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From Natural to Artificial: The Transformation of the Concept of Logical Consequence in Bolzano, Carnap, and Tarski

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Abstract: Our standard model-theoretic definition of logical consequence is originally based on Alfred Tarski's (1936) semantic definition, which, in turn, is based on Rudolf Carnap's (1934) similar definition. In recent literature, Tarski's definition is described as a *conceptual analysis* of the intuitive 'everyday' concept of consequence or as an *explication* of it, but the use of these terms is loose and largely unaccounted for. I argue that the definition is not an analysis but an explication, in the Carnapian sense: the *replacement* of the inexact everyday concept with an exact one. Some everyday intuitions were thus brought into a precise form, others were ignored and forgotten. How exactly did the concept of logical consequence change in this process? I suggest that we could find some of the forgotten intuitions in Bernard Bolzano's (1837) definition of 'deducibility', which is traditionally viewed as the main precursor of Tarski's definition from a time before formalized languages. It turns out that Bolzano's definition is subject to just the kind of natural features—paradoxicality of everyday language, Platonism about propositions, and dependence on the external world—that Tarski sought to tame by constructing an artificial concept for the special needs of mathematical logic.

Keywords: logical consequence; deducibility; explication; formalization; semantics; Alfred Tarski; Rudolf Carnap; Bernard Bolzano

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1. Introduction

Although Alfred Tarski [1] is usually credited as the father of the modern (model-theoretic) concept of logical consequence (see, e.g., [2], p. 52; [3], p. 1), the basic idea had already been conceived by Rudolf Carnap in his 1934 book *Logical Syntax of Language* [4]. Tarski himself, in his 1936 article, cites Carnap's definition as the first attempt to formulate the "proper" concept of logical consequence. Douglas Patterson wonders why "pretty much nobody has bothered to follow up the article's many references to Carnap's *Logical Syntax* in any detail" ([5], p. 181). For instance, John Etchemendy, in his book *The Concept of Logical Consequence*, does not even include Carnap on his list of Tarski's precursors ([3], p. 7; [5], p. 187). Patterson argues that Tarski's account "intends nothing more than an improvement on Carnap's conception ... and that all of the important claims and arguments in the article are directed at Carnap" ([5], p. 181). I will go a step further and argue that Tarski's 'philosophy of definition' is essentially Carnapian as well. I will also take a step back and place his definition within a broader historical context. What kind of a definition did Tarski see himself as giving, and how did it change the concept of logical consequence from what it was before modern logic? This paper is an attempt to answer these questions.

The paper is made up of three parts, the first of which is a general introduction to Carnap's and Tarski's definitions of logical consequence. The second part deals with Tarski's Carnapian 'philosophy of definition'. This second part is ultimately motivated by Etchemendy's [3] claim that Tarski's definition is a failed attempt to give a *conceptual analysis* of the intuitive 'everyday' concept of logical consequence. Etchemendy argues that, depending on the language to which the definition is applied, it sometimes renders invalid arguments valid and valid arguments invalid. Even when the analysis gets the extension right, it does so just by accident, with no "conceptual justification" for getting it right (p. 81), whereas a proper conceptual analysis, like Dedekind's definition of the natural numbers, "obviously captures the essential feature of the intuitive notion" so that "its extensional adequacy is apparent from the definition itself" (p. 9; cf. [6], sec. 5). Therefore, Etchemendy claims "that Tarski's analysis is wrong, that his account of logical truth and logical consequence does not capture, or even come close to capturing, any pretheoretic conception of the logical properties" (p. 6).

Subsequent commentators disagree on what exactly the 'everyday' concept of logical consequence is and on whether or not Tarski's account fails to capture it, but many seem to maintain that the account is supposed to capture the everyday concept, in one way or another. In this discussion, Tarski's original definition is not always distinguished from its later model-theoretic development, and the focus often is on vindicating a broadly Tarskian approach rather than on getting clear on what Tarski actually aimed to achieve (see, e.g., [7], nn. 2, 24; [8]). The definition is sometimes called an analysis (e.g., [5,6,9–11]) and sometimes an *explication* ([11–14]), with not much discussion about the meaning of these labels or about their faithfulness to Tarski's thinking. An exception is Ray [15], who makes it clear that Tarski's definition is not an analysis, although he is not as clear on what it is. Similarly, Hodges [16] claims that Tarski's definition is neither an analysis nor an explication, but is, instead, an example of "intuitionistic formalism". However, in his detailed account of Tarski's intuitionistic formalism, Patterson [5] suggests that Tarski's definition of logical consequence marked a shift from intuitionistic formalism to conceptual analysis (p. 184).

It would be both historically accurate and philosophically illuminating, I think, to make a clear distinction among 'intuitionistic formalization', conceptual analysis, and explication here. In contrast to his definition of truth, which is best understood as an 'intuitionistic formalization' of the 'classical' analysis of truth (see [5,16]), I argue that Tarski's definition of logical consequence is a paradigm example of what Carnap would later call an explication: "the transformation of an inexact, prescientific concept ... into a new exact concept" ([17], p. 3). As an explication, it is not meant to coincide with, but merely be similar to, the everyday concept, and it is to be judged on its pragmatic utility rather than on getting the extension right. It is misguided to ask, in the vein of Etchemendy, whether Tarski's definition is 'right', because an explication is not something that can be right or wrong in the first place.

Once we understand the explicative nature of Tarski's definition, we are in a better position to fully appreciate its metamathematical, metalogical, and philosophical merits. In this paper, however, I am not asking what Tarski achieved; instead, I am asking what price he paid to achieve what he did. While Tarski's explication undoubtedly renders precise many everyday intuitions about logical consequence, it is part of the nature of an explication that some everyday intuitions are lost in the process: this is what Reck [18] calls the "philosophical residue" of explication, and what Bays [9], arguably, is also after when he notes that Tarski's "analysis" has a "provisional character" and that "it may have hidden inadequacies which would be revealed if we considered additional intuitions" beyond those that Tarski has set down in his condition of intuitive adequacy (p. 1702). Philosophers of logic have been busy assessing Tarski's definition against this condition, but what about those intuitions that did not even make their way to the condition? I suggest that there may have been some philosophical residue left behind in the past, namely everyday intuitions that were upheld before Tarski's definition and then forgotten. This, in turn, raises further questions, such as: what exactly was this residue? What was the concept of logical consequence like before explication?

Tarski was not very clear on the everyday intuitions that he intended to explicate. Some commentators [1,5] think that his everyday concept of logical consequence is a general notion used in everyday language; others [6,13,19] view it as a more restricted notion used by mathematicians and logicians. As noted by Jané [6], the first paragraph of Tarski's paper suggests that he is dealing with the general notion, but, for the rest of the paper, he seems to focus on the restricted one. To resolve the tension, Jané downplays the importance of the first paragraph and argues that Tarski's everyday concept is the restricted one, which he identifies with the one at play in late 19th- and early 20th-century axiomatics. Like Gómez-Torrente [20], I think that this move obscures part of the historical and philosophical background of Tarski's definition. Gómez-Torrente (pp. 290–294) does a good job of broadening the picture of Tarski's metamathematical and metalogical motivations, but his ideas about the more general everyday intuitions do not go much beyond those that Tarski explicitly appeals to in his paper.

I think that there is more to be said on the relationship between Tarski's precise definition and the everyday intuitions that did not fit in. To fully appreciate Tarski's everyday concept, it is worthwhile to go back to a time when logic had not yet been completely formalized and look for a 'natural' account of consequence untouched by the 'metamathematician's mould'. In the third part of the paper, I suggest that such an account is found in Bernard Bolzano's [21] definition of 'deducibility' (*Ableitbarkeit*), which is traditionally regarded as the main precursor of Tarski's definition from a time before formalized languages. It turns out that Bolzano's definition is subject to just the kind of natural features—paradoxicality of everyday language, Platonism about propositions, instability of the division between logical and non-logical terms, relevance of the premises to the conclusion, and dependence on the external world—that Tarski sought to tame by constructing an artificial concept for the special needs of mathematical logic. Contrasting the two definitions in this way sheds light not only on the philosophical residue of Tarski's definition, but also on the applicability of the Carnapian method of explication in general.

2. Carnap and Tarski on Logical Consequence

While the concept of logical consequence as we know it has been significantly shaped by Carnap's and Tarski's hands, they certainly did not start from scratch. There is a long history, too long to be told here, of the development of related ideas in the era between Frege's *Begriffschrift* (1879) and Carnap's *Logical Syntax*. For one thing, the inchoate idea of 'truth in a structure' appears in the investigation of properties of axiomatic systems by Peano's school (e.g., [22,23]), Hilbert's school (e.g., [24,25]), and by the American postulate theorists (e.g., [26,27]). The idea that a sentence is (universally) *valid* or *analytic* if it is true in all structures was well established by the end of the 1920s and played an important role in the study of completeness (see, e.g., [28–30]).[1] Rudimentary formulations of a semantic notion of logical consequence can be found at least in Pieri ([35], pp. 109, 248), Padoa ([23], pp. 122–123), Veblen ([26], p. 346; [36], p. 28), Łukasiewicz ([37], p. 17; [38], pp. 66–67), and Fraenkel ([39], pp. 347–349).[2] Similarly, Wittgenstein says in his *Tractatus* [40] that a proposition is a *tautology* (or *analytical*) if it is true for all truth valuations of elementary propositions, and he also says that a proposition 'follows from' other propositions if it is true in all truth valuations where the other propositions are true (4.46, 5.11, 5.132, 6.1). In this section, I elaborate on the most proximate motivation for the explication of a semantic notion of consequence, namely the quest for complete axiomatizations of deductive theories and the inadequacy of the syntactic notion of consequence as shown by Kurt Gödel's incompleteness theorems.

Until the 1930s, formal logicians by and large equated logical consequence with *derivability* in an axiomatic system of a deductive theory. The question of whether a sentence X 'follows logically' from, or is a 'logical consequence' of, a class K of sentences was reduced to the question of whether X could be derived from K using the rules of a given system.[3] Nowadays, we are accustomed to contrasting such a syntactic notion of logical consequence with a semantic one and saying that a system is complete if all semantic consequences are derivable in it. Back in the early 1930s, there was a different notion of completeness based not on semantic consequence, but on (mathematical or logical) truth: a system was considered complete in this sense if all true (valid, analytic) sentences were derivable from its axioms (cf. [29,30,45]). Then, in 1931, Gödel's incompleteness theorems revealed that every sound system, apart from the most elementary ones, was doomed to be incomplete in the sense that it was always possible to construct a true sentence that was not derivable from the axioms [46]. In the wake of this result, Tarski initially thought that the concept of logical consequence could never deliver all the truths that it should ([47], p. 295; cf. [48], pp. 550–551). Carnap took the other route and set out to develop an alternative, more adequate definition of logical consequence, and Tarski soon followed him. What resulted from their work is a *semantic* concept of consequence that is just as mathematically precise as its syntactic sister but is free from the latter's limitations.[4]

The lesson that Carnap and Tarski learned from Gödel's proof may be described as follows. As part of his proof, Gödel had shown that, for any consistent theory strong enough to express a certain amount of arithmetic, it is possible to construct a formula φ such that its instances $\varphi(0)$, $\varphi(1)$, $\varphi(2)$, . . . are all derivable from the theory—that is, $\varphi(n)$ is derivable for all natural numbers n—and yet the universally quantified sentence

$$\text{for all natural numbers } n,\ \varphi(n) \tag{1}$$

is not derivable (cf. [50], p. 130). The sentence (1) is thus not derivable from $\varphi(0)$, $\varphi(1)$, $\varphi(2)$, . . . and so does not 'follow from' them in the syntactic sense, although it arguably follows from them in the intuitive sense ([1], 1.2).

The obvious solution would be to add a new rule of inference, known as the 'ω-rule' or rule of infinite induction, equipping the theory with just the sort of deductive power it was lacking: by this rule, one would be permitted to conclude (1) from the totality of the premises $\varphi(0)$, $\varphi(1)$, $\varphi(2)$, ... The problem with this rule, according to Tarski, is its "'infinitistic' character" ([47], p. 295). In order to use it in a derivation, one should first have derived all the infinitely many premises. While such a situation can be imagined as a metamathematical possibility, it can never occur in reality: at every stage of the construction of a derivation, "only a finite number of sentences is 'effectively' given to us" ([47], p. 295; [1], 1.3.3).[5] Moreover, Gödel's result entails that, no matter how many rules of inference we add, it is always possible to construct a sentence which follows from the theory in the intuitive sense but which cannot be derived from it using the rules ([1], 1.4.2, F.1). Tarski concludes that the syntactic approach fails to capture the "proper" concept of logical consequence ([1], 1.4.3). Indeed, his commitment to Husserl's and Leśniewski's theory of semantic categories originally led him to believe that the proper concept could not be consistently defined at all for such theories. Apparently, this was because the definition should be given in a metalanguage of order greater than ω, but the theory of semantic categories required that all meaningful languages be at most of order ω.[6]

Carnap, by contrast, adopted the ω-rule when defining a concept of 'consequence' (*Folge*) for his Language I in *Logical Syntax*, and it is precisely the "infinitistic character" of this rule that sets the resulting concept of consequence apart from the concept of derivability or 'derivation' (*Ableitung*) ([4], sec. 14; p. 126). Tarski later complained that this definition could not be applied to more complicated languages ([1], F.2; cf. [50], pp. 138–140). In defining consequence for the more expressive Language II, Carnap took a different approach, but, unlike Tarski, he was not weighed down by the theory of semantic categories and was free to pursue the mission that Tarski had deemed impossible ([48], p. 564; n. 17). As it turns out, Tarski soon abandoned both the theory of semantic categories ([51], p. 268ff.) and his pessimism about defining the proper concept of consequence ([1], E). Although Tarski does not say so explicitly, we may guess that Carnap played a role in his change of mind ([48], n. 19; [50], p. 122; [5], chap. 6).

Carnap's strategy was to define, first, what it is for a sentence to be 'analytic' and 'contradictory'. This involved converting the sentence to prenex form and evaluating the resulting quantifier-free matrix for different valuations of its value-bearing signs. The details are convoluted due to Carnap's antisemantic attitude. He insisted that a valuation consists in numerals rather than numbers, and that the evaluation of a matrix consists not in its satisfaction or truth under a given valuation, but in its transformation into either '$0 = 0$' or '$0 \neq 0$' through a series of reductions (see [53], pp. 5–6; [5], sec. 6.2). Then, based on these definitions, he proposed the following definition of logical consequence:

> We say that the sentence X *follows logically* from the class of sentences K if and only if the class consisting of all sentences of the class K and of the negation of the sentence X is contradictory. ([4], p. 89; as rephrased in [1], 2.1.2)

Tarski acknowledged this definition as the "first attempt at the formulation of a precise definition for the proper concept" of consequence ([1], 2.1). The weakness of the definition, according to Tarski, is that it depends on the specifics of Language II (i.e., the inclusion of negation in the vocabulary) and on an unnecessarily complicated definition of 'contradictory'.[7] It seems that Carnap's verificationist and antimetaphysical convictions made him go to great pains to avoid semantic methods, not realizing how fundamentally semantic his methods actually were ([48]; cf. [5], pp. 87, 176). Like his friends in the Vienna Circle, Carnap regarded the concept of truth with suspicion (cf. [4], p. 164; [54], p. 216) and failed to see that it was possible to define it relative to a given interpretation of non-logical constants. Indeed, he had almost given that definition himself as part of his definition of 'analytic' and 'contradictory' ([48], pp. 565–568; [50], pp. 122–124; [5], p. 178). It was not until he learned about Tarski's definition of truth that he opened his eyes to the prospects of semantics ([55], p. x; [56], p. 60ff.; [48], p. 568) and set out to develop his own semantic account of analyticity [57,58].

Tarski, by contrast, could build on his earlier work on semantics, in particular his monograph 'The Concept of Truth in Formalized Languages' (CTFL) [51], and give a simple definition of logical consequence in just a few pages in his article 'On the Concept of Logical Consequence' (CLC).[8] Unlike Carnap, who defined logical consequence for his particular Languages I and II, Tarski outlined a general method for defining logical consequence for a large class of formalized languages. The definition proceeds as follows. We first transform a sentence X into a *sentential function* X′ by replacing all extra-logical constants with variable symbols. We are here presupposing a classification of the constants of the language into logical and extra-logical. Suppose, for the sake of illustration, that individual constants are all and only extra-logical constants. From, e.g., the sentence '1 + 2 = 3', we thus obtain the sentential function 'x + y = z'. We then say that a sequence of objects is a *model* of a sentence X if it *satisfies* (see [51]) the sentential function X′, just as, e.g., the sequence (1, 2, 3) satisfies the sentential function 'x + y = z'. Similarly, a sequence of objects is a model of a class K of sentences if it is a model of each sentence in K. In contrast to modern model-theoretic semantics, these objects were assumed to come from a fixed domain, namely a hierarchy of types [11,63,64]. Finally, we define logical consequence as follows:

> We say that the sentence X *follows logically* from the sentences of the class K if and only if every model of the class K is at the same time a model of the sentence X. ([1], 2.6.1)

In a similar vein, we say that a sentence X (class K of sentences) is *analytic* if every sequence of objects is its model, and we say that it is *contradictory* if it has no model ([1], 2.8.2). Moreover, provided that our formalized language includes negation, we say that a sequence of objects is a model of the negation of X if and only if it is not a model of X ([1], 2.8.3). It follows that Tarski's definition of logical consequence is equivalent to Carnap's definition for any formalized language that includes negation ([1], 2.8.4).

In sum, we can view the three definitions discussed above as successive stages of development from a syntactic notion of consequence to a semantic one. First, Carnap's definition of consequence for Language I extended the syntactic notion in the obvious way. In the corresponding definition for Language II, he employed semantic methods that were contorted to look like syntactic ones. Tarski's definition reorganized the previous definition into a transparently semantic form, which was then further developed by Tarski's students. The now standard model-theoretic account of logical consequence, where models are constructed out of sets and domains are variable, was not completed in its full form until the 1950s [32,64–66]. In this paper, however, we restrict ourselves to Tarski's original (1936) formulation, which is close enough to the final one for our purposes and more aptly displays the methodological shift that took place.

3. Carnap and Tarski on Explication

In this section, I abstract from the specifics of Tarski's definition in order to study its general character. I explain how Carnap describes the method of explication, how Tarski describes his approach to logical consequence, and why we should understand the latter in terms of the former.[9] Let us begin by explaining what Carnapian explication is about and how it fits into our overall story of Tarski's intellectual development.

3.1. Explication and Intuitionistic Formalism

Carnap first introduced his method of 'explication' in an article on probability (1945) [67] and further developed it in subsequent works [17,68], but he had effectively adopted this approach already in the early 1930s. Along with his famous Principle of Tolerance, Carnap abandoned the search for 'correct' analyses of concepts and declared himself free to propose all sorts of conventions for different purposes [69]. Indeed, Carus claims that *Logical Syntax* is where the ideal of explication makes its first appearance, albeit in a restricted form, namely the idea of "'translation' from the material into the formal mode of speech" ([70], p. 256; cf. [71], pp. 180–181). This is how Carnap later describes explication in his *Meaning and Necessity* (1947):

> The task of making more exact a vague or not quite exact concept used in everyday life or in an earlier stage of scientific or logical development, or rather of replacing it by a newly constructed, more exact concept, belongs among the most important tasks of logical analysis and logical construction. We call this the task of explicating, or of giving an *explication* for, the earlier concept; this earlier concept, or sometimes the term used for it, is called the *explicandum*; and the new concept, or its term, is called an *explicatum* of the old one. ([68], pp. 7–8)

Here, as well as elsewhere ([17], p. 5; [72], p. 66), Carnap mentions Tarski's definition of truth as a paradigmatic example of an explication. One of these passages occurs in the context of Carnap's semantic explication of "analyticity" (now understood in the broad sense, which includes both logical truths and truths like 'No bachelor is married'). In this passage, Carnap contrasts natural languages with artificial ones and points out that his explication of analyticity, like Tarski's explication of truth and "most of the explications of philosophically important concepts given in modern logic", is based on artificial languages, whereas attempts to explicate logical concepts for natural languages face considerable challenges ([72], p. 66; cf. [73], p. 108). This aspect of explication will prove to be particularly important for our discussion on Tarski's definition of logical consequence below.

In his *Logical Foundations of Probability* (1962), Carnap further elaborates on the requirements for an adequate *explicatum*: it must be similar to the *explicandum*, exact, fruitful, and as simple as possible ([17], pp. 5–7). We will return to these later; for now, it suffices to note that the *explicatum* is not supposed to give necessary and sufficient conditions for the *explicandum*, to be extensionally equivalent to the *explicandum*, or anything of the sort.

Given that Carnap regarded Tarski's definition of truth as an explication, one might believe it to be obvious that Tarski's definition of logical consequence is an explication too. As it turns out, the matter is far from obvious. For one thing, it has been questioned whether Carnap was right in interpreting Tarski's definition of truth as an explication. Hodges ([16], p. 113) bluntly claims that he was not, and Wagner [73] also warns against concluding "that Tarski's motivation for defining truth has anything to do with Carnap's principle of tolerance or his method of explication" (p. 108).

Rather than explication, Tarski's definition of truth has to do with intuitionistic formalism [5,16]. Tarski expressly subscribes to this doctrine in 1930 and resigns from it in 1956 ([74], p. 62). How and when he changed his mind is not immediately clear, but we can construct a plausible timeline.

An early example of intuitionistic formalism is found in Tarski's article on definability from 1930 [75]. There, he is concerned with the problem of defining the term 'definable set of real numbers'. He describes it as a term "of which we can give an account that is more or less precise in its intuitive content, but the meaning of which has not at present been rigorously established, at least in mathematics" (p. 111). The goal is to "construct a definition of this term which, while satisfying the requirements of methodological rigour, will also render adequately and precisely the actual meaning of the term" (p. 112). Having set down his definitions, "the formal rigour of which raises no objection", he asks whether they "*are also adequate materially*; in other words *do they in fact grasp the current meaning of the notion as it is known intuitively?*" (pp. 128–129) and concludes that they do (p. 132).

Although it can be questioned whether Tarski was right in this conclusion, his aim is clear: to express the intuitive content of 'definable set of real numbers' by a term in a formal theory ([5], chap. 3.3). Tarski compares this task to the "problems that the geometers solved when they established the meaning of the terms 'movement', 'line', 'surface', or 'dimension' for the first time" (p. 112), but then adds the following disclaimer:

> Strictly speaking this analogy should not be carried too far. In geometry it was a question of making precise the spatial intuitions acquired empirically in everyday life, intuitions which are vague and confused by their very nature. Here we have to deal with intuitions more clear and conscious, those of a logical nature relating to another domain of science, metamathematics. To the geometers the necessity presented itself of choosing one of several incompatible meanings, but here arbitrariness in establishing the content of the term in question is reduced almost to zero. ([75], p. 112)

Carnap, by contrast, uses Karl Menger's definition of 'dimension' as an example of what he means by explication. He takes it for an explication precisely because it extends the ordinary usage of the term and "cannot help being arbitrary" ([17], p. 7). Roughly, we can say that intuitionistic formalization and explication both introduce an informal concept into a formal theory, but the former merely *expresses* an intuitive content that was clear to begin with (cf. [5]), whereas the latter *replaces* intuitions that are typically too unclear and too conflicting to admit of direct formalization.

Intuitionistic formalism is further developed in Tarski's monograph of truth (CTFL), originally published in 1933 [5,16]. We will see that the intuitionistic formalization of the concept of truth is preceded by an *analysis* of the concept of truth, which Tarski simply takes from Kotarbiński without discussion [5]. Carnapian explication, by contrast, is not necessarily preceded by any such analysis. Carnap does say that the *explicatum* "is in many cases the result of an analysis of the explicandum . . .; in other cases, however, it deviates deliberately from the explicandum but still takes its place in some way" ([17], p. 3).

In the following subsections, we see Tarski gradually distancing himself from intuitionistic formalism. When he moves on to define logical consequence in the mid-'30s, his approach looks similar to that in CTFL, but one can tell from small details that something has changed below the surface. When he later (1944) clarifies and defends his definition of truth and of other semantic concepts, he appears to be further backing off from CTFL, and, when he proposes a definition of 'logical notion' (1966), he makes his new, explicative approach very clear from the outset. On the whole, we will see that Tarski's attitude toward definition undergoes a shift after the early 1930s and that his definition of logical consequence is the watershed where intuitionistic formalization gives way to explication. Thus, it seems that Tarski's definition of logical consequence is actually a better example of Carnapian explication than his definition of truth.

As for Carnap's own definition of logical consequence (for Language II) in *Logical Syntax*, it could be viewed as an explication too [50]; whether it is a good one is another matter. According to de Rouilhan [50], Carnap's rejection of semantic concepts resulted in an unnecessarily complicated and cumbersome explication far from the "standard informal explanation" of logical consequence. Tarski's definition, by contrast, is designed exactly to yield a precise counterpart for the "everyday" concept of logical consequence.

3.2. Tarski on Adequate and Correct Definition

Let us now elaborate on the relationship between Tarski's definition of truth in CTFL and his definition of logical consequence in CLC. In CTFL, Tarski's attitude toward intuitions is as straightforward as in the earlier paper on definable sets. He begins by formulating the "semantical definition" of truth according to the "classical" view:

> (1) *a true sentence is one which says that the state of affairs is so and so, and the state of affairs indeed is so and so.* ([51], p. 155)

While this formulation leaves "much to be desired" in terms of "formal correctness, clarity, and freedom from ambiguity", nevertheless, "its intuitive meaning and general intention seem to be quite clear and intelligible" ([51], p. 155; see also [76], secs 3, 17; cf. [75], p. 112). According to Patterson [5], Tarski takes the concept of truth to be perfectly analyzed in (1), refers the reader to Kotarbiński for such an analysis, and devises his own definition merely to express this concept by the term 'true sentence' in the metalanguage of a formalized language (see [51], pp. 153, 155). In the first paragraph of CLC, he is singing quite a different tune:

> The concept of *following logically* belongs to the category of those concepts whose introduction into the domain of exact formal investigations was not only an act of arbitrary decision on the side of this or that researcher: in making precise the content of this concept, efforts were made to conform to the everyday 'pre-existing' way it is used. This task was accompanied by the difficulties usual in such situations: the concept of following is not distinguished from other concepts of everyday language by a clearer content or more precisely delimited denotation, the way it is used is unstable, the task of capturing and reconciling all the murky, sometimes contradictory intuitions connected with that concept has to be acknowledged a priori as unrealizable, and one has to reconcile oneself in advance to the fact that every precise definition of the concept under consideration will to a greater or lesser degree bear the mark of arbitrariness. ([1], sec. 0)

Tarski, here, from the outset, commits himself to the view that the concept of logical consequence or "following logically" (*logische Folgerung* in German, *wynikanie logiczne* in Polish) originates in everyday language and yet cannot be given a precise definition that would exactly correspond to its usage in that language. In contrast to CTFL, the everyday concept of logical consequence is not formulated as anything like (1), and the 'messiness' about it apparently pertains not only to grasping its intuitive content through a precise definition, but also to the intuitive content itself (cf. [5], p. 184; [9], p. 1702). A few years later, Tarski describes the concept of truth as somewhat messy as well,[10] but he also describes the concept of logical consequence as one "whose intuitive content is more involved and whose semantic origin is less obvious" compared to the concept of truth ([76], p. 354).

Another difference between CTFL and CLC concerns the role of everyday language. In CTFL, Tarski devotes the first chapter to considering the possibility that truth be defined directly for everyday (colloquial, natural) language. In CLC, he does not even mention such a possibility, but appears to proceed from the outset toward a definition of logical consequence for formalized languages. One can only conjecture that he is ruling out natural language in CLC for reasons similar to those in CTFL (cf. [15], p. 623), such as its lack of an exactly specified structure or its universality:

> A characteristic feature of colloquial language (in contrast to various scientific languages) is its universality. It would not be in harmony with the spirit of this language if in some other language a word occurred which could not be translated into it; it could be claimed that 'if we can speak meaningfully about anything at all, we can also speak about it in colloquial language'. If we are to maintain this universality of everyday language in connexion with semantical investigations, we must, to be consistent, admit into the language, in addition

> to its sentences and other expressions, also the names of these sentences and expressions, and sentences containing these names, as well as such semantic expressions as 'true sentence', 'name', 'denote', etc. But it is presumably just this universality of everyday language which is the primary source of all semantical antinomies, like the antinomies of the liar or of heterological words. ([51], p. 164)

Considerations like this lead Tarski to doubt "the very possibility of a consistent use of the expression 'true sentence' which is in harmony with the laws of logic and the spirit of everyday language" ([51], p. 165; cf. [76], sec. 8). He consequently abandons the attempt to construct a "formally correct and materially adequate" definition of truth for everyday language and restricts himself to formalized languages, which he describes as "artificially constructed languages in which the sense of every expression is uniquely determined by its form" ([51], pp. 165–166, 208–209). In the final chapter, Tarski revisits the contrast between "natural" and "artificial" language:

> Philosophers who are not accustomed to use deductive methods in their daily work are inclined to regard all formalized languages with a certain disparagement, because they contrast these 'artificial' constructions with the one natural language—the colloquial language. [...] In my opinion the considerations of § 1 prove emphatically that the concept of truth (as well as other semantical concepts) when applied to colloquial language in conjunction with the normal laws of logic leads inevitably to confusions and contradictions. Whoever wishes, in spite of all difficulties, to pursue the semantics of colloquial language with the help of exact methods will be driven first to undertake the thankless task of a reform of this language. [...] It may, however, be doubted whether the language of everyday life, after being 'rationalized' in this way, would still preserve its naturalness and whether it would not rather take on the characteristic features of the formalized languages. ([51], p. 267)

If we include the concept of logical consequence among the "other semantical concepts"—disregarding the fact that, at this point, Tarski was still treating logical consequence as a syntactic notion (see [5], sec. 4.2.3)—we can read this passage as an explanation of why Tarski restricts himself to formalized languages in CLC. Here, the juxtaposition between the semantics of natural language and that of formalized languages is explicitly characterized. A decade later, Tarski makes a similar point by saying that there could be "languages which have an exactly specified structure without being formalized" and that the definition of truth for a natural language could be approximated by replacing it with a language "whose structure is exactly specified" and which differs from it "as little as possible" ([76], p. 347). That is, even if Tarski prefers to define semantic concepts for formalized languages and holds that they cannot be precisely defined for natural languages, he recognizes the middle way of doing so for a "rationalized" language, or a non-formalized language with an exactly specified structure, as a possible approach that someone else might prefer.

Let us return to CLC. Having bypassed any informal explanation of the everyday concept of logical consequence in the first paragraph, Tarski goes on to criticize previous logicians who thought they had managed, by purely syntactic means, "to capture almost precisely the everyday content of the concept of following, or rather to define a new concept which with respect to its denotation would coincide with the everyday concept" ([1], 1.1.1). He argues that "the formalized concept of following, which until now was generally used in the construction of deductive theories, by no means coincides with the everyday concept" ([1], 1.2.3). He then considers a possible response, extending the formalized concept so that "we would succeed finally in capturing the 'essential' content of the concept of following". Having refuted this response, he concludes that the syntactic approach fails to capture the "proper" concept of logical consequence: "In order to obtain the proper concept of following, essentially close to the everyday concept, one must resort in its definition to other methods altogether and use a quite distinct conceptual apparatus" ([1], 1.4.3).[11] While the "old" concept of consequence, "generally used until now by mathematical logicians", will continue to be useful for practical purposes, "one should put the proper concept of following in the foreground in considerations of a general theoretical character" ([1], 1.4.4; cf. [47], p. 293ff.). The former corresponds to Carnap's notion of 'derivability', and the latter to his notion of 'consequence' ([1], E.4).

Like Jané [6], I want to stress the ternary distinction among the "everyday" concept, the syntactic ("formalized", "old") concept, and the "proper" concept of consequence. However, I agree with Gómez-Torrente [20] that Jané is wrong to claim that the proper concept would exactly correspond to the everyday concept. The proper concept captures the everyday concept more extensively than the syntactic concept, but even the proper concept is only "essentially close", not equivalent, to the everyday concept (cf. [20], pp. 285–290; [19], p. 55), which is too vague and contradictory to coincide with any precise definition. The everyday concept is the concept of consequence actually used within natural language; the syntactic concept is what mathematical logicians have used as its exact counterpart in the methodology of the deductive sciences until now; and the proper concept is what they should use from now on.

The means for defining the proper concept is provided by "scientific semantics", the set of concepts and techniques introduced in CTFL. Tarski sets out to develop a "general method" which makes it possible to "construct a formally correct and materially adequate definition of the concept of following for an extensive category of formalized languages" ([1], 2.2.1) in parallel to CTFL, the goal of which is to come up with a formally correct and materially adequate definition of truth for formalized languages (see also [75], pp. 128–129). That the definition is formally correct means that it is "logically in order" (exact, rigorous, free from paradoxes); that it is materially adequate means that it captures certain intuitive aspects of the concept defined ([16], pp. 114–117; [5], pp. 109–111).[12]

Tarski emphasizes that his definition will make "no exaggerated claims to complete originality", that it involves intuitions which "will undoubtedly be sensed by many a logician who considered the concept of following logically and attempted to characterize it more closely as something well-known or even as something of his own"[13] and that he will only put these intuitions, with the tools of scientific semantics, "into a form which is exact and not subject to reproach" ([1], sec. 2.2). He goes on to formulate two requirements "from the point of view of everyday intuitions" for a sentence X to follow logically from sentences of a class K, jointly expressed in the following condition:

> *(F) If in the sentences of the class K and in the sentence X we replace the constant terms which are not general-logical terms correspondingly by arbitrary other constant terms (where we replace equiform constants everywhere by equiform constants) and in this way we obtain a new class of sentences K′ and a new sentence X′, then the sentence X′ must be true if only all sentences of the class K′ are true*. ([1], 2.3.5).[14]

This condition is clearly analogous to Convention T in CTFL ([51], pp. 187–188). "T" is short for "truth", "F" for "following" ([1], pp. 167–168). Both are necessary conditions that a definition should meet in order to be materially (intuitively) adequate. The condition T acts as a bridge that leads from the intuitive, everyday concept of truth to the definition of truth for a formalized language (Def. 23 in [51], p. 195); the condition F, likewise, leads from the intuitive, everyday concept of logical consequence to the definition of logical consequence for a formalized language. The difference is that the intuitive concept of truth is given an informal explanation in the "semantic definition" of truth ([51], p. 155; already cited above), whereas the intuitive concept of logical consequence is given no such explanation (cf. [5], p. 182). Another difference, according to Patterson ([5], p. 184; cf. [15], p. 624), is that condition T was taken to be sufficient for intuitive adequacy, but condition F was explicitly noted by Tarski to be insufficient.

Having formulated the final definition of logical consequence, which is both necessary and sufficient for intuitive adequacy (the technical details need not concern us here), Tarski states: "I have the impression that everyone who understands the content of the above definition will admit that it captures many intuitions manifested in the everyday usage of the concept of following" ([1], 2.7.1). Note his cautious choice of words. The definition captures "many intuitions", but no claim of complete equivalence is made (cf. [20], pp. 286, 289).[15] A definition for a natural language might have been intuitively 'more adequate' than that for a formalized language, but such a definition would not have been formally correct; in order to obtain a formally correct definition, some everyday intuitions have to be sacrificed. Thus, it is the *proper* concept of logical consequence that Tarski has defined, not the everyday one. In contrast to commentators who have questioned whether his definition really is intuitively adequate by his own criteria, I focus on the gap between these criteria and the everyday intuitions on which they are based.

3.3. Tarski on Conceptual Analysis

The main change from CTFL to CLC, according to Patterson [5], is that CLC takes a step away from intuitionistic formalism. Unlike the definition of truth, which merely enables the expression of an already analyzed concept, the definition of logical consequence "appears to carry the weight of conceptual analysis itself" ([5], p. 184). However, in his expository paper from 1944, Tarski makes it quite clear that he does not believe in conceptual analyses. Consider the following "general remarks" that, while focused on his semantic conception of truth, are clearly meant to apply more generally:

> I do not have the slightest intention to contribute in any way to those endless, often violent discussions on the subject: "What is the right conception of truth?" I must confess I do not understand what is at stake in such disputes; for the problem itself is so vague that no definite solution is possible. In fact, it seems to me that the sense in which the phrase "the right conception" is used has never been made clear. In most cases one gets the impression that the phrase is used in an almost mystical sense based upon the belief that every word has only one "real" meaning (a kind of Platonic or Aristotelian idea), and that all the competing conceptions really attempt to catch hold of this one meaning; since, however, they contradict each other, only one attempt can be successful, and hence only one conception is the "right" one. ([76], p. 355; cf. [77], p. 145)

Tarski says that such futile disputes "occur in all domains where—instead of an exact, scientific terminology—common language with its vagueness and ambiguity is used" and that the only rational solution to them would be the following:

> We should reconcile ourselves with the fact that we are confronted, not with one concept, but with several different concepts which are denoted by one word; we should try to make these concepts as clear as possible (by means of definition, or of an axiomatic procedure, or in some other way); to avoid further confusions, we should agree to use different terms for different concepts; and then we may proceed to a quiet and systematic study of all concepts involved, which will exhibit their main properties and mutual relations. ([76], p. 355; cf. [77], p. 145; [78], p. 27)

"Referring specifically to the notion of truth", Tarski recognizes that, in addition to the "classical" (correspondence-theorist) conception of truth on which his semantic conception is based, there are other conceptions of truth (pragmatist, coherence-theorist) to be found in philosophical and everyday speech and that these conceptions might also be put in a clear and precise form ([76], pp. 355–356; cf. [51], p. 153). Would he be prepared to take a similar stance on his conception of logical consequence? In CLC, Tarski had written of "the" everyday concept, conflating various incompatible intuitions that cannot be captured by a single precise definition, but, perhaps, after a few years, he would have expressed the same point by saying that there is not one but many everyday concepts of logical consequence (classical, intuitionist, relevantist, etc.), each of which can be given a definition that is "as clear as possible".[16]

One may argue that such a definition could very well be a conceptual analysis. Even if Tarski did not believe in one correct analysis of logical consequence, he might still believe that all the different concepts of logical consequence admit of analysis individually. However, in response to complaints that his definition of truth does not grasp the "essence" of the concept, Tarski bluntly states that he has "never been able to understand what the 'essence' of a concept is" and that he must therefore "be excused from discussing this point any longer" ([76], p. 361; see also [79], pp. 30–31). This, if anything, is clear evidence that Tarski's talk of defining a concept of logical consequence "essentially close to the everyday concept" should not be mistaken for an attempt at conceptual analysis ([15], p. 643). His definition is merely meant to capture certain everyday intuitions about logical consequence (those expressed in his condition of material adequacy) while giving up on others, and the passages quoted are part of his "final attempt to get philosophers to stop bothering him" ([5], p. 232). His unsuspecting references to everyday intuitions in CTFL and CLC are just harmless vestiges of the intuitionistic formalism of his philosophically oriented teachers (cf. [5]). As a mathematical logician, Tarski could very well do without them, and, indeed, that is what he chose to do for the rest of his career.

3.4. Tarski's Definition as a Carnapian Explication

Let us sum up. On one hand, Tarski grants that it is impossible to fully capture the everyday concept of logical consequence with a precise definition. On the other hand, some precise definitions may capture it better than others; in particular, Tarski's semantic definition captures it better than any syntactic definition. However, not even Tarski's definition claims to exhaust the everyday concept. The definition is not an intuitionistic formalization of the everyday concept, nor is it an analysis of it. The concept defined is the proper concept, not the everyday concept. What, then, is the relationship of the everyday concept to the proper concept?

This is where the Carnapian notion of explication enters the stage. Even if Tarski himself does not describe his approach in terms of explication, what he does say about it is conspicuously similar to what Carnap says about explication. The everyday concept is Tarski's *explicandum*, the proper concept is his *explicatum*, and the point of his paper is to replace the former with the latter. While such phrasing does not essentially add anything to Tarski's text, it enables us to bring important aspects of his definition into sharper focus.

First, it is noteworthy that Tarski does not make much effort to clarify his *explicandum*, but rather concentrates on formulating the *explicatum*. For this reason, it is not very clear whether his "everyday" concept of logical consequence refers to the practices of ordinary people or those of mathematicians. In the literature, one finds arguments for either interpretation. The former is supported by his recurring talk of "everyday language", "everyday usage", and the like (cf. [5], pp. 219–220). In a lengthy discussion on the extent to which his definition of truth reflects "common-sense and everyday usage", Tarski suggests settling the matter through a statistical survey on the intuitions of laymen ([76], p. 360). On the other hand, the metamathematical background of his definition of logical consequence (discussed at the beginning of Section 2), his active involvement in the "methodology of the deductive sciences", as he calls it, and his explicit references to intuitions of logicians working in that field suggest that the definition is first and foremost based on mathematical practice (see [6,13]). To some extent, the controversy may be due to terminological differences between the German and the Polish version of the paper, which, in turn, can be explained by Tarski's tailoring his vocabulary to suit his respective audiences. The German paper was read at a congress to an audience of logical positivists, whereas the Polish paper appeared in a journal allegedly read by phenomenologists ([1], pp. 166–167). Plausibly, readers of the German text (and the earlier English translation based on it) tend to conclude that Tarski is concerned with mathematical practice, whereas the readers of the Polish text (and the new English translation, which is primarily based on it and only secondarily on the German text) are more likely to read him as referring to everyday reasoning of ordinary people (cf. [1], pp. 165–166; [6], p. 14).

What is important for our purposes, however, is that either reading is compatible with Carnapian explication. According to Carnap, the *explicandum* "may belong to everyday language or to a previous stage in the development of scientific language" ([17], p. 3). The method of explication makes perfect sense of Tarski's vagueness about the *explicandum*. If the main point is not to analyze a pre-existing concept but to create a new one, it does not matter so much whether the pre-existing concept (which is assumed to be inexact anyway) is the prescientific concept used by ordinary people or its scientific counterpart used by mathematicians and logicians, or its *philosophical* counterpart, as Tarski seems to suggest when he mentions that:

> [T]he concept of an analytic sentence—in the intention of some contemporary logicians—is to be a precise formal correlate of the concept of *tautology* as a sentence which "says nothing about the real world", a concept which to me personally seems rather murky but which played and still plays a prominent role in the philosophical speculations of L. Wittgenstein and almost the whole Vienna Circle. ([1], 3.2.1)

Here, Tarski acknowledges that one could view his definition of analyticity as "a precise formal correlate"—i.e., an explication—of the informal concept of tautology. The fact that he personally regards that concept as "murky" only speaks in favor of this interpretation, for, at the beginning of the article, he uses that very same term (*mętny* in Polish) to describe the everyday concept of logical consequence.[17] Even if Tarski may have been mostly interested in the intuitions of mathematicians and logicians, he saw the relevance of his explication for the intuitions of philosophers as well.

That takes us to the second implication of reading Tarski's definition as an explication. The idea of an explication is not to analyze the essence of a concept already used, but rather to *produce* a concept to be used. If we accept that the *explicatum* is meant to play a normative rather than a descriptive role, as a proposed definition that should be employed by mathematical logicians rather than a faithful account of the ways in which the everyday concept is actually used, it follows that Tarski's definition cannot be "right" or "wrong", only more or less "satisfactory" ([17], p. 4; cf. [76], p. 341; [18], pp. 103, 109). One of Carnap's [17] requirements for a satisfactory *explicatum* is that it be "fruitful", i.e., useful for the formulation of general laws. Tarski, in turn, says that definitions of 'logical term' and 'logical truth' "prove fruitful, and this is really the most important" ([79], p. 29). In 1941, he expresses a similar point concerning the difference between the use of implications in everyday language and in mathematical logic:

> If a scientist wants to introduce a concept from everyday life into a science and to establish general laws concerning this concept, he must always make its content clearer, more precise and simpler, and free it from inessential attributes; it does not matter here whether he is a logician concerned with the phrase "*if* . . ., *then* . . ." or, for instance, a physicist establishing the exact meaning of the word "*metal*". In whatever way the scientist realizes his task, the usage of the term as it is established by him will deviate more or less from the practice of everyday language. If, however, he states explicitly in what meaning he decides to use the term, and if he consistently acts in conformity to this decision, nobody will be in a position to object that his procedure leads to nonsensical results. ([78], pp. 27–28)

Tarski then mentions that some logicians (such as C. I. Lewis) have been developing a theory of implication that would be more faithful to everyday usage than that of material implication. However, he is convinced that the latter "will surpass all other theories in *simplicity*" (another one of Carnap's criteria for an explication) and points out that "logic, founded upon this simple concept, turned out to be a *satisfactory* basis for the most complicated and subtle mathematical reasonings" (p. 28, my emphases).

Fruitfulness is thus more important to Tarski than faithfulness to everyday intuitions. Fruitfulness also provides an additional explanation of his ambiguous *explicandum*. As suggested above, Tarski was personally more concerned with metamathematical matters, but he also wanted to point out the philosophical significance of his explication. In this respect, he is no different from an engineer who designs a saw to cut wood and notes that one can make music with it as well. Explications are just such tools that can be useful for multiple purposes.

Finally, the distinction between the *explicandum* and the *explicatum* provides a neat explanation of why Tarski is happy to settle for capturing only some everyday intuitions. While Carnap requires that the *explicatum* be "similar" to the *explicandum*, he also emphasizes that "close similarity is not required, and considerable differences are permitted" ([17], p. 7). Indeed, the *explicatum* is bound to differ from the *explicandum* to some degree, simply because the latter is too vague to ever coincide completely with an exact concept, and the point of explication is precisely to replace the vague concept with an exact one ([17], pp. 3–5). Sometimes, the *explicatum* deliberately deviates from the *explicandum* even more than the latter's vagueness requires. As an example, Carnap notes that zoologists excluded whales, seals, and the like from their explication of the prescientific concept 'fish' so as to have a more fruitful concept that allows for more universal statements ([17], pp. 3, 5–6).

Tarski, as we have seen, was perfectly happy to compromise some everyday intuitions about logical consequence and settle for mere "adequacy" (see note 12) so as to get a formally correct concept for the study of formalized languages. It is little wonder, then, that his definition falls short of the kind of conceptual analysis that many philosophers have mistakenly expected from it. While explicating an inexact concept formally has many benefits, "it tends to leave a 'philosophical residue', one that has to be addressed informally in the end" ([18], p. 114). Whether Tarski's *explicatum* successfully clarifies the *explicandum* or not, we will likely be left with some everyday intuitions about logical consequence that do not fit within the sharp mould of formalized languages.

Some of this 'philosophical residue' is already acknowledged by Tarski himself. Having formulated his definition of logical consequence, Tarski devotes the last chapter of his paper to the problem of making a distinction between the logical and non-logical terms of a language. This, as he indicates, was of great importance to the Tractarian demarcation between logic and empirical science in the Vienna Circle ([1], 3.2.1; cf. [5], sec. 7.5.5). He admits that he has been presupposing such a distinction all along and yet can give no criterion to ground it, apart from remarking that such a criterion would have to accord with "everyday intuitions" ([1], 3.1.4) and "the dividing line traced by tradition between logical and extra-logical terms" (3.3.1). He leaves the issue open and prepares for the possibility that no objective grounds for the division could ever be found, in which case:

> [I]t would turn out to be necessary to treat such concepts as following logically, analytic sentence or tautology as relative concepts which must be related to a definite but more or less arbitrary division of the terms of a language into logical and extra-logical; the arbitrariness of this division would be in some measure a natural reflection of that instability which can be observed in the usage of the concept of following in everyday speech. ([1], sec. 3.3.2)

That is to say, instead of one true explication of 'logical consequence', based on one true explication of 'logical term', there would be different explications of 'logical consequence' for different explications of 'logical term'.[18] Yet that is not to say that any selection of logical terms would be just as good as the others. Indeed, the explication of 'logical term' is "certainly not entirely arbitrary" (3.1.4). While Tarski allows that one could "without expressly violating everyday intuitions . . . count among the logical terms also terms which logicians do not usually count among this category" (3.1.6),[19] he maintains that "if we did not count among the logical terms e.g., the implication sign or the quantifiers, the definition provided of following could lead to consequences manifestly contradictory to everyday intuitions" (3.1.4). Even if the concept of logical consequence may not be uniquely determined by everyday intuitions, it is nevertheless constrained by them.

As a final piece of evidence of Tarski's commitment to explication, I point to his lecture 'What are logical notions?', originally delivered in 1966 and edited for posthumous publication by John Corcoran in 1986 [77]. Here, Tarski can be seen proposing a criterion for the distinction between logical and extra-logical terms, thus offering a solution to the problem that he had left open 30 years before, although he now speaks of objects rather than of terms that denote them ([77], p. 143; cf. [80]; [5], sec. 7.5.2).

At the beginning, Tarski lists different approaches to the question 'What is logic' or 'What is such and such science?'. Some of them are descriptive, trying to give an account of the actual usage of the term, others are normative, suggesting a sense in which the term should be used, and some are mysterious attempts at "catching the proper, true meaning of a notion, something independent of actual usage, and independent of any normative proposals, something like the platonic idea behind the notion" (p. 145). Tarski dismisses the last approach as something so "foreign" and "strange" that he cannot say "anything intelligent" on it and goes on to formulate his favored approach:

> Let me tell you in advance that in answering the question 'What are logical notions?' what I shall do is make a suggestion or proposal about a possible use of the term 'logical notion'. This suggestion seems to me to be in agreement, if not with all prevailing usage of the term 'logical notion', at least with one usage which actually is encountered in practice. I think the term is used in several different senses and that my suggestion gives an account of one of them. ([77], p. 145)

Corcoran sums this up by saying that Tarski is proposing an "explication" that "shares features both with nominal (or normative) definitions and with real (or descriptive) definitions" but "is not arbitrary" and "is not intended to 'catch the platonic idea'" (p. 143). Corcoran also refers to Tarski's definitions of truth and logical consequence as explications (see also [12]), and suggests that Tarski's remarks on those earlier explications be compared to what he says about the present one. Above, we have seen that Tarski was devoted to intuitionistic formalism when he defined truth in the early 1930s, but abandoned it when he defined logical consequence in the mid-'30s. In his later work in the '40s and '60s, we see him making it more and more explicit that he only means to propose explications. The shift to explication is subtle, no doubt, but it did happen, and it does make a difference for how we should evaluate Tarski's achievements today.

4. Bolzano and Tarski on Logical Consequence

As an explication, Tarski's definition of logical consequence is bound to have left some 'philosophical residue'. To illustrate this residue, we will turn to Bernard Bolzano, whose definition of 'deducibility' (*Ableitbarkeit*) from 1837 is generally recognized as the closest precedent of Tarski's definition that can be found in the literature before Carnap and Tarski ([81], sec. 6.4).

I will not, however, discuss the one obvious candidate for Tarski's 'philosophical residue'—the necessity of logical consequence—that has received so much attention in the literature (see, e.g., [3,5–8,13,15]). It is a matter of dispute what kind of necessity Tarski aimed to capture through his definition and whether he succeeded, whereas Bolzano admittedly did not give any special role to necessity [82,83]. To recover the full modal force of logical consequence, we would have to go further than Bolzano. As it turns out, necessity was a fundamental feature of medieval theories of consequence and can be traced back, ultimately, to Aristotle's definition of a syllogism [8,84].

4.1. Introduction to Bolzano's Definition

Bernard Bolzano (1781–1848) was a reformer of traditional logic just some decades before the rise of modern formal logic. In his monumental magnum opus, *Theory of Science* (*Wissenschaftslehre*, 1837), Bolzano seeks to lay an objective foundation for sciences in general. He conceives of each science as a body of truths about a certain domain that is to be organized into a treatise. The rules for dividing the total domain of truth into individual sciences and for writing their respective treatises are given by the theory of science, namely logic, which is itself a body of truths to be organized: it is the science of presenting other sciences in their respective treatises ([21], vol. 1, secs 1, 6).

The basic building blocks of Bolzano's logic are *propositions in themselves* (*Sätze an sich*) and *ideas in themselves* (*Vorstellungen an sich*)—I will, henceforth, simply call them *propositions* and *ideas*. Propositions are states of affairs that we state when we utter a sentence or think about when we make judgements in our mind, and yet they themselves are independent of both our language and our mind. Ideas are components of propositions that correspond to terms in sentences or constituents of judgements but are objective in the same sense as propositions. For instance, the sentence '4 is even' expresses the proposition that 4 is even, which combines the ideas of 4 and evenness. A propositional or *sentential form* (*Satzform*) is a proposition (or rather, a sentence expressing that proposition) where (expressions of) certain ideas are considered variable: e.g., the proposition (expressed by the sentence) '4 is even', where '4' is considered variable, has the same form as '5 is even'.[20]

Ideas and propositions are logical objects in the logical world, distinct from the spatial and temporal 'real' world. Some ideas have extensions in the real world, and some propositions may describe states of affairs that actually obtain—if they do, they are true, otherwise they are false—but the ideas and propositions themselves are abstract entities that do not require real counterparts for their being. This is an essential departure from Kantian psychologism, i.e., the view of logical objects as mental entities, and an important move toward a more mathematical logic—a similar view of logical objects was adopted decades later by Frege, Husserl, Church, and others ([81], sec. 4).

For Bolzano, the most important concept of logic is 'deducibility' (*Ableitbarkeit*), which, despite the syntactic connotation of its name,[21] is in fact a semantic consequence relation between propositions ([81], sec. 6.3). The concept of deducibility, in turn, depends on that of 'compatibility' (*Verträglichkeit*). Propositions A, B, C, D, . . . are mutually *compatible* with respect to variable ideas i, j, . . . if there is a sequence of ideas whose substitution for i, j, . . . makes all of A, B, C, D, . . . true. Propositions M, N, O, . . . are *deducible* from A, B, C, D, . . . with respect to variable ideas i, j, . . . if the propositions A, B, C, D, . . . are compatible and if every sequence of ideas whose substitution for i, j, . . . makes all of them true also makes all of M, N, O, . . . true. For instance, the propositions 'Caius is dead' and 'Caius is alive' are incompatible with respect to the variable idea 'Caius', because no substitution of this idea can make both propositions true, whereas the propositions:

'Caius is either a plant or an animal' (2)

'Caius is not capable of photosynthesis' (3)

are compatible with respect to the variable ideas 'Caius' and 'animal', which can be seen by substituting these ideas for themselves. The proposition

'Caius is an animal' (4)

is deducible from (2) and (3) with respect to the variable ideas 'Caius' and 'animal', since (2) and (3) are compatible with respect to them and (4) is true whenever (2) and (3) are under any substitution of 'Caius' and of 'animal'. However, (4) is not deducible from (2) and (3) with respect to the variable ideas 'plant' and 'animal', which can be seen by replacing them with 'human being' and 'mushroom', respectively.

At certain points in his writings, Bolzano hints at an important special case of deducibility that resembles Tarski's logical consequence. As Siebel ([82], pp. 588–589) shows, Bolzano is aware of the distinction between formal and material validity (as is Tarski) and characterizes 'formal deducibility' as follows:

> A proposition is formally deducible from a given proposition X if it is deducible when all those parts of X that the logicians do not look upon as belonging to its *form* are taken as variable. ([21], vol. 1, p. 105)

Elsewhere in the text, Bolzano characterizes propositions like 'A is A', 'A which is B is B', 'Every object is either B or non-B' as "*logically* analytic" propositions on the grounds that:

> [i]n order to appraise the analytic nature of [these propositions], nothing but logical knowledge is needed, since the concepts which form the invariable part of these propositions all belong to logic. ([21], vol. 2, p. 59)

From passages like these (see also [21], vol. 1, secs 12, 81, 116; vol. 2, secs 186, 223, 254), we can extract a notion of 'logical deducibility' (cf. [82]), the special case of deducibility where the variable ideas are the non-logical ones. However, Bolzano does not believe in a sharp line between logical and non-logical:

> This distinction, I admit, is rather unstable, as the whole domain of concepts belonging to logic is not circumscribed so sharply that controversies could not arise at times. ([21], vol. 2, p. 59)

Tarski was not aware of Bolzano's definition when he developed his own.[22] The similarity of Tarski's logical consequence with Bolzano's deducibility was first pointed out by Heinrich Scholz [87] in 1937, a year after Tarski's paper had appeared in print. Tarski reacted by adding a footnote to the English translation of his paper in 1956, mentioning that Scholz had pointed out a "far-reaching analogy" between the two definitions ([1], n. I). Since then, this analogy has been commented on by various authors, many of whom have agreed that Bolzano 'anticipated' Tarski's definition. Hodges [2] even calls the model-theoretic definition of logical consequence "the Bolzano–Tarski definition" on the grounds that it "was first set down by Tarski . . ., though it only makes precise what Bolzano . . . and Hilbert . . . already understood" (pp. 52–53).[23] Other commentators have been more reluctant to draw parallels between the two definitions and have pointed out various interesting differences between them [81–83,90–92]. In the remainder of this section, I will highlight some of these differences to illustrate the 'philosophical residue' of Tarski's explication.

4.2. The Relationship Between Language and Reality

In contrast to Tarski's logical consequence, which is a relation among sentences of formalized languages, Bolzano's deducibility is a relation among propositions and ideas expressed in a natural language. As noted by Berg ([92], p. 21), this difference "is of vital importance for the study of the relationships between consequence and other logical notions", such as syntactic derivability. Only after logic had been "completely formalized by Frege" could a notion of syntactic derivability be defined, and only after that notion had been defined could the notion of semantic consequence be distinguished from it ([92], p. 21). Part of Tarski's motivation for defining logical consequence was precisely to develop a semantic alternative to the syntactic notion. However, that was not his only motivation: as discussed above, Tarski was also concerned with the difficulties of defining semantic concepts for natural languages. He thus aimed to create a new, artificial concept of logical consequence for formalized languages—the 'proper' concept—which was to replace the messy 'everyday' concept. He did not, in fact, provide an account of any particular consequence relation, but rather a general strategy or technique that could be applied to define such a relation for a given formalized language. The definition itself is formulated in a separate, more expressive metalanguage, equipped with some type theory, set theory, or "any sufficiently developed system of mathematical logic" ([51], p. 170; cf. [16], p. 118).

Bolzano's deducibility originated in a completely different conceptual environment. There was no such thing as 'formalized languages' at the time Bolzano was writing, and, instead of type or set theory, he was building on a logic of classes from a 1793 booklet by J. G. E. Maaß ([86], sec. 4.2). He does not separate between object language and metalanguage but works entirely within a natural language with unlimited expressive power. This point has not always been appreciated in the literature. Etchemendy ([3], p. 27), for instance, associates Bolzano's definition of deducibility with Tarski's criterion of material adequacy. Tarski found this criterion insufficient, because it only works if there are constant symbols in the object language for all referents in the domain, and replaced it with a definition based on the notion of satisfaction ([1], sec. 2.4). By associating Bolzano with the failed "substitutional" definition, Etchemendy ([3], pp. 162–163, notes 2, 5) deliberately does "some injustice" to Bolzano's intentions. As already mentioned, Bolzano does not separate between object language and metalanguage and, therefore, does not recognize the possibility of there not being a linguistic expression available for an idea (cf. [15],

p. 632; [82,83]). He does not define deducibility in terms of constants in the first place but in terms of ideas, and he believes that there is an idea for every object.

This takes us to another point which is easily overlooked from the modern point of view. The relation of deducibility does not hold between sentences but between propositions, i.e., the 'thoughts' expressed by sentences (cf. Frege's *Gedanken*). Propositions are composed of ideas, which are expressed by linguistic expressions and yet fundamentally independent of them. Tarski, by contrast, deliberately rejects "propositions" as unclear and ambiguous and rather postulates "sentences" understood as classes of concrete, physical objects ([76], p. 342, n. 5). Far from being a mere terminological nuance, this distinction has important implications for our comparison.

As noted by Siebel ([82], pp. 593–594), Bolzano does not discriminate between synonymous expressions, such as 'Cicero' and 'Tully', for they both express the same idea, whereas Tarski always replaces different expressions with different symbols, whether they are synonymous or not. Consequently, the sentences 'Cicero speaks' and 'Tully speaks' express the same proposition for Bolzano (cf. [41], sec. 3), but, for Tarski, the two sentences are logically independent in the sense that neither is a logical consequence of the other. This point can be further generalized to the level of languages. Bolzano's deducibility (as soon as the sequence of variable ideas is specified) is completely independent of the language wherein the ideas and propositions are expressed and could just as well be expressed in an alternative language, whereas Tarski's logical consequence is always relative to a particular formalized language. For instance, the deducibility of the proposition expressed by 'Socrates is mortal' from the propositions expressed by 'Socrates is human' and 'Every human is mortal' is one and the same as the deducibility of the proposition expressed by 'Socrates est mortalis' from the propositions expressed by 'Socrates est homo' and 'Omnis homo mortalis est', for the same propositions are expressed in both cases, only in different languages (English and Latin). Logical consequence, by contrast, holds among sentences of a particular language, depends on the vocabulary of that language, and is not conceivable as an abstract relation independent of any language.[24] Because of this, Tarski avoids all controversial talk of ideas and propositions, which is beneficial from a mathematician's point of view, but not necessarily from a philosopher's (say, a logical realist's). This is exactly the kind of 'philosophical residue' that is characteristic of explication: formal rigor at the cost of substance.

On the other hand, even if Bolzano's deducibility is independent of any language in this sense, there is a sense in which he falls prey to just the kind of problems that Tarski associated with natural languages. It has been pointed out that Bolzano's ignorance of set theory and his assumption that there is an idea for everything expose him to paradoxes, such as the Liar's paradox, Russell's paradox, or a contradiction with Cantor's theorem [85,93]. For instance, Bolzano explicitly argues that there is an idea for any multitude of objects ([21], vol. 1, sec. 101), but he also treats an idea as an abstract object itself. It follows that there are at least as many objects as there are multitudes of objects, which is impossible according to Cantor's diagonalization argument ([85], p. 39). Unlike Tarski, who blocked paradoxes by moving to formalized languages and by defining their semantic concepts in a separate metalanguage, Bolzano is led into trouble by freely exploiting the unlimited expressive power of natural language. This is precisely the "spirit" of everyday language which Tarski describes in § 1 of CTFL and which he may have in mind in CLC, too, when he talks of the "murky, sometimes contradictory intuitions" about the everyday concept of consequence ([1], 0.2). In replacing the everyday concept with the proper one, Tarski secures formal correctness at the cost of losing some everyday intuitions. This cost is yet another 'philosophical residue' of explication deliberately cut off by the explicator.

4.3. Demarcation of Logical Constants

Another key difference between Tarski and Bolzano concerns the distinction between logical and non-logical terms or ideas. Tarski is specifically interested in logical consequence and presupposes a distinction between logical and extra-logical terms from the outset.

Bolzano, by contrast, is generally interested in all kinds of consequence: for him, there are as many relations of deducibility as there are sequences of variable ideas. This difference is typically summarized by saying that Tarski's logical consequence is a dyadic (two-place) relation between premises and conclusions, whereas Bolzano's deducibility is a triadic (three-place) relation among premises, conclusions, and variable ideas [94].[25]

As noted above, Bolzano does not believe in the possibility of drawing a sharp line between logical and non-logical ideas. Tarski, too, admits that there may be no decisive grounds for a distinction between logical and non-logical terms and that the concept of logical consequence may have to be treated as relative to a given distinction. Even so, Tarski's account is an account of *logical* consequence and so depends on there being a collection of *logical* terms. Bolzano is concerned with the methodology of all sciences, not just deductive ones, and is, therefore, interested in all the different kinds of form, not only in the *logical* form (cf. [81], sec. 8; [86], sec. 3). A physicist, for instance, might consider physical terms fixed and all other terms variable; a biologist might, instead, have biological terms fixed, and so on. In this way, Bolzano shows a way to validate a great deal of actual scientific reasoning, such as causal inferences or arguments based on the laws of nature (cf. [82], p. 587). He even makes room for inductive and probabilistic reasoning, as we shall see below.

After Tarski ([1], 2.3.4, 3.1.7), we tend to single out logical form (constituted by a standard collection of logical terms) as the one true form and accordingly equate logical with formal consequence, with the result that most cases of Bolzano's deducibility turn out to be *material* consequences by our standards (cf. [82]). Tarski understands formal consequence in the narrow, logical sense to get a useful explication for deductive sciences, but it is material consequence, in its various forms, that is much more important for all the other sciences. Excluding it from the scope of logic proper appears to be a particularly substantial part of Tarski's philosophical residue—a drastic move, the repercussions of which we are still recovering from to this day (cf. [95]).

4.4. Compatibility of Premises

Lastly, I focus on a very particular and yet significant difference, namely Bolzano's condition that the premises be compatible. According to Tarski's definition of logical consequence, anything follows from contradictory premises ([1], sec. 2.8; cf. note 16), but, according to Bolzano's definition of deducibility, nothing follows. This concerns not only contradictory premises, but all incompatible premises, such as 'Caius is dead' and 'Caius is alive', where 'Caius' is considered variable. It has been pointed out that Bolzano's compatibility condition, together with his definition of 'exact deducibility' which requires that there are no redundant premises ([21], vol. 2, sec. 155.26),[26] guarantees that there is at least one idea that appears both in the premises and the conclusions and thus that the premises are, in this sense, relevant to the conclusion ([97], pp. 313–315; [98]).

Tarski is well aware that, in common language, the premises of a consequence are expected to be relevant to the conclusion. In his introductory textbook, there is a lengthy discussion on the ways in which the use of implications in logic and mathematics deviates from everyday usage ([78], pp. 23–32). For example, "*the hypothesis that* $2 \cdot 2 = 4$ *has as a consequence that New York is a large city*" sounds paradoxical in everyday language but is a meaningful and true sentence in logic (p. 31). Here, Tarski is clearly concerned with conditional sentences—namely material implications—that belong to the object language, and not with consequence relations among sentences of the object language. As for the latter, he notes that:

> [T]here are situations—though not in logic itself, but in a field closely related to it, namely the methodology of deductive sciences (cf. Chapter VI)—in which we talk about sentences and the relation of consequence between them, and in which we use such terms as "*implies*" and "*follows*" in a different meaning more closely akin to the ordinary one. (p. 31)

Indeed, the foregoing example would hardly qualify as a logical consequence in this latter sense on any reasonable selection of logical terms. Even so, it is a fact that Tarski's definition of logical consequence does allow for cases where the premises are inconsistent (or where the conclusion is logically true) regardless of whether there is any connection of sense between the premises and the conclusion. While Tarski does not explicitly say that such cases differentiate his definition from everyday usage, it is suggested by everything that he says about the paradoxes of material implication and its less paradoxical alternatives (pp. 23–32; cf. [20], pp. 290–291).

Bolzano's compatibility condition also makes it possible, in some cases, to calculate the *relative validity* or *probability* of an argument ([21], vol. 2, sec. 161; [81], sec. 6.7). Bolzano defines this as the ratio of the number of sequences of ideas that make both the premises and the conclusion true to the total number of sequences of ideas that make the premises true (here, certain restrictions are imposed to make sure that the numbers are finite). However, if the premises are incompatible, there are no sequences of ideas that make them true: one would then have to divide zero by zero, and the relative validity would be undefined. Provided that the premises are compatible, the relative validity will be 0 whenever the conclusion is incompatible with the premises, 1 whenever the conclusion is deducible from the premises, and a rational number between 0 and 1 in all the other cases. Thus, the compatibility condition ultimately allows Bolzano to subsume deductive logic under inductive logic ([21], vol. 2, sec. 161; [81], sec. 6.7). This link to inductive reasoning is a remarkable achievement in the history of logic and further highlights the contrast with Tarski's exclusive focus on deductive reasoning. Bolzano's logic is designed to organize all knowledge, including empirical facts, whereas Tarski's 'everyday intuitions' about logical consequence specifically exclude any dependence on "empirical knowledge" or "knowledge of the external world" ([1], 2.3.4; cf. p. 169).

As I see it, the question of whether anything is deducible from a contradiction is analogous to the question of whether zero really is a natural number, or whether the empty set really is a set: in all cases, the initially counterintuitive limit case is ultimately included under a general definition in order to simplify a mathematical theory.[27] Tarski, who rejects this condition, acts just like any modern mathematician who wants to generalize his notions as far as possible and subsume trivial special cases under those general notions. Bolzano's definition, by contrast, is certainly more faithful to everyday usage. Plausibly, we are not naturally inclined to accept inferences like 'Tarski is a mushroom and Tarski is not a mushroom; therefore, I am the Queen of Sheba' as intuitively valid, and we only come to accept them among the class of logical consequences by convention when we take our first course in classical logic—or then we revolt and turn to relevance logic. Indeed, it seems that relevance logic emerged when formal logicians recognized the received classical logic as inadequate for a faithful account of everyday reasoning (cf. [98]). This is typical of explications: you come up with a rough-and-ready formalization that cuts a lot of corners, and, before too long, the stuff that you cut out—the 'philosophical residue'—comes up again calling for accommodation.

5. Conclusions

Let me recapitulate the main steps in the development of the concept of logical consequence discussed in this paper. First, Bolzano defined his concept of deducibility, which applies to a broad range of reasoning in everyday language and is not based on any formalized languages, nor is it based on any demarcation of logical constants. Second, when logic was formalized by Frege and others, the notion of derivability in a formalized system of rules was taken to explicate the everyday concept of consequence in the construction of deductive theories. Third, Gödel's incompleteness theorems revealed that the notion of derivability does not exhaust the everyday concept of consequence and pointed out the need for a more adequate explication of the latter. Fourth, Carnap proposed such an explication for his Language II, but did this in a somewhat language-specific, convoluted, and latently semantic manner. Fifth, Tarski defined the same concept in a more generalizable, simpler, and manifestly semantic fashion that later evolved into our standard model-theoretic definition of logical consequence.

My main emphasis in this paper has been on the contrast between the first and the last step: the shift from Bolzano's deducibility to Tarski's logical consequence. I have argued that this shift is best understood in terms of Carnapian explication: the replacement of the inexact 'everyday' concept (*explicandum*) with the exact 'proper' concept (*explicatum*). While Tarski's explication has many technical benefits, Bolzano is, in many ways, more faithful to everyday intuitions—or, if you like, we could perhaps say that Bolzano and Tarski explicated everyday intuitions for different purposes (Bolzano for all sciences, Tarski for deductive sciences).[28] In short, we may conclude that Bolzano's deducibility accounts for many natural features of everyday and scientific reasoning, whereas Tarski's logical consequence is an artificial construct designed for the special needs of mathematical logic.

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Notes

1. For more background and references, see, e.g., [6,31–34], [3] (p. 7), and [20] (pp. 290–294).
2. See [33] (pp. 18, 22), [6] (pp. 22–30), [34] (p. 328), and [5] (pp. 184–185, 200–201).
3. Cf. [6] (p. 29). The concept of derivability was inspired by Frege [41], further developed by Whitehead and Russell [42], and given full proof-theoretic treatment by Gentzen [43]. For a detailed account of the history of formalization and derivability, see [44].
4. For an overview of Gödel's, Carnap's, and Tarski's mutual dynamics, see [49]. Detailed studies of Carnap's and Tarski's work on logical consequence can be found in [5,48,50].
5. Note that the rule of infinite induction is different from the ordinary rule of induction, which only has two premises (the base step and the inductive step).
6. Tarski adopted the theory of semantic categories in § 4 of [51] (p. 215), but resigned from it in the postscript that was added later on (p. 268ff.). Relatedly, he first rejected the possibility of defining the proper concept of consequence and then recanted ([47], p. 295; [1], E.1). For a discussion on Tarski's relation to the theory of semantic categories, see [48] (pp. 551, 562–563; n. 4) and [5] (p. 125ff.; secs 5.1.3, 6.1). For the origins of semantic categories in Leśniewski and Husserl, see [52].
7. Carnap also had a third definition of logical consequence and related concepts, based on what he called 'general syntax' ([4], sec. 48ff.). This attempt, too, was criticized by Tarski ([1], H). However, even if he did not embrace it as a definition of logical consequence, it may have inspired his criterion of material adequacy ([5], pp. 213–214; cf. note 14 below).
8. Tarski's article is based on a speech given at a congress in Paris in 1935. It was originally published in 1936 in both Polish [59] and German [60]. The German version was translated into English and published in Tarski's collected papers in 1956 and, again, in 1983, under the title "On the concept of logical consequence" [61]. A new translation from both Polish and German, entitled "On the concept of following logically", was published in 2002 [1]. I will keep to the established term 'logical consequence', but will otherwise rely on the new translation, which I take to be more accurate and authoritative than the old one, the reliability of which was questioned by Tarski himself ([62], pp. xiii–xiv; cf. [1], p. 156).
9. To be clear, I am not arguing that Tarski's use of explication was particularly deliberate or articulate, nor am I here intending to establish that it was due to Carnap's influence or that Carnap himself classified Tarski's definition of logical consequence as an explication. For the time being, I am merely suggesting that what Tarski says about his approach and what he actually does in CLC perfectly fits Carnap's description of explication.

10 Tarski remarks that "the common meaning of the word '*true*'—as that of any other word of everyday language—is to some extent vague", that "its usage more or less fluctuates", and that every solution of "the problem of assigning to this word a fixed and exact meaning . . . implies necessarily a certain deviation from the practice of everyday language" ([76], p. 360; cf. [5], pp. 229–232). These remarks are clearly aligned with the opening paragraph of CLC.

11 Compare this with what Tarski says about the concept of consequence in CTFL: "When, in everyday life, we say that a sentence follows from other sentences we no doubt mean something quite different from the existence of certain structural relations between these sentences [i.e., syntactic derivability]. In the light of the latest results of Gödel it seems doubtful whether this reduction has been effected without remainder." ([51], p. 252, n. 1).

12 The term 'adequate' has an interesting history. Tarski himself chose this term in 1944 [76] to translate the Polish term *trafny*. Hodges [16] points out that the original term (and its German translation *zutreffend*) literally means 'on target' or 'accurate', suggesting extensional correctness. He complains that, although 'adequate' (Latin *adaequatus*) was used in the medieval Aristotelian tradition to express extensional correctness, the sense it has in modern English is "far too vague" for this purpose ([16], pp. 114–115). However, we will see in the next subsection that Tarski rejects Aristotelian attempts at extensional correctness in [76]. For this reason, I think it is plausible that he would have deliberately weakened the sense of the term.

13 For a tentative list of logicians whose intuitions Tarski might be referring to, see the introduction to Section 2 above, and [20] (p. 292) in particular.

14 Patterson suggests that Tarski's grounds for this criterion are "straight out of [Carnap's] *Logical Syntax*" ([5], p. 213).

15 Compare this with Patterson's distinction between "intuitive adequacy" and "complete intuitive adequacy" in the context of CTFL ([5], p. 124).

16 I am grateful to an anonymous referee for drawing my attention to the various everyday concepts and 'non-classical' everyday intuitions (see also [19], p. 56; [20], pp. 290–291). There is some evidence that Tarski might have been sensitive to such variation. Already before CLC, Tarski notes that the formalized concept of consequence cannot exhaust the idea of "all purely structural operations, which unconditionally lead from true statements to true statements", but hesitates whether this idea exhausts the everyday concept ([47], pp. 294–295). It has been suggested that he has the intuitionists in mind when he speaks of the "sceptics" of the formalized concept in CLC ([1], 1.1.6; [20], p. 291). Some indication of non-classical intuitions may also be found in his discussion about less paradoxical alternatives to material implication ([78], pp. 25–28; cf. [20], pp. 290–291) and in his speculation that "we decided for some reasons to weaken our system of logic so as to deprive ourselves of the possibility of deriving every sentence from any two contradictory sentences" ([76], p. 368), although he is here concerned with the syntactic notion of consequence, not the semantic one.

17 In the German text [60], Tarski uses the term "murky" (*verworren*) at the beginning and the term "vague" (*vage*) in the later passage.

18 On this point, Tarski appears to be even more 'tolerant' than Carnap, who continues to emphasize a principled distinction between logical and non-logical terms (see [55], p. xi; cf. [5], sec. 7.5.5).

19 In a letter to Morton White, Tarski explains that he sometimes likes to include mathematical terms, such as the membership predicate $\in$, among the logical terms, and sometimes prefers to restrict himself to terms of "elementary logic" ([79], p. 29; cf. [76], n. 12; [75], p. 114).

20 Following Bolzano, I loosely speak of variation of ideas in propositions, although I really mean variation of *linguistic expressions* of ideas in *sentences* that express propositions (see [81], sec. 6.1).

21 Indeed, Carnap [4] uses the very word *Ableitbarkeit* (along with *Ableitung*) for his syntactic notion of consequence, but it is his *Folge* that corresponds to Bolzano's *Ableitbarkeit*. To avoid confusion, I prefer to translate Carnap's *Ableitbarkeit* as 'derivability' (as in [54]) and Bolzano's *Ableitbarkeit* as 'deducibility' (as in [21]).

22 However, it has been speculated that Tarski might have been indirectly influenced by Bolzano through, e.g., Husserl's *Logical Investigations*, which is known to have influenced Tarski through his teachers (recall the discussion of semantic categories above) and which includes a passage praising Bolzano ([85], pp. 14–15; cf. [86], sec. 8).

23 Similar claims are made by [88] (p. 18), [89] (p. 95), [12] (p. 97), and [3] (p. 27).

24 According to Tarski, "we must always relate the notion of truth, like that of a sentence, to a specific language; for it is obvious that the same expression which is a true sentence in one language can be false or meaningless in another" ([76], p. 342). The same applies to logical consequence: it simply makes no sense to ask whether some logical consequence holds without specifying the language, just as it makes no sense to ask whether some string of symbols is a sentence without specifying the language.

25 To be exact, Tarski uniformly varies every occurrence of a variable term, whereas Bolzano allows for variation of only some occurrences ([83], p. 20).

26 Originally, Bolzano also required that not even any parts of the premises should be redundant, but he later dropped this requirement with good reason (see [96], p. 54; [97], pp. 312–313).

27 The analogy with the empty set actually applies to Bolzano's logic of classes. In contrast to the standard set theory of our day, Bolzano does not allow ideas which represent no objects (such as the idea of a round square) to have empty collections as their extensions, but, instead, holds that they have no extensions at all ([86], sec. 4.2).

28 For an account of 'Bolzanian explication', see [99].

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