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Review

Action Functional for a Particle with Damping

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Abstract: In this brief report we discuss the action functional of a particle with damping, showing that it can be obtained from the dissipative equation of motion through a modification which makes the new dissipative equation invariant for time reversal symmetry. This action functional is exactly the effective action of Caldeira-Leggett model but, in our approach, it is derived without the assumption that the particle is weakly coupled to a bath of infinite harmonic oscillators.

Keywords: quantum tunneling; dissipation; effective action

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1. Introduction

The number of different systems and physical variables displaying damped dynamics is vast. A dissipative equation of motion can be found in various models, where the degree of freedom undergoing a damped evolution can be the spatial coordinate of a classical particle moving inside a fluid [1], but also, for instance, the phase difference of a bosonic field at a josephson junction [2–4], or a scalar field in the theory of warm inflation [5–7]. Obviously, the issue of dynamical evolution displaying dissipation has been analyzed also in the quantum realm, by making use of a great variety of techniques, spanning from the quasiclassical Langevin equation and stochastic modelling [8,9] to a refined Bogoliubov-like approach for the motion of an impurity through a Bose-Einstein condensated [10].

The presence of a dissipative term in the equation of motion makes the formulation of a variational principle and the derivation of an action functional quite problematic. On the other hand, the knowledge of the effective action of a particle with damping is crucial to study the role of dissipation on the quantum tunneling of the particle between two local minima of its confining potential [11].

There are various approaches to the construction of an action fuctional for a particle with damping. In 1931 Bateman [12] derived it by exploiting the variational principle with a Lagrangian involving the coordinates of the particle of interest and an additional degree of freedom. The price to pay in this doubling of the variables is a complicated expression for the kinetic energy, which does not have a simple quadratic form. In 1941 another approach was suggested by Caldirola [13], who wrote an explicitly time-dependent Lagrangian, whose Euler-Lagrange equation gives exactly the equation of motion with a dissipative term. In 1981, Caldeira and Leggett [14] considered a particle weakly coupled to the environment, modelled by a large number of harmonic oscillators. Integrating out the degrees of freedom of the harmonic oscillators, they found an effective action for the particle with dissipation and then they used it to calculate the effect of the damping coefficient on the quantum tunneling rate of the particle [14–16]. This framework started a deep theoretical effort devoted to understand quantum dynamics in a dissipative environment with its related features such as the localization transition [17] and the diffusion in periodic potential [18–20].

In this review paper we discuss a shortcut of the treatment made by Caldeira and Leggett [14,16]. We show that the Caldeira-Leggett effective action can be obtained, without the assumption of an environmental bath, directly from the dissipative equation of motion through a modification which makes the new dissipative equation of motion invariant for time reversal symmetry. Our approach is somehow similar to the one recently proposed by Floerchinger [21] via analytic continuation. The paper is organized as follows. First, in Section 2 we briefly review the quantum tunneling probability of the particle on the basis of the saddle-point approximation of the path integral with imaginary time. Then, in Section 3 we discuss the Caldeira-Leggett approach, where the effective action of a particle with damping is obtained assuming that the particle is weakly coupled to a bath of harmonic oscillators. Finally, in Section 4 we show this effective action can be obtained, without the assumption of an environmental bath, directly from a modified dissipative equation of motion.

2. Path Integral Formulation of Quantum Tunneling

We consider a single particle of mass m and coordinate $q(t)$ subject to an external potential $V(q)$ with a metastable local minimum at q_I.

In this section we discuss the quantum tunneling of the particle from the metastable local minimum in the absence of dissipation. The transition amplitude for the particle located at the metastable minimum q_I at time t_F to propagate to the position q_F at time t_F can be computed using the Feynman path integral

$$\langle q_F, t_F | q_I, t_I \rangle = \int_{q(t_I)=q_I}^{q(t_F)=q_F} D[q(t)]\, e^{\frac{i}{\hbar} S[q(t)]} \,, \tag{1}$$

where

$$S[q(t)] = \int_{t_I}^{t_F} dt\, L(q(t)) \tag{2}$$

is the action functional of the system and $L(q(t))$ is its Lagrangian, which has the form

$$L(q) = \frac{1}{2} m \dot{q}^2 - V(q) \,. \tag{3}$$

To obtain an analytical approximation of (1) one would like to use the saddle-point approximation, expanding the action around the "classical trajectory", that is the one minimizing the action. The problem is that, without the help of some external energy, there is no classical trajectory starting from $q(t_I) = q_I$ with $\dot{q}(t_I) = 0$ and ending at $q(t_F) = q_F$. What we can do is evaluate the imaginary time path integral, obtained by performing the change of variable $t = -i\tau$. Such operation is simply a $\pi/2$ rotation in the complex t plane which, given that no poles are encountered during the rotation, does not change the results of the integral even if the time integration is performed in the real domain after the change of variables. Thus, Equation (1) becomes

$$\langle q_F, \tau_F | q_I, \tau_I \rangle = \int_{q(\tau_I)=q_I}^{q(\tau_F)=q_F} D[q(\tau)]\, e^{-\frac{1}{\hbar} S_E[q(\tau)]} \,, \tag{4}$$

where $S_E[q(\tau)]$ is the so-called Euclidean action

$$S_E[q(\tau)] = \int_{-\infty}^{+\infty} d\tau\, L_E(q(\tau)) \,, \tag{5}$$

and $L_E(q(\tau))$ the corresponding Euclidean Lagrangian which, in the usual conservative case (3) is simply the Lagrangian with inverted potential $V(q) \to -V(q)$

$$L_E(q) = \frac{1}{2}m\dot{q}^2 + V(q) \ . \tag{6}$$

Because of this, the minima of $V(q)$ behave like maxima in the Euclidean action and so there exists a trajectory which minimizes $S_E[q(\tau)]$ starting from q_I and ending at q_F.

Then, to calculate an approximate expression for the transition amplitude, one finds the "imaginary-time classical trajectory" $\bar{q}(\tau)$, which minimizes the Euclidean action with the chosen boundary conditions, and one eventually gets

$$\langle q_F, \tau_F | q_I, \tau_I \rangle \simeq A\, e^{-S_E[\bar{q}(\tau)]/\hbar} \ , \tag{7}$$

where

$$A = \Big(\frac{S_E[\bar{q}(\tau)]}{2\pi\hbar}\Big)^{\frac{1}{2}} \Big(\frac{\det\big(-\partial_\tau^2 + V''(q_I)\big)}{\det'\big(-\partial_\tau^2 + V''(\bar{q})\big)}\Big)^{\frac{1}{2}} \tag{8}$$

with $\det'(\cdot)$ the determinant computed by excluding the null eigenvalues [22,23].

Our interest is to study how a particle subject to friction can escape from a metastable state driven by quantum tunneling. The action with Lagrangian (3), in such case, will not be useful, as it cannot describe a dissipative system. We want to study how the presence of friction influences the form of (7), and to do so we need to know the form of the Euclidean action of a dissipative system. It can be derived by means of the Caldeira-Leggett model, which is presented in the following section.

3. The Caldeira-Leggett Model

The Caldeira-Leggett model describes the motion of a particle in one dimension in a heat bath made of N decoupled harmonic oscillators, each with a characteristic frequency ω_j, with $j \in \{1, 2, \ldots, N\}$; their coordinates will be labeled as $x_j(t)$. The particle, described by the coordinate $q(t)$, is subject to an external potential $V(q)$ and coupled to the *j-th* harmonic oscillator via the coupling constant g_j.

The Lagrangian describing this system is

$$L = \frac{1}{2}m\dot{q}^2 - V(q) + \sum_{j=1}^{N}\left(\frac{1}{2}m_j\dot{x}_j^2 - \frac{1}{2}m_j\omega_j^2 x_j^2\right) + q\sum_{j=1}^{N} g_j x_j - q^2 \sum_{j=1}^{N}\frac{g_j^2}{2m_j\omega_j^2} \ . \tag{9}$$

The last term is simply a counterterm not depending on the oscillator coordinates. The physical reason for the introduction of such term is to let the minimum of the Hamiltonian, and thus of the energy, correspond to the minimum of the external potential $V(q)$.

Given Equation (9), the action immediately reads

$$S[q(t)] = \int_{t_I}^{t_F} dt \left(\frac{1}{2}m\dot{q}^2 - \frac{\partial V(q)}{\partial q} + \sum_{j=1}^{N}\frac{1}{2}m_j\dot{x}_j^2 - \sum_{j=1}^{N}\frac{m_j}{2}\left(\omega_j x_j - \frac{g_j}{m_j\omega_j}q\right)^2\right) \ , \tag{10}$$

while the transition amplitude for the particle to propagate from position q_I at time t_I to position q_F at time t_F and for the j-th harmonic oscillator to propagate from coordinate $x_{j,I}$ at time t_I to $x_{j,F}$ at t_F can be written as

$$\langle q_F, \{x_{j,F}\}, t_F | q_I, \{x_{j,I}\}, t_I \rangle = \int_{q(t_I)=q_I}^{q(t_F)=q_F} D[q] \left(\prod_{j=1}^{N} \int_{x_j(t_I)=x_{j,I}}^{x_j(t_F)=x_{j,F}} D[x_j] \right) e^{\frac{i}{\hbar} S[q;\{x_j\}]} \,. \tag{11}$$

The degrees of freedom of the environment are of no actual interest, and from (11) we would like to obtain a theory involving only the degrees of freedom of the particle, as suggested by Feynman and Vernon [24]. We will build an effective theory for the system, with effective action $S^{eff}[q(t)]$, by integrating out all the degrees of freedom of the environment, such that

$$\left(\prod_{j=1}^{N} \int D[x_j] \right) e^{\frac{i}{\hbar} S[q;\{x_j\}]} = e^{\frac{i}{\hbar} S^{eff}[q(t)]} \,, \tag{12}$$

where the initial conditions $\{x_{j,I}\}$ and the final conditions $\{x_{j,F}\}$ can assume any real value. Luckily, this is a doable task thanks to the fact that in the action the highest polinomial degree in x_j is x_j^2 and that the coupling with the particle is bilinear.

The path integral over the coordinates of the environment can be decoupled from the one over the paths of the particle of interest. This procedure is discussed in detail in Refs. [11,23]. We then obtain an effective action

$$\begin{aligned} S^{eff}[q(t)] = &\int_{t_I}^{t_F} dt \left(\frac{m}{2}\dot{q}^2 - V(q) - q^2 \sum_{j=1}^{N} \frac{g_j^2}{2m_j\omega_j^2} \right) + \\ &+ i \sum_{j=1}^{N} \frac{g_j^2}{4m_j\omega_j} \int_{t_I}^{t_F} dt \int_{-\infty}^{+\infty} dt' \, q(t)\, q(t')\, e^{-i\omega_j|t-t'|} \,, \end{aligned} \tag{13}$$

which can also be re-written as

$$S^{eff}[q(t)] = \int_{t_I}^{t_F} dt \left(\frac{m}{2}\dot{q}^2 - V(q) + \frac{1}{4} \int_{-\infty}^{+\infty} dt' \, K(t-t') \left(q(t) - q(t') \right)^2 \right) \,, \tag{14}$$

where

$$K(t-t') = -i \sum_{j=1}^{N} \frac{g_j^2}{2m_j\omega_j} e^{-i\omega_j|t-t'|} \xrightarrow[N\to+\infty]{} -i \int_0^{+\infty} d\omega \frac{g^2(\omega)}{2m\omega} n(\omega) e^{-i\omega|t-t'|} \,, \tag{15}$$

assuming that the spectrum of frequencies of the bath is continuous with $n(\omega)$ the number of harmonic oscillators with frequency ω and that the masses of the oscillators are all the same. We can now use Equation (14) to express the particle propagator as

$$\langle q_F, t_F | q_I, t_I \rangle = \int_{q(t_I)=q_I}^{q(t_F)=q_F} D[q] \, e^{\frac{i}{\hbar} S^{eff}[q(t)]} \,. \tag{16}$$

The equation of motion obtained by extremizing (14) is (see also [23])

$$- m\ddot{q}(t) - \frac{\partial V(q)}{\partial q} + \int_{-\infty}^{+\infty} dt' K(|t-t'|) q(t) - \int_{-\infty}^{+\infty} dt' K(|t'-t|) q(t') = 0 \,, \tag{17}$$

which in frequency space reads

$$- m\omega^2 \tilde{q}(\omega) + (\tilde{K}(\omega) - \tilde{K}(0))\tilde{q}(\omega) + \mathcal{F}\left[\frac{\partial V}{\partial q}\right](\omega) = 0 \,, \tag{18}$$

where by $\mathcal{F}[\cdot]$ we denote the Fourier transform. For the sake of simplicity, we now assume the following low frequency behaviour of the kernel:

$$\tilde{K}(\omega) = -i\gamma|\omega|. \tag{19}$$

Equation (18), then, becomes

$$-m\omega^2\tilde{q}(\omega) - i|\omega|\gamma\tilde{q}(\omega) = -\mathcal{F}\left[\frac{\partial V}{\partial q}\right](\omega) \,. \tag{20}$$

By taking (20) to real-time space the equation becomes

$$m\ddot{q} + \frac{i}{\pi}\gamma\int_{-\infty}^{+\infty} dt' \,\frac{q(t')}{(t-t')^2} = -\frac{\partial V(q)}{\partial q} \,, \tag{21}$$

which describes a particle of mass m subject to a nonlocal friction, with damping coefficient γ. A detailed discussion of this real-time equation and the associated real-time action functional (14) can be found in Ref. [23].

4. Direct Derivation of the Action Functional for a Particle with Damping

The discussion of the last part of the previous section is useful for our scope of deriving the action functional for a particle subject to friction directly from its equation of motion, without the need for the introduction of a bath made of harmonic oscillators.

Let us consider a particle of mass m and coordinate $q(t)$ in the presence of an external conservative force

$$F_c = -\frac{\partial V(q)}{\partial q} \tag{22}$$

with $V(q)$ the corresponding potential energy, and also under the effect of a dissipative force

$$F_d = -\gamma\dot{q} \,, \tag{23}$$

with $\gamma > 0$ the damping coefficient. The equation of motion for the particle is given by

$$m\ddot{q} + \gamma\dot{q} = -\frac{\partial V(q)}{\partial q} \,. \tag{24}$$

The Newton equation is clearly not invariant for time reversal $(t \to -t)$ due to the dissipative term, which contains a first order time derivative.

The Fourier transform of Equation (24) reads

$$-m\omega^2\tilde{q}(\omega) - i\gamma\omega\tilde{q}(\omega) = -\mathcal{F}\left[\frac{\partial V}{\partial q}\right](\omega) \,, \tag{25}$$

where $\tilde{q}(\omega) = \mathcal{F}[q(t)](\omega)$ is the Fourier transform of $q(t)$ and i is the imaginary unit. This equation is clearly not invariant under frequency reversal $(\omega \to -\omega)$ due to the dissipative term, which has a linear dependence with respect to the frequency ω.

Thus, we have seen that the absence of time-reversal symmetry implies the absence of frequency-reversal symmetry, and vice versa. The frequency reversal symmetry can be restored modifying Equation (25) as follows

$$-m\omega^2\tilde{q}(\omega) - i\gamma|\omega|\tilde{q}(\omega) = -\mathcal{F}\left[\frac{\partial V}{\partial q}\right](\omega) \,, \tag{26}$$

where in the dissipative term we substituted ω with $|\omega|$. This equation is clearly equal to Equation (20) of the previous section.

The inverse Fourier transform of this modified equation gives

$$m\ddot{q}(t) + \int_{-\infty}^{+\infty} dt' K(t-t')q(t') = -\frac{\partial V(q)}{\partial q} \,, \tag{27}$$

where

$$K(t-t') = i\frac{\gamma}{\pi(t-t')^2} \tag{28}$$

is the nonlocal kernel of our modified Newton equation (27) in the time domain. Unfortunately this equation, which is exactly Equation (21) of the previous section, depends explicitly on the imaginary units i and this means that the coordinate $q(t)$ must be a complex number evolving in real time. Quite formally, Equation (27) can be seen as the Euler-Lagrange equation of this complex and nonlocal action functional

$$S = \int_{t_I}^{t_F} dt \left(\frac{m}{2}\dot{q}^2 - V(q)\right) + \frac{1}{4}\int_{t_I}^{t_F} dt \int_{-\infty}^{+\infty} dt'\, K(t-t')\left(q(t)-q(t')\right)^2 \,, \tag{29}$$

that is indeed the Caldeira-Leggett effective action (14) we have obtained in the previous section, integrating out the degrees of freedom of the environmental bath.

Performing a Wick rotation of time, i.e., setting $t = -i\tau$, this action functional can be written as

$$S = i\, S_E, \tag{30}$$

where

$$S_E = \int_{\tau_I}^{\tau_F} d\tau \,\left(\frac{m}{2}\dot{q}^2 + V(q)\right) + \frac{1}{4}\int_{\tau_I}^{\tau_F} d\tau \int_{-\infty}^{+\infty} d\tau'\, K_E(\tau-\tau')\left(q(\tau)-q(\tau')\right)^2 \tag{31}$$

is the Euclidean action, namely the action with imaginary time τ, and

$$K_E(\tau-\tau') = -\frac{1}{\pi}\frac{\gamma}{(\tau-\tau')^2} \tag{32}$$

is the Euclidean nonlocal kernel. It is important to stress that, contrary to the action S, the Euclidean action S_E can be considered a real functional, assuming that the coordinate $q(\tau)$ is a real number evolving in imaginary time τ.

We can use the action functionals (29) and (31) to determine the probability amplitude that the particle of our system located at q_I at τ_I arrives in the position q_F at time τ_F. This is given by

$$\langle q_F, \tau_F | q_I, \tau_I\rangle = \int_{q(\tau_I)=q_I}^{q(\tau_F)=q_F} D[q(\tau)]\, e^{-\frac{1}{\hbar}S_E[q(\tau)]} \,. \tag{33}$$

This formula, with the Euclidean action $S_E[q(\tau)]$ given by Equation (31) and the dissipative kernel $K_E(\tau-\tau')$ given by Equation (32), is exactly the one used by Caldeira and Leggett [14] to find the effect of dissipation on the tunneling probability between two local minima q_I and q_F of the potential $V(q)$.

There is, however, a remarkable difference between our approach and the one of Caldeira and Leggett in [16]: we have derived Equations (31)–(33) directly from the dissipative equation of motion (24), while Caldeira and Leggett derived these equations starting from the Euclidean Lagrangian of the particle coupled to a bath of harmonic oscillators.

The effective action we have derived can then be used to compute directly (33), using the approximation (7). With such action, though, A and $S_E[\bar{q}(\tau)]$ will have to depend on γ, too. In particular, the coefficient A, following [16], becomes

$$A = \Big(\frac{S_E[\bar{q}(\tau)]}{\pi\hbar}\Big)^{\frac{1}{2}} \Big(\frac{\det\big(-\partial_\tau^2 + K_E(\tau-\tau') + V''(q_I)\big)}{\det'\big(-\partial_\tau^2 + K_E(\tau-\tau') + V''(\bar{q})\big)}\Big)^{\frac{1}{2}}\,, \tag{34}$$

As reported in [16], the contribution of the friction coefficient to $S_E[\bar{q}(\tau)]$ is positive, so that friction always tends to suppress the tunneling rate.

At this point, we have to consider a broader point of view on dissipative processes and the dynamical evolution of a quantum system coupled to the external environment. Certainly, a vast literature [23,25,26] has clarified that the usual Feynman path integration is not suited to deal, in principle, with dissipation and, in general, with non-equilibrium dynamics. In order to develop a meaningful microscopic approach, the Schwinger-Keldysh closed time path integral appears to be a more reliable framework. Unfortunately, it is immediately realized that price to pay is a much more complex formalism than the one outlined in this paper.

However, for a wide range of problems it is possible to recover a Feynman formulation in terms of an effective action such as Equation (31). For instance, this is the case for the quantum tunneling in a dissipative environment or the transition to a localized state for a particle moving in a quasiperiodic potential [20]. In these situations, we are not interested to the full quantum dynamical evolution, but we actually restrict ourselves to study fluctuations around an equilibrium state [21]. Indeed, even when there is thermal equilibrium between the system and its environment, fluctuations are still present and may crucially affect correlation functions such as the position one [11]. In order to compute these important quantities rather than the dissipative equation of motion, it is fundamental to have a well-defined effective action with time-reversal invariance, such as the one in Equation (31).

5. Conclusions

In conclusion, we have reviewed different approaches to the derivation of an action functional for a particle with damping and its crucial role on quantum tunneling. In Section 4 we have also proposed a slightly new approach by changing the dissipative equation of motion of a particle to make it invariant for time reversal symmetry. This modified equation of motion is nonlocal and complex, and it can be considered as the Euler-Lagrange equation of a nonlocal action functional. We have shown that this action functional is exactly the one derived by Caldeira and Leggett to study the effect of dissipation on the quantum tunneling of the particle. We stress again that, contrary to the Caldeira-Leggett approach, our action functional has been derived without the assumption that the particle is weakly coupled to a bath of infinite harmonic oscillators.

In the end, it is worth remembering, as stated in the introduction, that the theoretical framework outlined in this review can be effectively used to deal with a broader class of problems, besides the modelling of dissipative quantum tunnelling. For instance, within cold atoms experiments, it is possible to engineer periodic or disordered potentials with an exquisite control over their characteristic parameters and the coupling with the external environment [27]. As a consequence, this has sprung a renewed effort to understand a quantum system towards a localized state [20,28–30]. While a full understanding of the non-equilibrium quantum dynamics may require more refined functional approaches [25], it has been shown that, by using the Feynman formulation of the path integral, one can understand this transition in great detail [20], where all the relevant physical information are basically encoded in the kernel $K_E(\tau)$ defined in Equation (32).

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