designs

MDPI

Technical Note

Clarification on Classical Fatigue Design for Biaxial Stress Fields

Paulo M. S. T. de Castro

Faculdade de Engenharia, Universidade do Porto, Rua Dr. Roberto Frias, 4200-465 Porto, Portugal; ptcastro@fe.up.pt

Abstract: Fatigue design is a compulsory topic in education for several engineering specialties, particularly mechanical engineering. Textbooks on the design of machine elements have included, since long ago, the treatment of high-cycle fatigue (HCF) according to a classical theory based on the pioneering work of many researchers such as Carl Richard Soderberg. Perusing textbooks published over a long period, as well as comparing current ones, leads to the conclusion that the classical approach is presented with some differences in many of these textbooks, and this raises the need for some clarification: what are the consequences of the differences found? The present technical note addresses this problem, starting by presenting the classical approach, emphasising the variations in presentation identified, and then making a systematic comparison in order to grade the conservatism of the several variants found.

Keywords: classical fatigue design; design of machine elements; shafts

1. Introduction

Throughout the years, the classical approach to design under high-cycle fatigue (HCF) has been presented in a vast number of publications, particularly machine design textbooks. Notwithstanding the development of many improved models in recent decades, the classical approach continues to be taught in courses such as machine element design, and Harris and Jur recall that "*the long-taught classical methodology is useful and accurate as both a design and an analysis tool*" [1]. A reason for the popularity of the classical approach is its expeditious use, as compared with more precise approaches developed in recent decades. This justifies its continued interest, particularly for preliminary design exercises. The classical fatigue methodology is widely presented in textbooks, e.g., the 2021 edition of Budynas et al.'s book (the Shigley treatise) [2], the 2021 edition of Childs' book [3], and the 2021 book by D'Angelo [4]; in sources more practically oriented such as the 2019 handbook by Childs [5] and publications by Beswarick [6,7]; and in papers such as [8–10]. It is widely used in industry, for design as well as for failure analyses, where the interpretation of failure causes and redesign of failed parts are the objectives to be pursued [11]. Early presentations of the subject are found, e.g., in the works of d'Isa [12], Hall et al. [13], and Spotts [14].

Milela [15] and Lee et al. [16] present comprehensive overviews of fatigue and discuss research on biaxial fatigue as experienced in situations of combined bending and torsion moments, typical of shafts. State-of-the-art approaches for multiaxial fatigue are discussed in articles by Papuga et al. [17] and Anes et al. [18], among many other sources of information; this is, however, out of the scope of the present technical note, which concentrates on a particular issue of the classical approach—specifically, the differences found in certain presentations—and their consequences.

Although multiaxial fatigue and, in particular, shaft fatigue are the object of continued research efforts, the present work is not focussed on those advances; instead, it is focussed on the classical approach for design under high-cycle fatigue (HCF), seeking to understand the diversity of presentations found in the literature and to evaluate the relative conservatism of the different presentations of the topic. This matter is of interest to many

Citation: de Castro, P.M.S.T. Clarification on Classical Fatigue Design for Biaxial Stress Fields. *Designs* **2024**, *8*, 125. https://doi.org/10.3390/designs8060125

Academic Editors: Tiago Pinto Ribeiro and Julian D. Booker

Received: 18 October 2024
Revised: 6 November 2024
Accepted: 21 November 2024
Published: 25 November 2024

practitioners worldwide who do apply the classical approach, since it is the approach they learned in their engineering education; they may eventually be puzzled by the differences found when consulting the literature.

This technical note is organised as follows: Firstly, a presentation of the classical approach is made, through extensive reference to the relevant literature, particularly several textbooks. Since the presentation may differ across several editions of a given textbook, in certain cases several editions will be explicitly mentioned. Secondly, since the problem of comparison is best carried out parametrically, graphical presentations of relevant parameters, such as safety factors, are presented for different relevant combinations of input parameters, using Matlab R2023a. Finally, conclusions are drawn from the presented analyses.

This work is motivated by the large numbers of practicing engineers who were educated in fatigue design according to classical formulations. Presentations of the topic in different sources, however, reveal some differences. The present work gives an analysis of the differences found and a parametric description of the consequences, particularly as regards the important question of the resulting safety factors.

This technical note aims to alert designers of machine elements to the differences in published classical fatigue methodologies for biaxial stress fields and help to evaluate and compare these approaches, particularly as regards the safety factors involved.

2. Review and Background of the Problem

Throughout the years, the classical approach to design under high-cycle fatigue (HCF) has been taught in courses on the design of machine elements, notwithstanding the emergence, in the last decades, of a variety of more precise approaches to high-cycle multiaxial fatigue. The reason for the enduring interest in the classical approach is its ease of application, and presently, even in cases where sophisticated approaches should be used for final design, the classical approach retains its interest for quick preliminary design evaluations. The classical fatigue approach is the object of interest, namely, for failure analyses, e.g., Refs. [1,19], and comparative studies, e.g., Refs. [20,21].

Fatigue behaviour is a vast area of science and technology, involving consideration of the initiation and propagation of damage and final failure. The present technical note concentrates on design for HCF using the classical approach. Textbooks such as that of Shigley with its many editions (now under the authorship of Budynas and Nisbett) [22,23] and Childs [3,5,24], among others such as [25,26], present the topic of classical fatigue design with a focus on design of machine elements.

As regards standards, the ANSI/ASME B106.1M:1985 standard [27] was withdrawn in 1994; nevertheless, some organisations continue to use it as a standard; see [28]. It does not prescribe safety factor (n) values but gives advice: the greater the uncertainties and the cost of failures, the greater n should be. This technical note, being concerned with safety factor comparisons, will mention some recommendations from the literature at the end of this section.

The equation of the straight line commonly known as Soderberg's criterion is

$$\frac{\sigma_a}{S_e} + \frac{\sigma_m}{S_y} = \frac{1}{n} \tag{1}$$

where σ_a and σ_m are amplitude and mean stress, S_y is yield stress, S_e is fatigue strength under load ratio $R = -1$, and n is the safety factor. In the above expression, no explicit reference is made to the parameters influencing fatigue behaviour, such as surface characteristics and especially stress concentration; in the present technical note, these effects are supposed to be accounted for in the value of S_e. In this way, a clearer presentation of the effects of nominal stresses will be possible. Soderberg's criterion (Equation (1)) is known as conservative in the sense that the straight line connecting fatigue strength (in the vertical axis) with yield strength (in the horizontal axis) may be distant from actual experimental fatigue fracture data points. Equation (1) is written for normal stresses but can be used for shear stresses, with relevant shear endurance and yield strengths.

Childs presents the following approach for shaft fatigue design in the second edition of *Mechanical Design* [24]. Consider the cross-section of a rotating shaft, where there is a bending moment and constant torsion moment. The equation for determining the diameter for a solid shaft, based upon fatigue considerations, is

$$d = \left[\frac{32n}{\pi} \cdot \sqrt{\left(\frac{M}{S_e}\right)^2 + \frac{3}{4}\left(\frac{T}{S_y}\right)^2} \right]^{1/3} \tag{2}$$

The above equation is commonly known as an ASME equation (e.g., Childs [3]). It is simple to identify the origin of that equation, through the following steps:

$$\frac{S_y}{n} = \frac{32}{\pi d^3} \cdot \sqrt{\left(\frac{S_y M}{S_e}\right)^2 + \frac{3}{4} T^2} = \sqrt{\left(\frac{S_y}{S_e} \cdot \sigma_a\right)^2 + 3 \cdot \tau^2} = \sqrt{\left(\sigma_{st}^{eq}\right)^2 + 3 \cdot \left(\tau_{st}^{eq}\right)^2} \tag{3}$$

Equation (3) represents the von Mises criterion for this situation, where normal stress σ is cyclic with $R = -1$ and shear stress τ is constant. Under the root sign of the last line of Equation (3), there are two summands: the first is the equivalent static normal stress, and the second is, by its nature, static (in the sense that it is not a function of time). Recall that multiplying Equation (1) by S_y leads to

$$S_y \frac{\sigma_a}{S_e} + S_y \frac{\sigma_m}{S_y} = \frac{S_y}{n} = \sigma_{st}^{eq} = \sigma_m + \frac{S_y}{S_e} \cdot \sigma_a \tag{4}$$

In the above case, $\sigma_m = 0$ (no normal/axial force is assumed); as regards shear stress,

$$\tau_{st}^{eq} = \tau_m + \frac{S_{Sy}}{S_{Se}} \cdot \tau_a \tag{5}$$

with, in this case of constant torsion moment, $\tau_a = 0$, i.e., $\tau_{st}^{eq} = \tau$ and S_{Sy} being the shear yield stress and S_{Se} the shear fatigue strength. As mentioned above, it will be assumed hereafter that stress concentration effects and other relevant parameters affecting fatigue behaviour (e.g., surface finish) are accounted for in the value of fatigue strength (S_e or S_{Se}).

Soderberg presents in [29] an analysis of fatigue under biaxial stress, which requires the consideration of a volume element in the shaft surface where stresses resulting from bending and torsion are higher. He considers a cut of that element by a plane whose normal makes an angle, α, with the shaft axis direction and considers the use of Equation (1) for the shear stress state in that plane, using the equivalent Tresca stress. After due analytical manipulation—see [29]—the resulting equation is

$$y = \left[\frac{\sigma_m \sin(2\alpha) + 2\tau_m \cos(2\alpha)}{S_y} + \frac{\sigma_a \sin(2\alpha) + 2\tau_a \cos(2\alpha)}{S_e} \right]^{-1} \tag{6}$$

with the safety factor (n) for a given combination of σ_a and τ_m being the minimum of function $y(\alpha)$. A concise presentation of the approach is also given in a subsequent paper by Soderberg [30].

Shigley presents in [22,31] a graphical interpretation of Soderberg's analytical formulation. As mentioned before, this is based upon the stress analysis of a volume element of the surface of a rotating shaft subjected to constant bending and torsion moments. In that surface element, the shear stress components $\tau_{\alpha a}$ and $\tau_{\alpha m}$ in a plane characterised by angle α are determined. It is found that, when plotted in a coordinate system with mean shear stress $\tau_{\alpha m}$ along the horizontal axis and alternating shear stress $\tau_{\alpha a}$ along the vertical axis, the values of $\tau_{\alpha a}$ and $\tau_{\alpha m}$ define a quarter of an ellipse. Consider a x, y cartesian coordinate system and an ellipse with semi-axes a along x (axis of mean shear stress) and b along y (axis of alternating shear stress):

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{7}$$

Consider the straight line $y = mx + c$ defined by the points ($x = 0$, $y = S_{Se}/n$) and ($y = 0$, $x = S_{Sy}/n$), which is tangent to the ellipse

$$y = -\frac{S_{Se}}{S_{Sy}}x + \frac{S_{Se}}{n} \tag{8}$$

But a tangent to an ellipse is

$$c = \pm\sqrt{a^2m^2 + b^2} \rightarrow \frac{S_{Se}}{n} = \pm\sqrt{a^2\left(-\frac{S_{Se}}{S_{Sy}}\right)^2 + b^2} \tag{9}$$

In this problem, a and b are shown to be $a = 16T/\pi d^3$ and $b = 16M/\pi d^3$ ([22,31]); therefore,

$$\begin{gathered}\frac{S_{Se}}{n} = \pm\sqrt{a^2\left(\frac{S_{Se}}{S_{Sy}}\right)^2 + b^2} \rightarrow \frac{S_{Se}}{n} = \pm\sqrt{\left(\frac{16T}{\pi d^3}\right)^2\left(\frac{S_{Se}}{S_{Sy}}\right)^2 + \left(\frac{16M}{\pi d^3}\right)^2} \\ d^3 = \sqrt{\frac{n^2}{S_{Se}^2}\left[\left(\frac{16T}{\pi d^3}\right)^2\left(\frac{S_{Se}}{S_{Sy}}\right)^2 + \left(\frac{16M}{\pi d^3}\right)^2\right]} \rightarrow d = \left\{\frac{16n}{\pi}\left[\left(\frac{T}{S_{Sy}}\right)^2 + \left(\frac{M}{S_{Se}}\right)^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{3}}\end{gathered} \tag{10}$$

Recall that S_{sy} and S_{se} are shear strengths. From the Tresca criterion, $S_{se} = S_e/2$ and $S_{Sy} = S_y/2$,

$$d = \left\{\frac{32n}{\pi}\left[\left(\frac{T}{S_y}\right)^2 + \left(\frac{M}{S_e}\right)^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{3}} \tag{11}$$

The above result (Equation (11))—presented, e.g., in [22]—could be obtained directly, using the concept of equivalent static stress. If the von Mises criterion was used, the above result would have the form of Equation (2), also presented in the ANSI/ASME B106.1M:1985 standard [27] among many other literature sources.

For a shaft subjected to a combination of steady torque T and alternating bending M, Equations (2) and (11) give the von Mises- and Tresca-based fatigue dimensioning. For shafts, recall the relationships $\sigma = 32\,M/\pi d^3$ and $\tau = 16\,T/\pi d^3$.

Let us now recall the generalisations of those studies (e.g., Ref. [22] using the Tresca criterion or Refs. [8–10] using the von Mises criterion). The generalisation is

$$d = \left\{\frac{32n}{\pi}\left[\left(\frac{T_a}{S_e} + \frac{T_m}{S_y}\right)^2 + \left(\frac{M_a}{S_e} + \frac{M_m}{S_y}\right)^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{3}} \tag{12}$$

which, if T_a and M_m are null, becomes Equation (11). It is interesting to check the relationship between Equation (12) and the concept of equivalent static stress (Equations (4) and (5)), assuming $S_y/S_{Sy} = S_e/S_{Se}$ and using the Tresca criterion:

$$\frac{\sigma_{eq}}{2} = \frac{S_y}{2n} = \sqrt{\left(\frac{\sigma_m + \frac{S_y}{S_e}\sigma_a}{2}\right)^2 + \left(\tau_m + \frac{S_{Sy}}{S_{Se}}\tau_a\right)^2}$$

leads to Equation (12), thus demonstrating its origin.

Once some basic aspects of classical fatigue design have been recalled, it matters to examine the safety factors involved. From Shigley [22], "… *The methods discussed give* *reliable and conservative results when appropriate factors of safety are used. They can be used for the great majority of shaft-design problems*. . .". In this reference the formulation "*when appropriate factors of safety are used*" seems vague and warrants further examination.

Safety factor (n) values are not prescribed in the withdrawn standard ANSI/ASME B106.1M:1985 [27], but it recommends that n should be considerably higher than 1 if there are great uncertainties and the consequences of failure are serious.

According to Zahavi [32], ". . . *the ASME equation for transmission shafting* [Equation (11), possibly rewritten to include also a constant axial load] *is as conservative as the original Soderberg diagram* [. . .] *and, in spite of the acknowledged conservative results, it is widely used in machine design*. . . .".

As mentioned previously, in the 2019 edition of [5] Childs suggests safety factor values for shaft design; quoting verbatim,

> "*1.25 to 1.5 for reliable materials under controlled conditions subjected to loads and stresses known with certainty,*
>
> *1.5 to 2.0 for well-known materials under reasonably constant environmental conditions subjected to known loads and stresses,*
>
> *2.0 to 2.5 for average materials subjected to known loads and stresses,*
>
> *2.5 to 3.0 for less well-known materials under average conditions of load, stress, and environment,*
>
> *3.0 to 4.0 for untried materials under average conditions of load, stress, and environment, and*
>
> *3.0 to 4.0 for well-known materials under uncertain conditions of load, stress, and environment*".

The above considerations were already included in earlier books by that author, e.g., in 2004, [24].

In 2021, Childs [3] states that ". . .. *The ASME* [. . .] *design code for the design of transmission shafting (ANSI/ASME B106.1M-1985) provided an approach but was suspended in 1995 by the ASME as improved understanding and more sophisticated methodologies have become available allowing more precise modeling. The ASME procedure aimed to ensure that the shaft is properly sized to provide adequate service life* . . ." and then recalls that the designer must attend to other aspects as stiffness, vibration, misalignment, etc., but those other considerations are out of the scope of the present technical note. Later on, ". . . *As noted, the ASME design code has now been suspended, but this equation is included here due to its relative simplicity, and usefulness on occasion to provide a starting estimate for a shaft diameter. It should be noted that* [Equation (2)] *tends to underestimate the diameter required* . . .".

The above excerpts concerning safety factors are sufficient to show that a variety of comments concerning safety factors may be found, sometimes expressing contradictory statements, which does not facilitate the task of the designer looking for an expeditious preliminary design.

3. The Problem

What is then the problem addressed in this technical note? The problem consists of an ambiguity in the implementation of the reasoning expressed in the previous sections, as regards the way to consider the amplitude and mean values of normal stress (σ_a, σ_m) and of shear stress (τ_a, τ_m). The procedure involves consideration of an equivalent stress, based upon the Tresca or von Mises criterion. Bending moments, torsion, and eventually axial loads must frequently be considered in shaft design, whereby amplitude and mean values of normal and shear stresses must be calculated for the critical region of the shaft. Following the classical presentation (e.g., textbooks such as D'Angelo's [4] and papers such as [8,10]), the definition of equivalent static stresses as in Equations (4) and (5), leads to the already-presented Equation (12) (using the Tresca criterion).

However, a different sequence is found in the literature, firstly by calculating $\sigma_a{}'$ and $\sigma_m{}'$ as follows (again in the case of the Tresca criterion),

$$\sigma_a{}' = \sqrt{\sigma_a^2 + 4\tau_a^2}\,;\, \sigma_m{}' = \sqrt{\sigma_m^2 + 4\tau_m^2} \tag{13}$$

and, only afterwards, by using the Soderberg diagram as follows:

$$\frac{1}{n} = \frac{\sigma_a{}'}{S_e} + \frac{\sigma_m{}'}{S_y} \tag{14}$$

A similar approach, using an elliptic equation instead of Equation (14), was adopted by AGMA for its standard procedure for the fatigue design of shafts [33]. Some other presentations include alternative criteria for the consideration of mean stress effects, as Goodman, Gerber and others have, but for conciseness this technical note uses the Soderberg criterion only. In the case of the Tresca criterion, this leads to (see, e.g., Childs ([3,5]) and Budynas and Nisbett [23])

$$d^3 = \frac{32n}{\pi}\left[\frac{1}{S_e}\left(M_a^2 + T_a^2\right)^{\frac{1}{2}} + \frac{1}{S_y}\left(M_m^2 + T_m^2\right)^{\frac{1}{2}}\right] \tag{15}$$

which is to be compared with the previously obtained Equation (12), repeated here for convenience:

$$d^3 = \frac{32n}{\pi}\sqrt{\left(\frac{M_m}{S_y} + \frac{M_a}{S_e}\right)^2 + \left(\frac{T_m}{S_y} + \frac{T_a}{S_e}\right)^2}$$

As a summary of the present section, we note that Equation (1) is a key feature of the classical analysis. The question here is as follows: at which point of the analysis does it intervene? In some presentations, it intervenes in the stage of defining the equivalent static σ and τ (e.g., Equation (12)). In other presentations, it intervenes only after amplitude and mean equivalent stress have been characterised (as was the case in Equation (15)). Clearly, Equations (12) and (15) are different, and the engineer perusing the literature to find a prompt answer to dimensioning may not be aware of this diversity. Experimental data evaluate the accuracy of any criterion. However, for these two widely published approaches, the aim of the present technical note is to evaluate how different they are, and which one is more conservative. In the first approach, Equation (1) (Soderberg's equation) is used to derive the equivalent normal and shear stresses, which are subsequently combined according to the von Mises or Tresca criterion, whereas in the second approach, the von Mises or Tresca criterion is used first to calculate alternating and mean normal stresses, and only afterwards, Equation (1) is used. The current technical note aims to clarify the consequences of this difference. This is a question of great practical interest, given the possible differences in the safety factors involved. The answer to this question is given through a parametric analysis performed using Matlab, as follows.

4. Parametric Analysis and Discussion

Again, stress concentration and the many other parameters that influence the fatigue behaviour will not be taken into account in the following analysis and are assumed to be accounted for in the value of S_e. This analysis seeks to identify, ceteris paribus, the influence of the fatigue model used, i.e., Equation (12) vs. Equation (15). Recall that stresses are the variables considered, and everything else is assumed constant. In terms of stresses, Equation (12) becomes

$$n = \left[\sqrt{\frac{\sigma_m^2 + 4\tau_m^2}{S_y^2} + \frac{\sigma_a^2 + 4\tau_a^2}{S_e^2} + \frac{2(\sigma_a\sigma_m + 4\tau_a\tau_m)}{S_yS_e}}\right]^{-1} \tag{16}$$

and Equation (15) becomes

$$n = \left[\frac{1}{S_e} \sqrt{\sigma_a^2 + 4\tau_a^2} + \frac{1}{S_y} \sqrt{\sigma_m^2 + 4\tau_m^2} \right]^{-1} \tag{17}$$

Both previous equations were derived using the Tresca criterion. When using the von Mises criterion, instead of Equation (16) we have

$$n = \left[\sqrt{\frac{\sigma_m^2 + 3\tau_m^2}{S_y^2} + \frac{\sigma_a^2 + 3\tau_a^2}{S_e^2} + \frac{2(\sigma_a\sigma_m + 3\tau_a\tau_m)}{S_y S_e}} \right]^{-1} \tag{18}$$

and instead of Equation (17) we have

$$n = \left[\frac{1}{S_e} \sqrt{\sigma_a^2 + 3\tau_a^2} + \frac{1}{S_y} \sqrt{\sigma_m^2 + 3\tau_m^2} \right]^{-1} \tag{19}$$

To evaluate the consequences of these differences, several cases were studied using Matlab. In order to cover a variety of relevant situations, several combinations of loads were considered, with special interest dedicated to those leading to values of safety factors approximately within the 1 to 3 range, where the consequences of differences are deemed more critical. Table 1 indicates the combinations analysed: C means constant, and V means quasi-continuously varying.

In Figures 1–3, the von Mises criterion was used.

The data for Case I are σ_m = 0, 50, or 100 MPa; σ_a varying from zero up to 300 MPa; τ_a = 0 MPa, τ_m = 100 MPa, and the assumed material properties S_e = 250 MPa and S_y = 350 MPa.

Table 1. Load combinations examined.

	Case I	**Case II**	**Case III**	**Case IV**
σ_m, [MPa]	C—three values: 0, 50, 100	C—three values: 0, 25, 50	V	C—0
σ_a, [MPa]	V	C—100	C—100	V
τ_m, [MPa]	C—100	C—50	C—50	C—100
τ_a, [MPa]	C—0	V	C—three values: 0, 25, 50	C—0
S_e, [MPa]	C—250	C—250	C—250	C—250
S_y, [MPa]	C—350	C—350	C—350	C—two values: 300, 400

Figure 1 shows, for this idealised situation, the safety factor obtained using both approaches. Clearly, in the region of interest (i.e., a safety factor greater than 1), the approach of Equation (18) [22] returns a higher safety factor, indicating that the approach of Equation (19) is more conservative because—for exactly the same circumstances (boundary conditions)—it associates with a lower safety factor. The greater conservativeness of Equation (19), in this example, is observed looking at the maximum σ_a value for which the safety factor is $\geq$1; the figure indicates that Equation (19) is more restrictive.

For the assumed material properties previously used, Figure 1d shows the relative difference, defined as

$$\text{relat.diff.} = (n_1 - n)/n_1 \tag{20}$$

where n is the safety factor for Equation (19) and n_1 is the safety factor for Equation (18), both derived using the von Mises criterion. In this case, after reaching a peak, the relative difference shows a decreasing value as σ_a increases.

Figure 1a presents the case of a rotating shaft with constant bending and torsion moments, i.e., $\sigma_m = \tau_a = 0$, a case often found in industrial practice. Equation (2) (which is the relevant particular case of Equation (18)), giving the higher safety factor in Figure 1a, is frequently found in the literature, including the ASME standard [27]. Figure 1a highlights that the alternative procedure, Equation (19), leads to more conservative assessments since, for the same circumstances, the safety factor is lower.

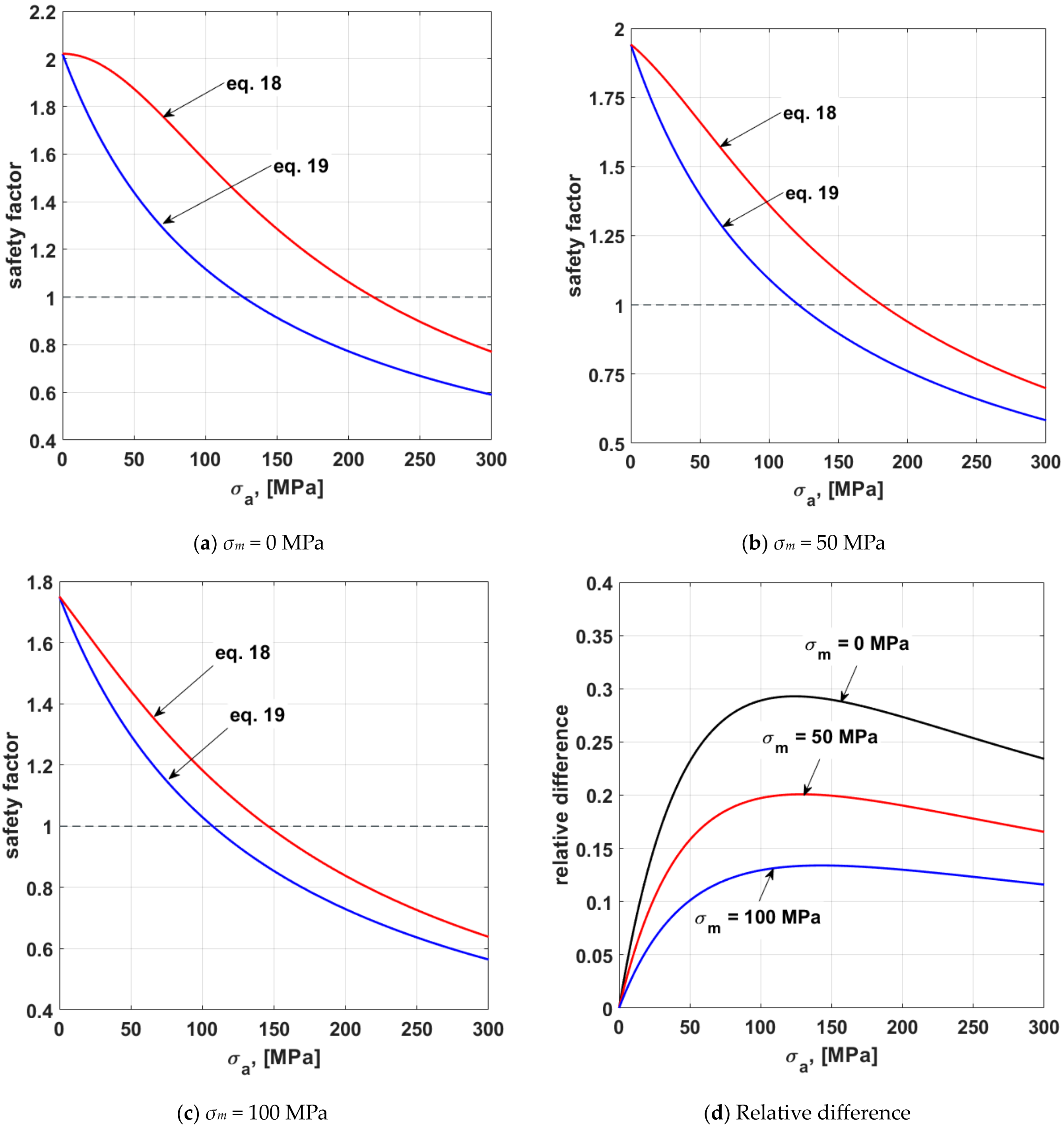


Figure 1. Case I of Table 1. Effect of σ_a on safety factor: comparison between Equations (18) and (19) (von Mises).

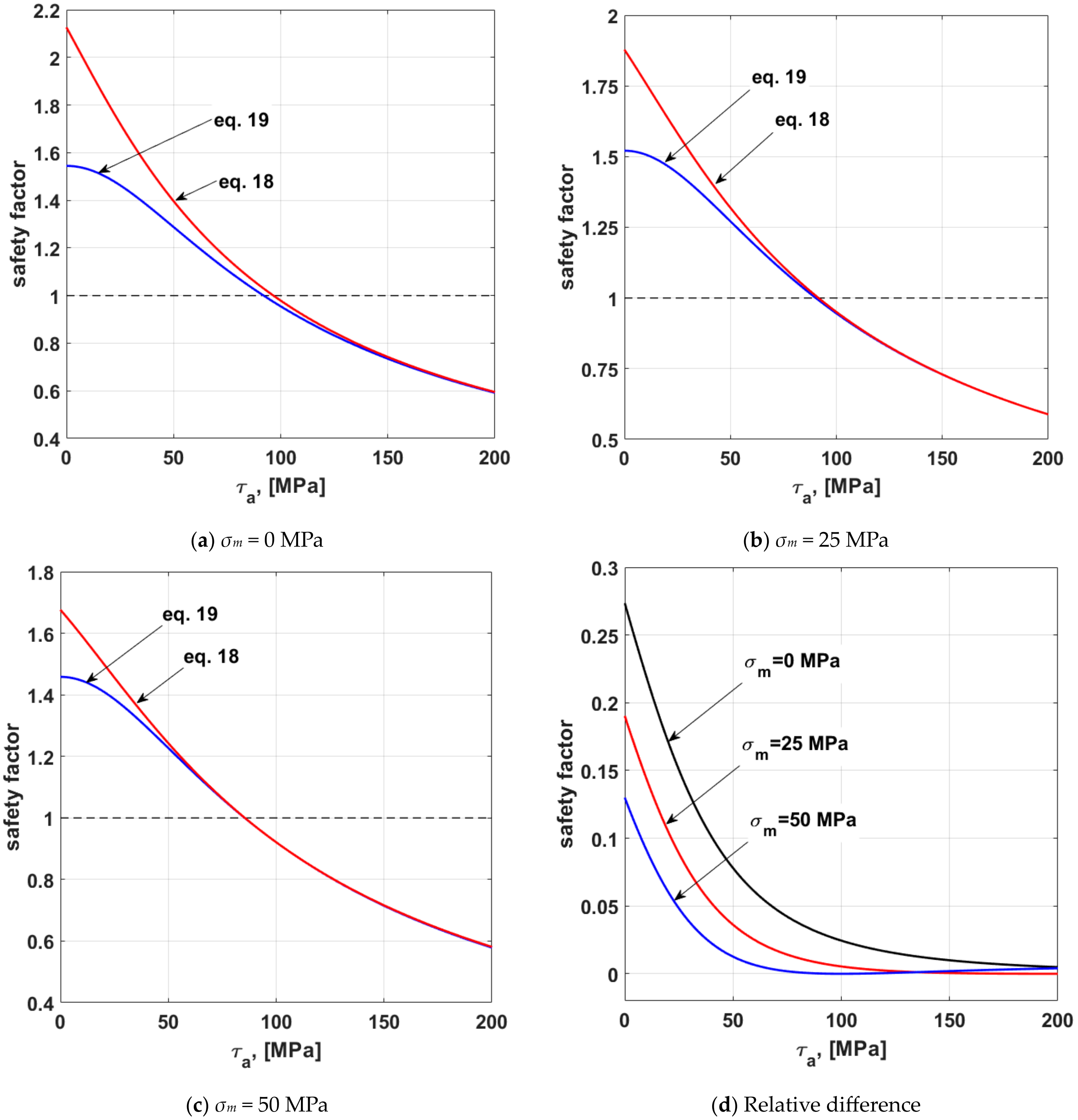
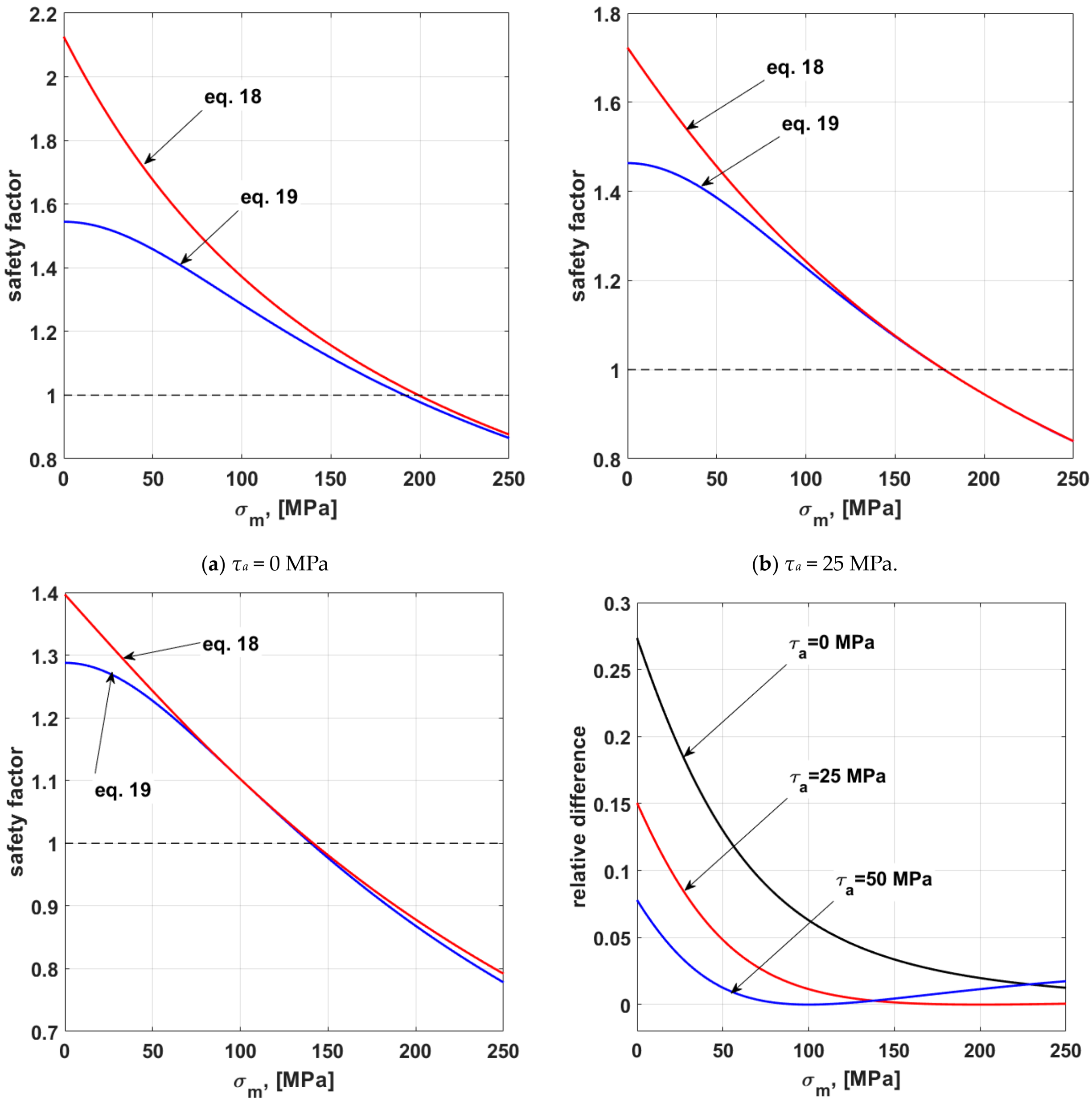


Figure 2. Case II of Table 1. Effect of τ_a on safety factor: comparison between Equations (18) and (19) (von Mises).

Figure 2 presents Case II of Table 1. The input data are σ_m = 0, 25, or 50 MPa; σ_a = 100 MPa, τ_a varying from zero up to 200 MPa; τ_m = 50 MPa, and, as before, the material properties S_e = 250 MPa and S_y = 350 MPa are assumed. As seen in Figure 2d, the relative difference in safety factors shows a decreasing value as τ_a increases (with a very minor exception of σ_m = 50 MPa, which is rather irrelevant because it already occurs in the region of n and $n_1 < 1$).

Again, assuming the material properties S_e = 250 MPa and S_y = 350 MPa, Figure 3 presents Case III, with τ_m = 50 MPa, σ_m varying from zero to 250 MPa, σ_a = 100 MPa, and τ_a = 0, 25, or 50 MPa.



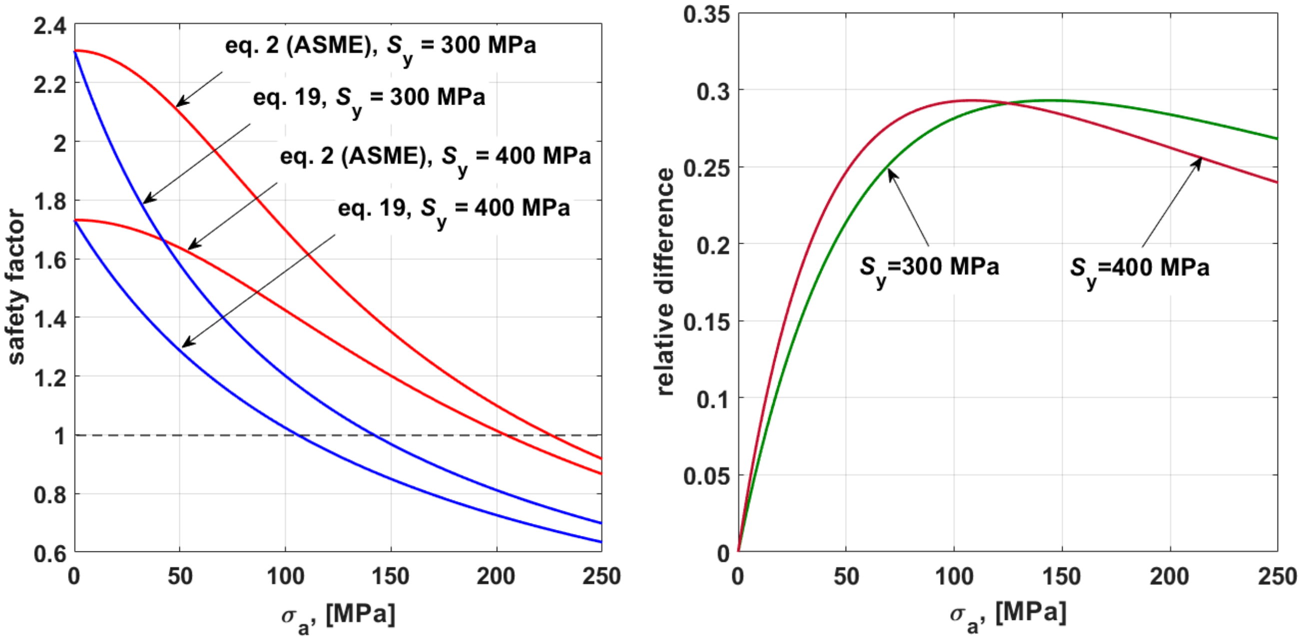
(**a**) τ_a = 0 MPa (**b**) τ_a = 25 MPa.

(**c**) τ_a = 50 MPa (**d**) Relative difference

Figure 3. Case III of Table 1. Effect of σ_m on safety factor: comparison between Equations (18) and (19) (von Mises).

In Cases II and III (Figures 2 and 3), the safety factors given by Equation (18) (n_1) and Equation (19) (n) tend to be approximately equal as the load increases. When safety values lower than one are reached (which is, of course, a situation without practical interest), n is almost identical to n_1. As in the previous cases, the relative difference in safety factors tends to decrease with increasing loading (which, in this case, means increasing τ_a). Again, a very minor exception is found for τ_a = 50 MPa, which is rather irrelevant because it already occurs in the region of $n < 1$).

Figure 4 deals with the situation where σ_m = 0 MPa with τ_a = 0 MPa and τ_m = 100 MPa, corresponding to a rotating shaft subjected to constant torsion and bending moments, in the absence of axial load. As a consequence, the stress ratio for σ is $R = -1$. In the analysis, σ_a varies from 0 to 250 MPa. Two values of yield strength were considered: 300 and 400 MPa (Case IV).



(**a**) Safety factor (**b**) Relative difference in safety factor

Figure 4. Case IV of Table 1. Effect of yield stress S_y on safety factor: comparison between Equations (2) and (19) (von Mises).

It is interesting to compare the influence of using the Tresca or von Mises criterion in the evaluation of the safety factor; recall Equations (16) and (18). This was performed for the following case: $\sigma_m = 0$, $\sigma_a = 100$ MPa, $\tau_a = 0$, τ_m varying from 25 to 225 MPa, $S_y = 350$ MPa and $S_e = 250$ MPa, as presented in Figure 5.

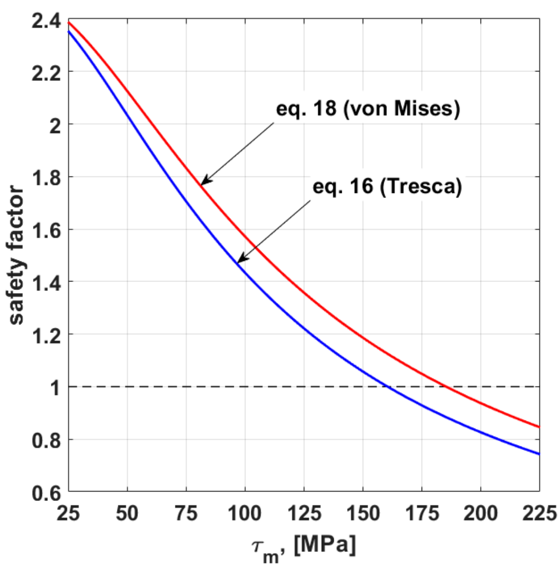


Figure 5. Influence of using the Tresca or von Mises criterion on the safety factor, for the case $\sigma_m = 0$, $\sigma_a = 100$ MPa, $\tau_a = 0$, τ_m varying from 25 to 225 MPa, $S_y = 350$ MPa and $S_e = 250$ MPa.

As mentioned before, Soderberg presents in [30] an analysis of the biaxial problem, which requires consideration of a volume element in the shaft surface (where stresses resulting from bending and torsion are higher). He considers a cut of that element by a plane whose normal makes an angle α with the shaft axis and considers the use of Equation (1) (now written in terms of shear stresses and strengths) for the shear stress state in that plane, using the equivalent Tresca stress. The resulting Equation (6) was mentioned before. In that equation, the safety factor is the minimum of function y. In a previous paper [29], the approach is thoroughly explained.

For the case $\sigma_m = 0$, $\sigma_a = 100$ MPa, $\tau_a = 0$, τ_m varying from 25 to 225 MPa, $S_y = 350$ MPa and $S_e = 250$ MPa, Figure 6 presents the function $y(\alpha)$ for several values of τ_m (50, 100, and 150 MPa). The minimum of each curve is the safety factor for the case considered. The values correspond exactly to the points of the Tresca curve of Figure 5.

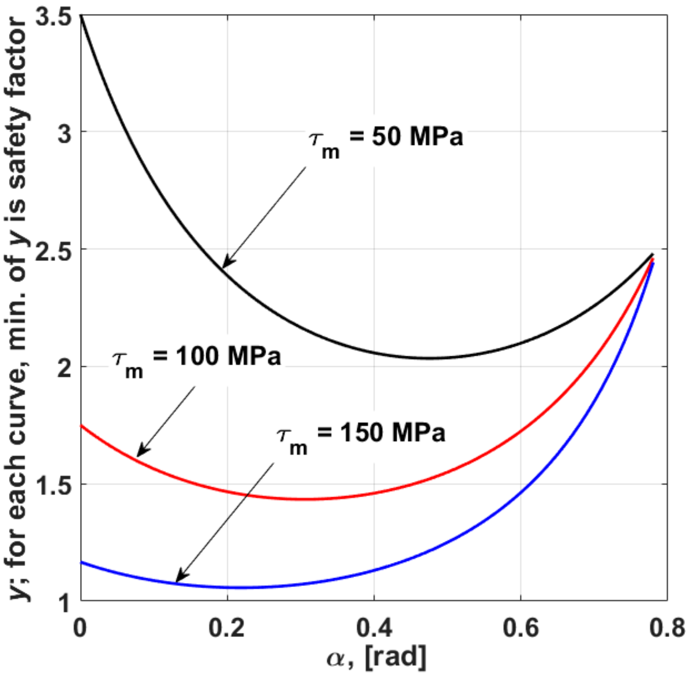


Figure 6. Examples of calculations of n as the minimum of $y(\alpha)$, Equation (6), following Soderberg's procedure [29] (Tresca criterion).

The work of Soderberg is widely mentioned in the technical literature, and the basic Equation (1) is universally known [34]. Ref. [30] is also included in an ASME book compiling design data [35]. The very recent (2023) inclusion of [29] in the ASME Digital Library makes it easily available now. As we saw before, the same content of Soderberg's paper [29] is presented by Shigley with a different (graphical) presentation, possibly simpler to follow [22,31].

We saw that some references state that the ASME approach to biaxial loading may be conservative (as the uniaxial Soderberg criterion clearly is); see Zahavi [32]. This is denied, e.g., by Childs [3], who states that the ASME approach—Equation (1)—"... *tends to underestimate the diameter required*". The diversity of opinions suggests that the classical approach should be used in more conservative formulations, and in any case, they should be taken as preliminary studies, reserving a more precise, detailed analysis using state-of-the-art methodologies, if advisable, for a subsequent final study.

5. Concluding Remarks

Introductory courses of the design of machine elements always include some treatment of fatigue design for shafts, which is, in its initial steps, a typical case of biaxial stress analysis. Although in recent decades several advanced criteria were developed to deal with that fatigue design problem, those introductory presentations are mostly based upon the use of some criterion to account for mean stress (Soderberg, Goodman, or other), the Tresca or von Mises concept of equivalent uniaxial stress, and a consideration of mean and alternating components of normal and of shear stresses. These considerations are typical parts of all textbooks on the design of machine elements. In those circumstances, two approaches to fatigue design in biaxial stress situations, both widely found in the literature, were compared in this technical note.

One approach consists of first (i) using the Soderberg line criterion to calculate a static equivalent to the normal stress and a static equivalent to the shear stress and (ii) finally use those equivalent normal and shear stresses in the Tresca or the von Mises criterion. The other approach involves (i) the calculation of mean and amplitude values of normal stress and of shear stress, (ii) the subsequent calculation of the equivalent Tresca or von Mises normal mean and alternating stress, and finally (iii) the use of the Soderberg line criterion, as presented in recent editions of Childs' or Shigley's books, for example. Under the same conditions (i.e., material properties, manufacturing conditions, loads, geometry, and dimensions), the second approach reveals lower safety factor values and is therefore more conservative than the first approach. This observation is of relevance in design for fatigue strength using the classical method as presented in the many sources of information currently available. Further work involving dedicated experiments may further elucidate this matter, but the results presented in this technical note, comparing the factor of safety implicit in two approaches, are of interest to practitioners seeking to achieve higher safety factors when designing according to the classical procedure.

Funding: This research received no external funding.

Data Availability Statement: All the data used are presented in this article.

Acknowledgments: The author thanks his past students of the Design of Machine Elements course at FEUP for their curiosity and inquisitive questions over the years.

Conflicts of Interest: The author declares no conflicts of interest.

Nomenclature

d	diameter
M	bending moment
M_a	amplitude of bending moment
M_m	mean value of bending moment
n	safety factor
n_1	safety factor
R	load ratio (max load/min load)
S_e	endurance limit ($R = -1$)
S_{se}	shear endurance limit ($R = -1$)
S_{sy}	shear yield strength
S_y	yield strength
T	torsion moment
T_a	amplitude of torsion moment
T_m	mean value of torsion moment
σ	normal stress
σ_a	normal stress amplitude (alternating normal stress)
σ_m	mean value of normal stress
τ	shear stress
$\tau_{\alpha a}$	alternating shear stress in a plane characterised by angle α
$\tau_{\alpha m}$	mean value of shear stress in a plane characterised by angle α
τ_a	shear stress amplitude (alternating shear stress)
τ_m	mean value of shear stress
Acronyms	
AGMA	American Gear Manufacturers Association
ANSI	American National Standards Institute
ASME	American Society of Mechanical Engineers
CEMA	Conveyor Equipment Manufacturers Association
HCF	high-cycle fatigue
NASA	National Aeronautics and Space Administration

References

1. Harris, D.; Jur, T. Classical fatigue design techniques as a failure analysis tool. *J. Fail. Anal. Prev.* **2009**, *9*, 81–87. [CrossRef]
2. Budynas, R.G.; Nisbett, J.K.; Tangchaichit, K. *Shigley's Mechanical Engineering Design. Eleventh Edition in SI Units*; McGraw Hill Education: New York, NY, USA, 2021.
3. Childs, P.R.N. *Mechanical Design: Theory and Applications*, 3rd ed.; Elsevier/Butterworth-Heinemann: Oxford, UK, 2021.
4. D'Angelo, A. *Machine Design for Technology Students. A Systems Engineering Approach*; (Synthesis Lectures on Mechanical Engineering); Springer Nature: Cham, Switzerland, 2021. [CrossRef]
5. Childs, P.R.N. *Mechanical Design Engineering Handbook*, 2nd ed.; Elsevier/Butterworth-Heinemann: Oxford, UK, 2019.
6. Beswarick, W.J. *Shaft with Fluctuating Loads*; SEED—Sharing Experience in Engineering Design; 1988; MPT 6.2.
7. Beswarick, W.J. Shaft with fluctuating load. In *Rotary Power Transmission Design*; Hurst, K., Ed.; McGraw Hill: London, UK, 1994; pp. 142–148.
8. Orthwein, W.C. Estimating fatigue due to cyclic multiaxial stress. *J. Vib. Acoust. Stress Reliab. Des.* **1987**, *109*, 97–102. [CrossRef]
9. Loewenthal, S.H. NASA RP-1123. Design of Power-Transmitting Shafts. NASA, July 1984. Available online: https://ntrs.nasa.gov/citations/19840018973 (accessed on 1 October 2024).
10. Loewenthal, S.H. Factors that affect the fatigue strength of power transmission shafting and their impact on design. *J. Mech. Transm. Autom. Design. Trans. ASME* **1986**, *108*, 106–114. [CrossRef]
11. de Castro, P.M.S.T.; Fernandes, A.A. Methodologies for failure analysis: A critical survey. *Mater. Des.* **2004**, *25*, 117–123. [CrossRef]
12. d'Isa, F.A. *Mechanics of Metals*; Addison-Wesley: Reading, MA, USA, 1968.
13. Hall, A.S.; Holowenko, A.R.; Laughlin, H.G. *Theory and Problems of Machine Design*; McGraw-Hill: New York, NY, USA, 1961.
14. Spotts, M.F. *Design of Machine Elements*, 3rd ed.; Prentice-Hall: Englewood Cliffs, NJ, USA, 1962.
15. Milella, P.P. *Fatigue and Corrosion in Metals*; Springer: Milan, Italy, 2013.
16. Lee, Y.L.; Pan, J.; Hathaway, R.; Barkey, M. *Fatigue Testing and Analysis. Theory and Practice*; Elsevier Butterworth–Heinemann: Oxford, UK, 2005.
17. Papuga, J.; Margetin, M.; Chmelko, V. Variable amplitude loading and their effect on fatigue life and fatigue life computation. *Fatigue Fract. Eng. Mater. Struct.* **2021**, *44*, 2890–2912. [CrossRef]
18. Anes, V.; de Freitas, M.; Reis, L. The damage scale concept and the critical plane approach. *Fatigue Fract. Eng. Mater. Struct.* **2017**, *40*, 1240–1250. [CrossRef]
19. da Silva, F.A.C.; de Castro, P.M.S.T. Redesign of a failed hoisting shaft of a vertical transfer device. *Eng* **2023**, *4*, 1981–2002. [CrossRef]
20. Tavares, S.M.O.; de Castro, P.M.S.T. A comparison of methodologies for fatigue analysis of shafts: DIN 743 vs. approaches based on Soderberg criterion. *Cienc. Tecnol. Dos Mater.* **2017**, *29*, e76–e81. [CrossRef]
21. Henriques, B.; Carvalho, M.; Tavares, S.M.O.; de Castro, P.M.S.T. A comparison of safety factor values for Soderberg and DIN 743 fatigue analyses. *U. Porto J. Eng.* **2021**, *7*, 11–21. [CrossRef]
22. Shigley, J.E. *Mechanical Engineering Design*, 3rd ed.; McGraw-Hill Kogakusha: Tokyo, Japan, 1977.
23. Budynas, R.G.; Nisbett, J.K. *Shigley's Mechanical Engineering Design*, 9th ed.; McGraw-Hill: New York, NY, USA, 2011.
24. Childs, P.R.N. *Mechanical Design*, 2nd ed.; Elsevier: Amsterdam, The Netherlands, 2004.
25. Juvinall, R.C.; Marshek, K.M. *Fundamentals of Machine Component Design*, 6th ed.; John Wiley & Sons: Hoboken, NJ, USA, 2017.
26. Pilkey, W.D.; Pilkey, D.F. *Peterson's Stress Concentration Factors*, 3rd ed.; John Wiley & Sons: Hoboken, NJ, USA, 2008.
27. *ANSI/ASME B106.1M:1985*; Design of Transmission Shafting. ANSI: Washington, DC, USA; ASME: Washington, DC, USA, 1985.
28. *ANSI/CEMA B105.1-2015*; Specifications for Welded Steel Conveyor Pulleys with Compression Type Hubs. ANSI: Washington, DC, USA; CEMA: Washington, DC, USA, 2015.
29. Soderberg, C.R. Factor of safety and working stress. *Trans. ASME* **1930**, *52*, 13–21. [CrossRef]
30. Soderberg, C.R. Working stresses. *J. Appl. Mech.* **1935**, *2*, A-106–A-108. [CrossRef]
31. Shigley, J.E. *El Proyecto en Ingeniería Mecánica*; Ediciones del Castillo SA: Madrid, Spain, 1970.
32. Zahavi, E. *Fatigue Design: Life Expectancy of Machine Parts*; CRC Press: Boca Raton, FL, USA, 1996.
33. *ANSI/AGMA 6001-D97*; Design and Selection of Components for Enclosed Gear Drives. ANSI: Washington, DC, USA; AGMA: Washington, DC, USA, 1997.
34. Shapiro, A.H. Carl Richard Soderberg 1895–1979. A biographical memoir by Ascher H. Shapiro. In *Biographical Memoirs*; The National Academies Press: Washington, DC, USA, 1998; Volume 75.
35. Lessels, J.M. (Ed.) *Design Data and Methods. Applied Mechanics*; The American Society of Mechanical Engineers: New York, NY, USA, 1953.