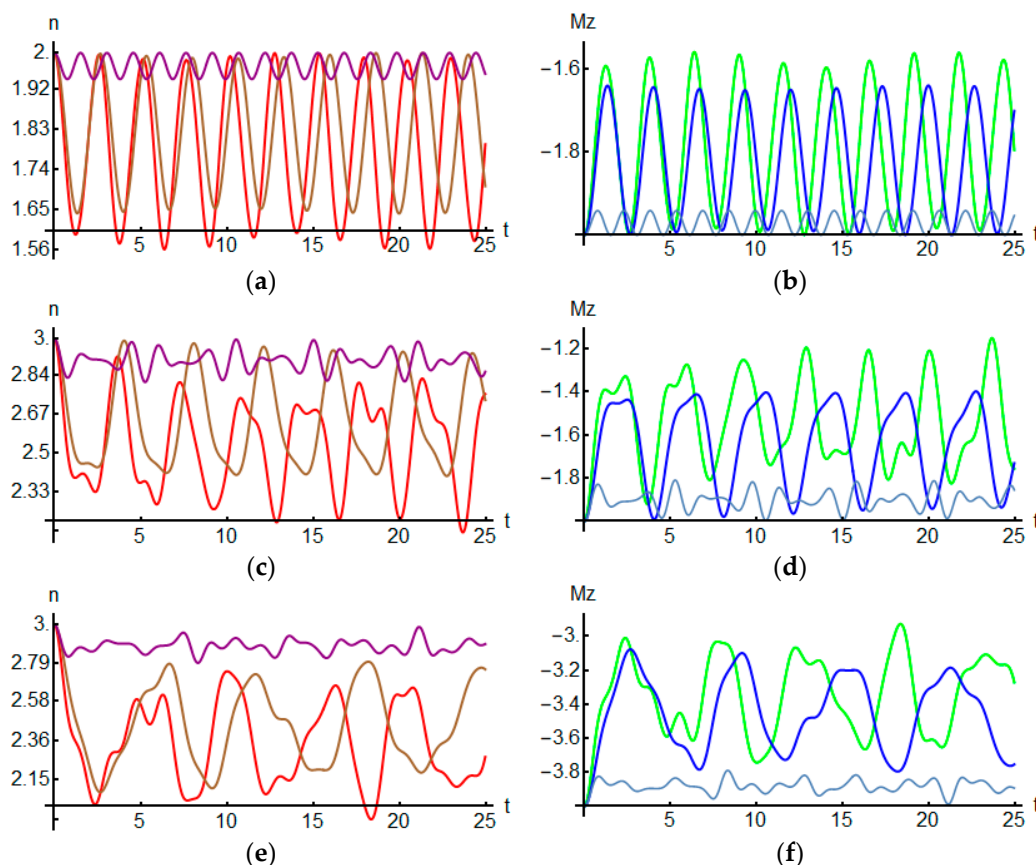
From a mathematical point of view  $\cos(\varphi)$  and  $\sin(\varphi)$  have the same value  $\sqrt{2}/2$  when  $\varphi = \pi/4$ , so that both  $G_1 = \cos(\varphi)$  and  $J = \sin(\varphi)$  have a similar contribution when only one photon interaction is present (as in the case of spin 1 and spin  $\frac{1}{2}$  chain). When there are only two photon absorptions,  $G_2 = \cos^2(\varphi)$  is much smaller than  $G_1 = \cos(\varphi)$ , but the contribution of the spin–spin interaction is the same as in the case when only two photon absorptions are present.

#### 2.2.4. Numerical Simulation of Magnetic Moment and the Number of Photons

The invariant value  $\langle \psi(t_i) | \hat{I} \hat{n} \nu | \psi(t_i) \rangle$  and the number of excitations do not change during evolution. They are simply offset from each other, and are spaced equal to  $\ln \nu = |n - N|$ . This is obtained from Equations (30) and (31) for the invariant of the system.

The difference between the magnetic moment behavior and the number of photons for spins  $\frac{1}{2}$  and 1 comes from the different invariant equations (6 and 30, respectively). The absorption of a photon by a particle chain with spins 1 changes the magnetic moment of the spin chain by half the difference from the magnetic moments at the two opposite spin states (Figure 7). In the case of a spin chain with spins  $\frac{1}{2}$ , upon absorption of a photon, the magnetic moment changes with the difference from the magnetic moments of the two opposite states (Figure 2).

Figure 7 shows a numerical simulation of the evolution of the number of photons (a, c, e) and the magnetic (b, d, f) moment of a spin chain, consisting of different numbers of spins, interacting with an electromagnetic field. In the case of more than two spins, the electromagnetic field interacts with the second spin. Figure 7a,b show the case where there is only one spin and two photons in the mode. Figure 7c,d show the case of a two-spin chain interacting with three photons. Figure 7e,f shows a four-spin chain interacting with three photons. The other parameters are  $c_0 = 0.01957$ ,  $c_1 = 0.026743$ ,  $c_2 = 2$ ,  $J = 0.86603$ ,  $G_1 = 0.5$ , and  $G_2 = 0.25$ . The initial state is all spins pointing downwards, and the number of excitations is equal to the number of photons.



**Figure 7.** (a,c,e) show the evolution of the number of photons,  $n$ , at time  $t$  during the evolution of the system; (b,d,f) show the magnetic moment,  $M_z$ , along the Z axis of the system at time  $t$  during the evolution of the system. The red line corresponds to the number of photons and the green to the magnetic moment in the case of single-photon and two-photon absorption simultaneously. The purple and teal lines correspond to the case where there is only two-photon absorption. The brown and blue lines correspond to the case where there is only a one photon absorption. The first two (a,b) correspond to an initial state  $|2, \downarrow\downarrow\rangle$ , the next two (c,d) to an initial state  $|3, \downarrow\downarrow\rangle$ . The remaining two correspond to an initial state  $|3, \downarrow\downarrow\downarrow\rangle$ .

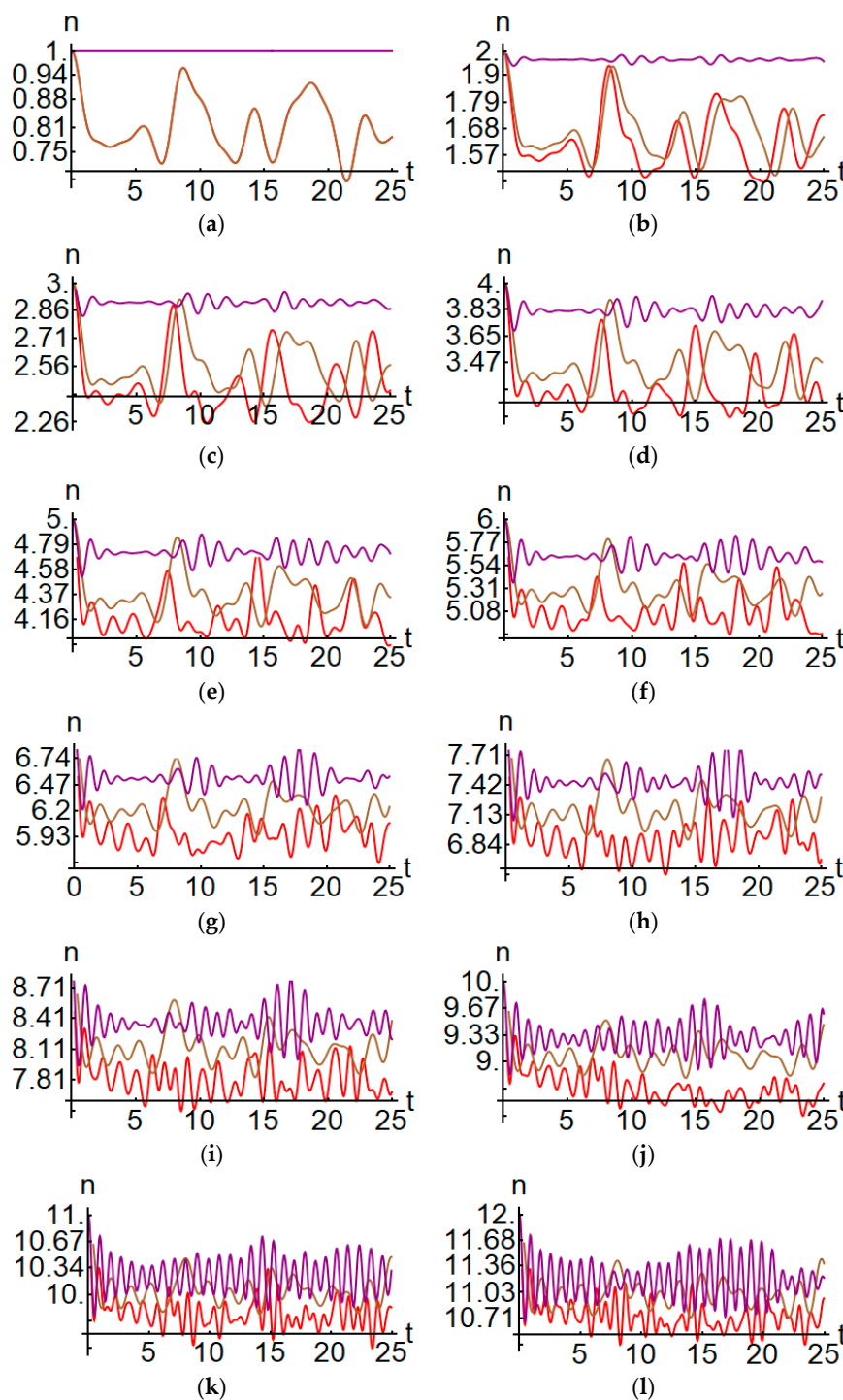
During evolution the Hamiltonian connects all states that have the same number of excitations. As in the case of spin  $\frac{1}{2}$ , the interaction between spins 1 does not change the number of photons or the magnetization. Only spin-mode interaction in the JC model changes the number of photons. However, the chain interferences occur again when the excitations “go through”. Increasing the number of spins increases the number of states connected through the evolution. The system can be in a superposition of states, and the magnitude of the spin–spin interaction gives a continuous spectrum magnetization and number of photons.

The term  $-G_1(\hat{a}^\dagger \hat{\zeta}_k^- - \hat{a} \hat{\zeta}_k^+)$  in the Hamiltonian changes the number of photons by  $\pm 1$ , and the term  $-G_2(\hat{a}^\dagger \hat{a}^\dagger \hat{\zeta}_k^- \hat{\zeta}_k^- - \hat{a} \hat{a} \hat{\zeta}_k^+ \hat{\zeta}_k^+)$  changes number of the photons by  $\pm 2$ .

For example, in the case of two spins and three photons in the mode, the states that are connected through evolution are  $|3, \downarrow\downarrow\rangle$ ,  $|2, \rightarrow\downarrow\rangle$ ,  $|2, \downarrow\rightarrow\rangle$ ,  $|1, \rightarrow\rightarrow\rangle$ ,  $|1, \downarrow\uparrow\rangle$ ,  $|1, \uparrow\downarrow\rangle$ ,  $|0, \uparrow\rightarrow\rangle$ , and  $|0, \rightarrow\uparrow\rangle$ .

In the case when only a two-photon spin–light interaction is included from the JC model, the excitation transfer between the mode and the field very much depends on the number of photons in the mode. The amplitude of the two photon interaction with the mode is much smaller compared to one photon interaction, because  $G_2 = G_1^2 < G_1$ . When the number of photons in the field is one, it plays no role in evolution. Since these terms change the number of photons only by two, there is no state connected with the state with one excitation in the mode. When the number of the excitations in the system is two or more, both spin–light interaction parts of the Hamiltonian govern the evolution.

As the number of photons in the field increases, the excitations transfer coming from this interaction also increases, as shown in Figure 8. The Hamiltonian part of the two photon JC model does not play any role in evolution when there is only one photon in the mode. That is the reason why, when the only spin–light interaction is through a one photon JC model and when both a one and two photon JC model interaction are present, they have the same evolution. This is shown in Figure 8a for a spin chain consisting of 5 spins.



**Figure 8.** The number of photons,  $n$ , in the system at time  $t$  during the evolution of the system is shown. The first spin of the system interacts with the electromagnetic field. Purple, brown, and red correspond to cases where there is only two-photon absorption, only one-photon absorption, and absorption in both cases, respectively. (a) corresponds to 1 photon, (b) corresponds to 2 photon, (c) corresponds to 3 photon, (d) corresponds to 4, (e–l) corresponds to 5–12 photon.



Evolution of the system when the both spin–light terms of the Hamiltonian are included, at the beginning is close to the evolution with only one photon interaction. With a longer evolution, however, the difference between them becomes significantly larger. As the number of photons in the mode increases, the two photon interaction part of the JC Hamiltonian has a bigger impact on the overall evolution of the system. This can be seen in Figure 8b for a spin chain consisting of 5 spins. The same is also true for a changing magnetization due to the interaction of the field with the spin chain.

Figure 8 shows the evolution of the number of photons for a 5-spin system. The first spin interacts with the photons. The other parameters  $c_0$ ,  $c_1$ ,  $c_2$ ,  $J$ ,  $G_1$ , and  $G_2$ , are the same as in the previous example. The initial state is when all the spins are down, and the number of excitations is equal to the number of photons. It can be seen that when there are a small number of photons, the single-photon absorption has a major influence on evolution. As the number of photons increases, the effect of two-photon absorption also increases.

### 3. Conclusions

This paper compares XY spin chains with a different spin;  $\frac{1}{2}$  and 1 interacting with a quantized electromagnetic field. In both cases, there is an invariant corresponding to the number of excitations. In the case of spin 1 three cases of Hamiltonian are shown: First, with only a one photon spin–light interaction between the mode and the chain. Second, only a two photon spin–light interaction is present. Third, both a one photon and two spin light interaction are present. The invariant is defined by the number of excitations, and can be obtained by using the number of photons in the mode and the number of excited spins, and it is the same in all three cases.

Using the invariant, the Hamiltonian can be diagonalized in blocks. The block structure is independent of the position of the interacting spin, and can be used to calculate the evolution of the state with the corresponding number of excitations. The different invariant of the chains with spin  $\frac{1}{2}$  and 1 leads to a difference in the evolution, and a corresponding difference in photon absorption and different magnetic moment.

The energy spectrum is obtained for a Hamiltonian of spin 1 chain, where energy is found for different values of spin–spin and spin–light interaction terms. By using the energy spectrum, it can be seen that when there is only one photon in the mode, the two photon interaction part of the Hamiltonian does not play any role. The contribution of the two photon spin–light interaction term increases with increasing the number of photons. The spin–light interaction in the Hamiltonian can be effectively reduced to single-photon absorption, such as in the case of a spin  $\frac{1}{2}$  chain; with different values of other parameters when there is only two-photon absorption and spin–spin interaction is not present. The addition of a spin–spin interaction leads to a different evolution, as the spins exchange only one excitation, but the spin–light interaction exchanges two.

The interaction of two photons with spin–light plays a much smaller role at the beginning of the evolution, when the amplitude of the two photon spin–light interaction is smaller than the amplitude of the one photon spin–light interaction and the number of photons in the mode is small. The role of the two photon spin–light interaction increases when the number of photons in the mode increases.

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