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Multi-State Synchronization of Chaotic Systems with Distributed Fractional Order Derivatives and Its Application in Secure Communications

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Abstract: This study investigates multiple synchronizations of distributed fractional-order chaotic systems. These systems consider unknown parameters, disturbance, and time delays. A robust adaptive control method is designed for multistage distributed fractional-order chaotic systems. In this paper, system parameters are changed step by step. Using Lyapunov's function, while the synchronization error convergence to zero is guaranteed, adaptive rules are designed to estimate the parameters. Then, a secure communication scheme is proposed using the new chaotic masking method. Finally, the simulations are performed on a chaotic system of distributed Duffing fractional order. The results show the high efficiency of the proposed synchronization scheme using robust adaptive control, despite the parametric uncertainties, external disturbance, and variable and unknown time delays. Then, the simulations were performed on the sinusoidal signals of the message in the application of secure communications. The results showed the success of the proposed masking scheme with synchronization in coding and decoding information.

Keywords: adaptive-robust synchronization; Lyapunovstability; time delay; secure communication; chaotic masking

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1. Introduction

Chaotic systems are nonlinear systems sensitive to their initial conditions. One of the applications of chaos is the synchronization of chaotic systems in which the state paths of a slave system follow the state paths of a master system. Synchronization of chaotic systems is used in secure communications [1]. As an advanced form of correct order calculations, fractional calculations date back to the 17thcentury. However, their application in control science dates back to recent centuries. Fractional differential equations give a more precise description than integer equations. Fractional order equations have memory and inheritance properties that provide better descriptions for different processes compared to correct order calculations. In recent years, the synchronization of fractional chaotic systems has received much attention [2–6]. For instance, Mirrezapour et al. (2021) used a new fractional slide mode controller based on the PID controller structure to synchronize the fractional-order chaotic system with uncertainty and disturbance [7]. Also, a new fractional-order integral sliding mode control has been used to stabilize and synchronize N-dimensional chaotic systems [8]. Comparative analysis between previous control techniques was performed through different time-domain functions such as sitting time, error indexes, and control energy measurement. In another study, Wu et al. [9] performed image synchronization of the inner layer of the finite-time derivative of the Caputo fraction of two-layer networks using a sliding mode. Yadav et al. (2021) investigated the exponential synchronization of fractional-order combined chaotic systems and their applications in digital cryptography [10].

A distributed order differential equation is a continuous generalization of the time-fractional order. If the density function takes a single impulse function in the distributed integral, the time-differential equation of the distributed order is reduced to a single order type. However, if the linear combination of several impact functions is considered a non-negative density function, then the multiple expression is retrieved [11]. This study presents distributed-order fractional calculus (DOFC) mathematics, including analytical and numerical methods. Then, the applications of DOFC were presented in the fields such as viscoelasticity, transfer processes, and control theory. Katsikadlis (2014) proposed a numerical solution for solving distributed fractional differential equations [12]. Later, Li et al. (2016) analyzed the solutions of a distributed time-fraction differential equation and its application to an inverse problem [13]. In [14], the numerical solution of the fractional equation with the Bagley-Torvik distributed order was presented. These authors offered two numerical methods for solving the Bagley-Torvik equation of distributed order fraction. In [15], a numerical solution is presented for distributed fractional order equations by hybrid functions. This method is based on the approximation of hybrid functions. The Riemann-Liouville operator was used for hybrid functions. Afterward, this operator was used to convert the distributed fractional differential equations to a system of algebraic equations. In [16], stability analysis of distributed fractional order differential equations was performed. The authors analyzed the stability of three classes of distributed order fractional equations according to the non-negative density function. In [17], the time variable distributed fractional-order equation was studied, analyzed, and approximated. In [18], the distributed fractional-order equation was studied in finite domains. This equation provides robust solutions and random analogs for distributed fractional order equations in finite domains with Dirichlet boundary conditions. Furthermore, the research [19] represent other studies on distributed fractional order.

Multi-mode synchronization is another type of synchronization that has attracted much attention among researchers. This technique is more complex and therefore more practical than conventional synchronization. In this type of synchronization, two or more slave systems follow a master system. In [20], chaotic control and anti-synchronization of the combined were performed on a new fractional chaotic system. Synchronization was performed on 12 fractional-order chaotic systems using a sliding mode. Then, the synchronization technique was implemented as an application of secure communications [21,22]. In their design, the state variables of the two master systems are synchronized with the state system variables of the slave system. Ricky Taki and Windy's chaotic systems were considered the master systems, and Chen's system was the slave system in the uncertain presence of the parameters [23].

In [24], chaotic absorbers that exist only in the fractional-order state have been investigated. Self-excited chaotic attractors were also considered. The results were supported by the calculation of the set of attractors, bifurcation, and Lyapunov exponent spectrum. Advanced applications of fractional differential operators in science and technology were studied in [25]. A new fractional-order chaotic system was introduced in [26]. Lyapunov exponent, Lyapunov spectrum, and bifurcation diagrams were calculated and chaotic attractors were investigated. Then, its application in linear control was explained. A non-standard finite difference scheme for unknown modelling and synchronization of a new fractional-order chaotic system including quadratic features was studied in [27]. In [28], the stability of Routh-Hurwitz and periodic Gaussian attractors in a fractional order model was investigated along with the application of N in the COVID-19 epidemic. In [29], dynamic analysis and adaptive synchronization were performed using a new fuzzy adaptive sliding mode control method. In [30], the control and synchronization problem of fractional-order chaotic satellite systems was studied using adaptive control and feedback techniques. The new Routh-Hurwitz stability conditions were applied for arbitrary orders. Also, the local stability of arbitrary orders was checked. The conditions for approximating the periodic solution in this model were discussed with Hopf's bifurcation theory [31]. The characteristics of nonlinear reverberation such as the presence of chaos were proved with the help of bifurcation diagrams, Lyapunov expressions, and Marotto's theorem [32].

Secure communications were implemented by Khan et al. (2020), using a parallel synchronization technique on a new fractional-order chaotic system [33]. Yu et al. (2019) presented an analysis and realization of a five-dimensional hyper-chaotic four-blade memristor system using active control synchronization with its application in secure communications [34]. Zhao et al. [35] performed an observer-based synchronization of chaotic systems with additional square constraints and evaluated its application in secure communications. In another study, observer-based synchronization of chaotic systems was studied by considering additional constraints and their application in secure communications. Samimi et al. (2020) provided secure communications based on chaotic synchronization using emotional learning [36]. Luo et al. (2019) used time-limited modified image synchronization of multi-mode unknown chaotic systems and their application in secure communications using DNA encoding [37]. Secure communication using matrix image synchronization was provided by Khan et al. [38].

According to previous studies, using adaptive control multi-mode synchronizations for fractional-order chaotic systems has not been considered, despite their uncertainties and time-varying parameters. The main part of this research is using distributed fractional-order derivatives instead of ordinary fractional-order derivatives, which have not been considered in previous similar studies. The stability of the distributed fraction order system was proved by Lyapunov's theorems and theory. This paper considers parametric and structural uncertainties, disturbance, and unknown delays. Using the obtained adaptive rules, the stability of the closed-loop system in the presence of disturbance and uncertainty is guaranteed. In addition, the chaotic masking method uses distributed fractional order synchronization in secure telecommunications to transmit the signal reliably.

The performance of the proposed method was tested by performing some simulations based on three distributed fractional-order duffing systems with unknown parameters, external disturbance, and unknown delay. Then, this distributed fraction order synchronization scheme is used in secure communications. Finally, the results of secure communications are tested on signals. The results show the robustness of the proposed adaptive control scheme. The results also indicate the stability of the system over synchronization time.

The remainder of this paper is organized as follows. Section 2 includes the formulation of the proposed method. In Section 3, the concepts related to the comparative synchronization of the multi-mode distributed fractional order are explained by considering the unknown parameters in the transition state. Then, the simulation results are given in Section 4. Finally, in Section 5, simulations related to secure communications are tested on message signals.

2. Problem Formulation

Distributed fractional-order derivatives are special and generalized forms of fractional-order models which are used to describe the equations of viscoelastic systems well. At first in this section, the first distributed fractional-order systems are presented. Then, multiple synchronizations of distributed fractional-order chaotic systems are proposed. The adaptive controller is designed to estimate unknown parameters using Lyapunov's theory of stability.

2.1. Distributed Fractional-Order Derivative

Several numerical definitions have been proposed to solve differential equations based on simple implementation and performance in research [39]. In this article, the definition of Caputo is used. The definition of the Caputo fraction derivative is as follows [39]:

$$D^q f(x) = I^{\beta-q} h^{(\beta)}(x), q > 0 \tag{1}$$

where $h^{(\beta)}$ represents the derivative of the order βth of $h(x)$, β = [q] is the integer component of first number that is less than q. The distributed fractional order is described as follows [40]:

$$I^q g(x) = \frac{1}{\Gamma(q)} \int_0^x (x-t)^{q-1} g(t)dt, q > 0 {}_{do}D^q x(t) = \int_0^1 m(q) D^q x(t) dq, \ \ m(q) > 0 \tag{2}$$

where $\Gamma(q)$ is a gamma function and the density function is defined on the interval [0, 1]. ${}_{do}D^q$ is a distributed fractional derivative operator. If q is constant, it becomes a normal fractional-order derivative, but if q is allowed to distribute, that is, it changes in a range, then it becomes a distributed fractional-order derivative which range is denoted by $m(q)$. Stability analysis of distributed fractional systems is guaranteed by Lyapunov's direct method and determination of necessary and sufficient stability conditions with the concept of Mittag-Leffler [41] and stability analysis based on convex Lyapunov functions for nonlinear systems is shown [42].

Lemma 1. [41–43]: *Suppose $h(t) \in R$ is a continuous and derivative function. Then we have for $t \geq t_0$:*

$${}_{do}D^q h^2(t) \leq 2h(t) \cdot {}_{do}D^q h(t) \tag{3}$$

Lemma 2. [41–43]: *Suppose $h(t) \in R^n$ is a derivative and continuous function. Then we have for $t \geq t_0$:*

$${}_{do}D^q h^T(t) h(t) \leq 2h^T(t) \cdot {}_{do}D^q h(t) \tag{4}$$

Theorem 1. [44,45]: *Assume that x = 0 is the equilibrium point of the distributed fractional order system (5) and the definition domain includes the origin. Assume that $V(t.x(t))$ is a derivative and continuous Lipschitzfunction*:

$${}_{do}D^q x(t) = f(x.t) \tag{5}$$

$$a_1 \|x\|^a \leq V(t.x(t)) \leq a_2 \|x\|^{ab} \tag{6}$$

$${}_{do}D^q V(t.x(t)) \leq -a_3 \|x\|^{ab} \tag{7}$$

where $0 < q < 1$ and a_1. $a_2.a_3.a.b$ are arbitrary and positive constants. Then x = 0 is stable in terms of Mittag-Leffler.

Theorem 2. [45,46]: *For a distributed fractional order system, Lyapunov $V(x)$ is as follows:*

$${}_{do}D^q V(t.x(t)) \leq \left(\frac{\partial V}{\partial x}\right)^T \cdot {}_{do}D^q x(t) = \left(\frac{\partial V}{\partial x}\right)^T \cdot f(x.t) \tag{8}$$

Since the implementation of Equation (2) is continuously difficult, the following special cases are considered:

(1) If $m(q) = \delta(q - q_0)$ (where $\delta(q - q_0)$ is the impact function), we have the impact function based on the screening property:

$${}_{do}D^q x(t) = \int_0^1 \delta(q - q_0) D^q x(t) dq = D^{q_0} x(t) \tag{9}$$

(2) If $m(q) = \sum_{i=1}^k \delta(q - q_i)$, then:

$${}_{do}D^q x(t) = \int_0^1 \sum_{i=1}^k \delta(q - q_i) D^q x(t) dq = \sum_{i=1}^k D^{q_i} x(t) \tag{10}$$

In this case, sincewe have several countable derivatives, it can be implemented.

2.2. Multi-State Adaptive Synchronizations

Figure 1 shows the multi mode synchronizations of a master system with several slave systems.

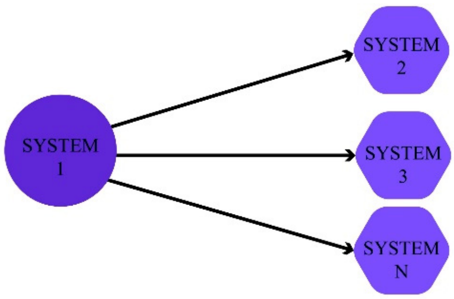

Figure 1. Multi-mode synchronizations of a master distributed system with several slave systems.

The chaotic system of the master distributed fractional-order is with unknown parameters as follows [43,44]:

$$ {}_{do}D^q x_1(t) = \int_0^1 m(q) D^q x_1(t) = f_1(x_1) + H_1(x_1)\theta_1(t) \tag{11} $$

where $x_1(t) = [x_{11}.x_{12}\cdots x_{1n}]^T$ are system state vectors, $f_1(x_1(t)) = [f_{11}.f_{12}\cdots f_{1n}]^T$ is a continuous function. $H_1(x_1(t)) = [H_{11}.H_{12}.\cdots .H_{1n}]^T$ is the matrix function and $\theta_1(t) = [\theta_{11}.\theta_{12}\cdots\theta_{1n}]^T$ is the master system parameters. They are unknown and variable.

The N-1 chaotic systems with control functions are as follows [45,46]:

$$ {}_{do}D^q x_i(t) = \int_0^1 m(q) D^q x_i(t) = f_i(x_i) + H_i(x_i)\theta_i(t) + u_{i-1}(t) \qquad i = 2.3\ \cdots N \tag{12} $$

where $x_i(t) = [x_{i1}.x_{i2}\cdots x_{in}]^T$ the state vector of ith system, $f_i(x_1(t)) = [f_{i1}.f_{i2}.\cdots .f_{in}]^T$ continuous function, $H_i(x_i(t)) = [H_{i1}.H_{i2}\cdots H_{in}]^T$ the matrix function, $\theta_i(t) = [\theta_{i1}.\theta_{i2}\cdots\theta_{in}]^T$ the basic parameters of ith slave system and $u_{i-1}(t) = [u_{i-1.1}(t).u_{i-1.2}(t)\cdots u_{i-1.n}(t)]^T$ are control function of ith slave system. Therefore, according to Equations (11) and (12), the synchronization of the chaotic system with the control function is stated as follows:

$$ \begin{cases} {}_{do}D^q x_1(t) = \int_0^1 m(q) D^q x_1(t)dq = f_1(x_1) + H_1(x_1)\theta_1 \\ {}_{do}D^q x_2(t) = \int_0^1 m(q) D^q x_2(t)dq = f_2(x_2) + H_2(x_2)\theta_2 + u_1(t) \\ \vdots \\ {}_{do}D^q x_N(t) = \int_0^1 m(q) D^q x_N(t)dq = f_N(x_N) + H_N(x_N)\theta_N + u_{N-1}(t) \end{cases} \tag{13} $$

In multi-mode synchronization form, the distribution synchronization error is given as follows:

$$ e_{i-1}(t) = x_i(t) - x_1(t)\ i = 2.3\cdots N $$

Definition 1. *For N chaotic systems of the distribution fractional order expressed by (13), if the adaptive controllers u_ (i − 1) (t) exist as follows, the error dynamics are defined as follows:*

$$ {}_{do}D^q e_{i-1}(t) = \int_0^1 m(q) D^q e_{i-1}(t)dq = f_i(x_i) - f_1(x_1) + H_i(x_i)\theta_i - H_1(x_1)\theta_1 + u_{i-1}(t) \qquad i = 2.3\ \cdots N-1 \tag{14} $$

The required conditions are:

$$ \lim_{t\to\infty} \|e_{i-1}(t)\| = \lim_{t\to\infty} \|x_i(t) - x_1(t)\|\ \to 0\ i = 2.3\cdots N $$

If established, then multiple synchronizations between N distribution chaotic systems with unknown parameters are achieved. The design of controllers and adaptive rules for achieving the goal based on the Lyapunov function is established by the synchronization of the transition mode. The control rule for $u_1(t).u_2(t).u_3(t)\cdots u_{N-1}(t)$ is designed as follows:

$$u_{i-1}(t) = -f_i(x_i) + f_1(x_1) - H_i(x_i)\hat{\theta}_i + H_1(x_1)\hat{\theta}_1 + k_{i-1}e_{i-1} \qquad i = 2.3\ \cdots N-1 \quad (15)$$

Therefore, errors of dynamics distribution chaotic system are given as follows:

$$_{do}D^q e_{i-1}(t) = H_i(x_i)\widetilde{\theta}_i - H_1(x_1)\widetilde{\theta}_1 + k_{i-1}e_{i-1} \qquad i = 2.3\ \cdots N-1 \quad (16)$$

where $\hat{\theta}_i$ is estimated and $\widetilde{\theta}_i(t) = \theta_i(t) - \hat{\theta}_i(t)$ is an approximate distribution system error as follows:

$$k_{i-1} < 0.$$

2.3. Synchronizations of Distributed Fractional Order in the Presence of Disturbance, Unknown Time Delay and Uncertaintyin Systems

The Distributed master and slave system with disturbance, time delay and uncertainty are as follows:

$$\begin{cases} {}_{do}D^q x_1(t) = f_1(x_1) + H_1(x_1)\theta_1 + F_1(x_1(t-\tau_1)) + \Delta f_1(x_1) + D_1(t) \\ {}_{do}D^q x_2(t) = f_2(x_2) + H_2(x_2)\theta_2 + F_2(x_2(t-\tau_2)) + \Delta f_2(x_2) + D_2(t) + u_1(t) \\ \vdots \\ {}_{do}D^q x_N(t) = f_N(x_N) + H_N(x_N)\theta_N + F_N(x_N(t-\tau_N)) + \Delta f_N(x_N) + D_N(t) + u_{N-1}(t) \end{cases} \quad (17)$$

It is assumed that $F_i(x_i(t-\tau_i))$ are Lipschitz, uncertainties and disturbances are bounded but with an unknown boundary:

$$\begin{array}{l} |\Delta f_i(x_i)| \le \gamma_i g_i(x_i)\,. \qquad |D_i(t)| \le d_i \qquad i = 1.2\ldots N \\ |F_i(x_i(t-\tau_i)) - F_i(x_i(t-p_i))| \le l_i|\tau_i - p_i| \end{array}$$

where γ_i, l_i and d_i are constant but unknown and $g_i(x_i)$ is definite and positive. The error dynamic ofdistributed system is described as follows:

$$\begin{aligned} {}_{do}D^q e_{i-1}(t) = & \int_0^1 m(q) D^q e_{i-1}(t)dq \\ & = f_i(x_i) - f_1(x_1) + H_i(x_i)\theta_i - H_1(x_1)\theta_1 + F_i(x_i(t-\tau_i)) - F_1(x_1(t-\tau_1)) \\ & + \Delta f_i(x_i) - \Delta f_1(x_1) + D_i(t) - D_1(t) + u_{i-1}(t) \qquad i = 2.3\ \cdots N-1 \end{aligned} \quad (18)$$

$$\begin{aligned} u_{i-1}(t) = & -f_i(x_i) + f_1(x_1) - H_i(x_i)\hat{\theta}_i + H_1(x_1)\hat{\theta}_1 + k_{i-1}e_{i-1} - F_i(x_i(t-\hat{\tau}_i)) \\ & + F_1(x_1(t-\hat{\tau}_1)) + \overline{u}_{i-1}(t) \quad i = 2.3\ \cdots N-1 \end{aligned} \quad (19)$$

where $\hat{\theta}_i, \hat{\tau}_i$ are estimations of θ_i, τ_i and $\overline{u}_{i-1}(t)$ is the part of the control function ofdistributed system, which is introduced below. By placing the control function in (18), the error dynamics are given as follows:

$$\begin{aligned} {}_{do}D^q e_{i-1}(t) = & H_i(x_i)\widetilde{\theta}_i - H_1(x_1)\widetilde{\theta}_1 + \Delta f_i(x_i) - \Delta f_1(x_1) + D_i(t) - D_1(t) + k_{i-1}e_{i-1} + F_i(x_i(t-\tau_i)) \\ & - F_1(x_1(t-\tau_1)) - F_i(x_i(t-\hat{\tau}_i)) + F_1(x_1(t-\hat{\tau}_1)) + \overline{u}_{i-1}(t), \qquad i = 2.3\ \cdots N-1 \end{aligned} \quad (20)$$

Theorem 3. *Distributed system error dynamics (20) under control law (42) and the update rules (36–41) are stable and the synchronization errors converge to zero despite uncertainty, time delay and disturbance.*

Proof. By explaining Lyapunov function of distributed system as follows:

$$V = \frac{1}{2}(V_e + V_\theta + V_\gamma + V_d + V_\tau) \quad (21)$$

where in:

$$V_e = \sum_{i=2}^{N} k_{i-1} e_{i-1}^T e_{i-1}, V_\theta = \sum_{i=2}^{N} \widetilde{\theta}_i^T \widetilde{\theta}_i + \widetilde{\theta}_1^T \widetilde{\theta}_1$$

$$V_\gamma = \sum_{i=2}^{N} \widetilde{\gamma}_i{}^2 + \widetilde{\gamma}_1{}^2 \,,\ V_d = \sum_{i=2}^{N} \widetilde{d}_i{}^2 + \widetilde{d}_1{}^2 \,, V_\tau = \sum_{i=2}^{N} l_i \widetilde{\tau}_i{}^2 + l_1 \widetilde{\tau}_1{}^2$$

$$\widetilde{\theta}_i = \theta_i - \hat{\theta}_i,\ \widetilde{\gamma}_i = \gamma_i - \hat{\gamma}_i,\ \widetilde{d}_i = d_i - \hat{d}_i,\ \widetilde{\tau}_i = \tau_i - \hat{\tau}_i\ .$$

□

By calculating the distributed fractional derivative of Lyapunov function and replacing control function (21):

$$\begin{aligned} {}_{do}D^q V \le \sum_{i=2}^{N} [e_{i-1}^T \Big(H_i(x_i)\widetilde{\theta}_i - H_1(x_1)\widetilde{\theta}_1 + \Delta f_i(x_i) - \Delta f_1(x_1) + F_i(x_i(t-\tau_i)) - F_i(x_i(t-\hat{\tau}_i)) - F_1(x_1(t-\tau_1)\,) \\ + F_1(x_1(t-\hat{\tau}_1)) + D_i(t) - D_1(t) + k_{i-1}e_{i-1} \big) + \widetilde{\theta}_i^T{}_{do}D^q\widetilde{\theta}_i + \widetilde{\gamma}_i{}_{do}D^q\widetilde{\gamma}_i + \widetilde{d}_i{}_{do}D^q\widetilde{d}_i + l_i\widetilde{\tau}_i{}_{do}D^q\tau_i \\ + \overline{u}_{i-1}(t)] + \widetilde{\theta}_1^T{}_{do}D^q\widetilde{\theta}_1 + \widetilde{\gamma}_1{}_{do}D^q\widetilde{\gamma}_1 + \widetilde{d}_1{}_{do}D^q\widetilde{d}_1 + l_1\widetilde{\tau}_1{}_{do}D^q\tau_1 \end{aligned} \tag{22}$$

If $\overline{u}_i{}^j$, $\Delta f_i{}^j, D_i{}^j,\ e_{i-1}^j$ is the component jth of the vectors $\overline{u}_{i-1}(t),\ \Delta f_i,\ D_i, e_{i-1}$, respectively, then:

$$\begin{aligned} {}_{do}D^q V \le \sum_{i=2}^{N}\sum_{j=1}^{n} e_{i-1}^j \big(\Delta f_i{}^j - \Delta f_1{}^j + D_i{}^j - D_1{}^j + F_i(x_i(t-\tau_i)) - F_i(x_i(t-\hat{\tau}_i)) - F_1(x_1(t-\tau_1)) + F_1(x_1(t-\hat{\tau}_1)) \\ + \overline{u}_{i-1}{}^j) + \sum_{i=2}^{N} [e_{i-1}^T \Big(H_i(x_i)\widetilde{\theta}_i - H_1(x_1)\widetilde{\theta}_1 \Big) + \widetilde{\theta}_i^T{}_{do}D^q\widetilde{\theta}_i] + \sum_{i=2}^{N} \Big(\widetilde{\gamma}_i{}_{do}D^q\widetilde{\gamma}_i + \widetilde{d}_i{}_{do}D^q\widetilde{d}_i + l_i\widetilde{\tau}_i{}_{do}D^q\tau_i \Big) \\ + \sum_{i=2}^{N} k_{i-1}e_{i-1}^T e_{i-1} + \widetilde{\gamma}_1{}_{do}D^q\widetilde{\gamma}_1 + \widetilde{d}_1{}_{do}D^q\widetilde{d}_1 + l_1\widetilde{\tau}_1{}_{do}D^q\tau_1 + \widetilde{\theta}_1^T{}_{do}D^q\widetilde{\theta}_1 \end{aligned} \tag{23}$$

Therefore:

$$\begin{aligned} {}_{do}D^q V \le \sum_{i=2}^{N}\sum_{j=1}^{n} [\left|e_{i-1}^j\right| \big(\left|\Delta f_i{}^j\right| + \left|\Delta f_1{}^j\right| + \left|D_i{}^j\right| + \left|D_1{}^j\right| + |F_i(x_i(t-\tau_i)) - F_i(x_i(t-\hat{\tau}_i))| \\ + |-F_1(x_1(t-\tau_1)) + F_1(x_1(t-\hat{\tau}_1))|\big) + e_{i-1}^j \overline{u}_{i-1}{}^j] + \sum_{i=2}^{N} \Big(\widetilde{\gamma}_i{}_{do}D^q\widetilde{\gamma}_i + \widetilde{d}_i{}_{do}D^q\widetilde{d}_i + l_i\widetilde{\tau}_i D^q\tau_i \Big) \\ + \sum_{i=2}^{N} k_{i-1}e_{i-1}^T e_{i-1} + \sum_{i=2}^{N} [e_{i-1}^T \Big(H_i(x_i)\widetilde{\theta}_i - H_1(x_1)\widetilde{\theta}_1 \Big) + \widetilde{\theta}_i^T D^q\widetilde{\theta}_i] + \widetilde{\gamma}_1{}_{do}D^q\widetilde{\gamma}_1 + \widetilde{d}_1{}_{do}D^q\widetilde{d}_1 \\ + l_1\widetilde{\tau}_1{}_{do}D^q\tau_1 + \widetilde{\theta}_1^T{}_{do}D^q\widetilde{\theta}_1 \end{aligned} \tag{24}$$

The disturbance and uncertainty boundary condition can be extended to the components of Δf_i and $D_i(t)$ as follows:

$$\left|\Delta f_i{}^j\right| \le \max_j \left|\Delta f_i{}^j\right| \le |\Delta f_i(x_i)| \le \gamma_i g_i(x_i)$$
$$\left|D_i{}^j(t)\right| \le \max_j \left|D_i{}^j(t)\right| \le |D_i(t)|\ \le d_i$$

which we have by substituting in (24):

$$\begin{aligned} {}_{do}D^q V \le \sum_{i=2}^{N}\sum_{j=1}^{n} \Big[\left|e_{i-1}^j\right| (\gamma_i g_i(x_i) + \gamma_1 g_1(x_1) + d_i + d_1 + l_i|\widetilde{\tau}_i|) + e_{i-1}^j \overline{u}_{i-1}{}^j\Big] + \sum_{i=2}^{N} \Big(\widetilde{\gamma}_i{}_{do}D^q\widetilde{\gamma}_i + \widetilde{d}_i{}_{do}D^q\widetilde{d}_i + l_i\widetilde{\tau}_i{}_{do}D^q\tau_i \Big) \\ + \sum_{i=2}^{N} k_{i-1}e_{i-1}^T e_{i-1} + \sum_{i=2}^{N} [e_{i-1}^T \Big(H_i(x_i)\widetilde{\theta}_i - H_1(x_1)\widetilde{\theta}_1 \Big) + \widetilde{\theta}_i^T{}_{do}D^q\widetilde{\theta}_i] + \widetilde{\gamma}_1{}_{do}D^q\widetilde{\gamma}_1 + \widetilde{d}_1{}_{do}D^q\widetilde{d}_1 \\ + l_1\widetilde{\tau}_1{}_{do}D^q\tau_1 + \widetilde{\theta}_1^T{}_{do}D^q\widetilde{\theta}_1 \end{aligned} \tag{25}$$

If $\overline{u}_{i-1}{}^j(t)$ (distributed system) is defined as follows:

$$\overline{u}_{i-1}{}^j(t) = -(\hat{\gamma}_i g_i(x_i) + \hat{\gamma}_1 g_1(x_1) + \hat{d}_i + \hat{d}_1 + \hat{l}_i + \hat{l}_1)\cdot sgn(e_{i-1}^j(t)) \tag{26}$$

Through the estimation of disturbance and uncertainty bounds in $\overline{u}_{i-1}{}^{j}(t)$, an effort control was made to eliminate the effects of disturbance, delay, and uncertainty as much as possible for distributed system. Therefore, Lyapunov function derivatives will be negative by selecting the proper update rules, and ultimately, the convergence of errors will guarantee zero.

Then

$$\begin{aligned} {}_{do}D^{q}V \leq & \sum_{i=2}^{N}\sum_{j=1}^{n}\left[\left|e_{i-1}^{j}\right|\left(\widetilde{\gamma}_i g_i(x_i)+\widetilde{\gamma}_1 g_1(x_1)+\widetilde{d}_i+\widetilde{d}_1+l_i|\widetilde{\tau}_i|\right)\right]+\sum_{i=2}^{N}\left(\widetilde{\gamma}_{i\,do}D^{q}\widetilde{\gamma}_i+\widetilde{d}_{i\,do}D^{q}\widetilde{d}_i+l_i\widetilde{\tau}_i D^{q}\tau_i\right) \\ & +\sum_{i=2}^{N}k_{i-1}e_{i-1}^{T}e_{i-1}+\sum_{i=2}^{N}[e_{i-1}^{T}\left(H_i(x_i)\widetilde{\theta}_i-H_1(x_1)\widetilde{\theta}_1\right)+\widetilde{\theta}_i^{T}{}_{do}D^{q}\widetilde{\theta}_i]+\widetilde{\gamma}_{1\,do}D^{q}\widetilde{\gamma}_1+\widetilde{d}_{1\,do}D^{q}\widetilde{d}_1 \\ & +l_1\widetilde{\tau}_{1\,do}D^{q}\tau_1+\widetilde{\theta}_1^{T}{}_{do}D^{q}\widetilde{\theta}_1 \end{aligned} \tag{27}$$

The rules updating estimation errors of distributed system are as follows:

$${}_{do}D^{q}\widetilde{\gamma}_i=-(g_i(x_i)\sum_{j=1}^{n}\left|e_{i-1}^{j}\right|+\alpha_i\widetilde{\gamma}_i) \qquad i=2.3\ldots N \tag{28}$$

$${}_{do}D^{q}\widetilde{\gamma}_1=-(\sum_{i=1}^{N}\sum_{j=1}^{n}\left|e_{i-1}^{j}\right|g_1(x_1)+\alpha_1\widetilde{\gamma}_1) \tag{29}$$

$${}_{do}D^{q}\widetilde{d}_i=-(\sum_{j=1}^{n}\left|e_{i-1}^{j}\right|+\beta_i\widetilde{d}_i) \qquad i=2.3\ldots N \tag{30}$$

$${}_{do}D^{q}\widetilde{d}_1=-(\sum_{i=1}^{N}\sum_{j=1}^{n}\left|e_{i-1}^{j}\right|+\beta_1\widetilde{d}_1) \tag{31}$$

$${}_{do}D^{q}\widetilde{\tau}_i=-(\sum_{j=1}^{n}\left|e_{i-1}^{j}\right|+\mu_i\widetilde{\tau}_i) \quad i=2.3\ldots N \tag{32}$$

$${}_{do}D^{q}\widetilde{\tau}_1=-(\sum_{i=1}^{N}\sum_{j=1}^{n}\left|e_{i-1}^{j}\right|+\mu_1\widetilde{\tau}_1) \tag{33}$$

$${}_{do}D^{q}\widetilde{\theta}_i=-(H_i{}^{T}(x_i)e_{i-1}+\sigma_i\widetilde{\theta}_i)i=2.3\ldots N \tag{34}$$

$${}_{do}D^{q}\widetilde{\theta}_1=-(\sum_{i=2}^{N}H_1{}^{T}(x_1)e_{i-1}+\sigma_1\widetilde{\theta}_1) \tag{35}$$

where $\alpha_i,\ \beta_i,\ \sigma_i,\ \mu_i$ are positive. By placing the above update rules in (27), we have:

$${}_{do}D^{q}V\leq\sum_{i=2}^{N}k_{i-1}e_{i-1}^{T}e_{i-1}-\sum_{i=1}^{N}(\alpha_i\widetilde{\gamma}_i{}^2+\beta_i\widetilde{d}_i{}^2+\mu_i\widetilde{\tau}_i{}^2)-\sum_{i=1}^{N}\sigma_i\widetilde{\theta}_i^{T}\widetilde{\theta}_i<-\mu V(t),$$

where $\mu=\underset{i}{min}(\alpha_i,\beta_i,\ \sigma_i,\mu_i,-k_{i-1})>0$. Therefore according to the theorems (1) and (2) and being Hurwitz $k_{i-1}<0$, the stability of the system according to Mittag-Leffler is also confirmed. The convergence of synchronization errors to zero is also guaranteed despite uncertainty and disturbance.

The estimations updating rules of distributed system are obtained as follows:

$${}_{do}D^{q}\hat{\gamma}_i=g_i(x_i)\sum_{j=1}^{n}\left|e_{i-1}^{j}\right|+\alpha_i\widetilde{\gamma}_i, \qquad i=2.3\ldots N \tag{36}$$

$${}_{do}D^{q}\hat{\gamma}_1=g_1(x_1)\sum_{i=1}^{N}\sum_{j=1}^{n}\left|e_{i-1}^{j}\right|+\alpha_1\widetilde{\gamma}_1. \tag{37}$$

$${}_{do}D^{q}\hat{d}_i=\sum_{j=1}^{n}\left|e_{i-1}^{j}\right|+\beta_i\widetilde{d}_i, \quad i=2.3\ldots N. \tag{38}$$

$${}_{do}D^{q}\hat{d}_1=\sum_{i=1}^{N}\sum_{j=1}^{n}\left|e_{i-1}^{j}\right|+\beta_1\widetilde{d}_1. \tag{39}$$

$${}_{do}D^{q}\hat{\tau}_i=\sum_{j=1}^{n}\left|e_{i-1}^{j}\right|+\mu_i\widetilde{\tau}_i, \qquad i=2.3\ldots N \tag{40}$$

$$ {}_{do}D^{q}\hat{\tau}_1 = \sum_{i=1}^{N}\sum_{j=1}^{n}\left|e_{i-1}^{j}\right| + \mu_1\widetilde{\tau}_1 \quad (41) $$

Therefore, the final control function of distributed system is as follows:

$$ \begin{array}{c} u_{i-1}(t) = -f_i(x_i) + f_1(x_1) - H_i(x_i)\hat{\theta}_i(t) + H_1(x_1)\hat{\theta}_1(t) + K_{i-1}e_{i-1}(t) - F_i(x_i(t-\hat{\tau}_i)) + F_1(x_1(t-\hat{\tau}_1)) \\ -sgn\left(e_{i-1}^{j}(t)\right).(\hat{\gamma}_i g_i(x_i) + \hat{\gamma}_1 g_1(x_1) + \hat{d}_i + \hat{d}_1) \qquad i = 1.2\cdots N-1 \end{array} \quad (42) $$

The presence of the sign function $(sgn(e_{i-1}^{j}(t)))$ in the control law leads to a discontinuity in the control function (42) of distributed system. To solve this problem, the symbol function can be replaced with the function $(tanh(e_{i-1}^{j}(t)))$.

3. Encryption Method with Chaotic Masking

Chaotic signals behave in a complex way that makes their behavior unpredictable. This paper uses chaotic signals as a carrier in the use of secure communications [45]. In this method, the message signal is collected as a nonlinear combination of the state vectors of the distributed master system. In other words, the message signal information is hidden by the chaotic states of the distribution system. This can establish security in the telecommunication channel. On the receiver side, the message signal can be recovered by a synchronization error between the master system and slave system of the distributed fractional order. The message signal is recovered on the receiver side, ensuring that the synchronization error goes to zero. The presence of parametric uncertainties and disturbance signals in the master and slave system of distributed systems can increase the security of the communication channel.

In the following, we present our proposed method for chaotic masking for a distributed fractional-order system.

Suppose $m(t)$ is a message. We encrypt this message with a proper map:

$$ m_0(t) = \Lambda(m(t), f(t), a) \quad (43) $$

where $\Lambda(m(t), f(t), a)$ is a definite and continuous function as the map, and $f(t)$ is a definite and continuous signal as a coder. For instance, we can define $\Lambda(m(t), f(t), a)$ as follows:

$$ \begin{array}{c} \Lambda(m(t), f(t), a) = tanh(a{\cdot}m(t) + f(t))\,, \\ f(t) = 0.2sin(10t) + 0.1sin(20\pi t) + 0.05cos(2\pi t), a \in R \end{array} \quad (44) $$

where a is a coefficient so that $|a{\cdot}m(t) + f(t)| \le 4$ will stand.

Signals $m_0(t)$ and $f(t)$ will be masked as follows and transmitted in two components different from the distributed fractional-order chaotic system:

$$ \begin{array}{c} \widetilde{m}(t) = m_0(t) + \sum_{i=1}^{n}\lambda_i x_i \\ \widetilde{f}(t) = f(t) + \sum_{i=1}^{n}\mu_i x_i \end{array} \quad (45) $$

The receiver initially obtains the estimation of signals $m_0(t)$ and $f(t)$. Then, it will calculate $\hat{m}_0(t)$, and ultimately, we will calculate $\hat{m}(t)$ as follows:

$$ \begin{array}{c} \hat{m}_0(t) = \widetilde{m}(t) - \sum_{i=1}^{n}\lambda_i y_i = \\ m_0(t) + \sum_{i=1}^{n}\lambda_i x_i - \sum_{i=1}^{n}\lambda_i y_i = m_0(t) + \sum_{i=1}^{n}\lambda_i e_i \to m_0(t) \end{array} \quad (46) $$

$$ \hat{f}(t) = \widetilde{f}(t) - \sum_{i=1}^{n}\mu_i y_i = f(t) + \sum_{i=1}^{n}\mu_i x_i - \sum_{i=1}^{n}\mu_i y_i = f(t) + \sum_{i=1}^{n}\mu_i e_i \to f(t) \quad (47) $$

To recover the signal of message $m(t)$, we can do as follows:

$$ \begin{array}{c} m_0(t) = \Lambda(m(t), f(t), a) \to \hat{m}_0(t) = \Lambda\left(\hat{m}(t), \hat{f}(t), a\right) = tanh\left(a{\cdot}\hat{m}(t) + \hat{f}(t)\right) \\ \Rightarrow \hat{m}(t) = \frac{1}{a}(tanh^{-1}\left(\hat{m}_0(t) - \hat{f}(t)\right) \end{array} \quad (48) $$

Figure 2 shows the masking in multiple synchronizations for the distributed fractional order system.

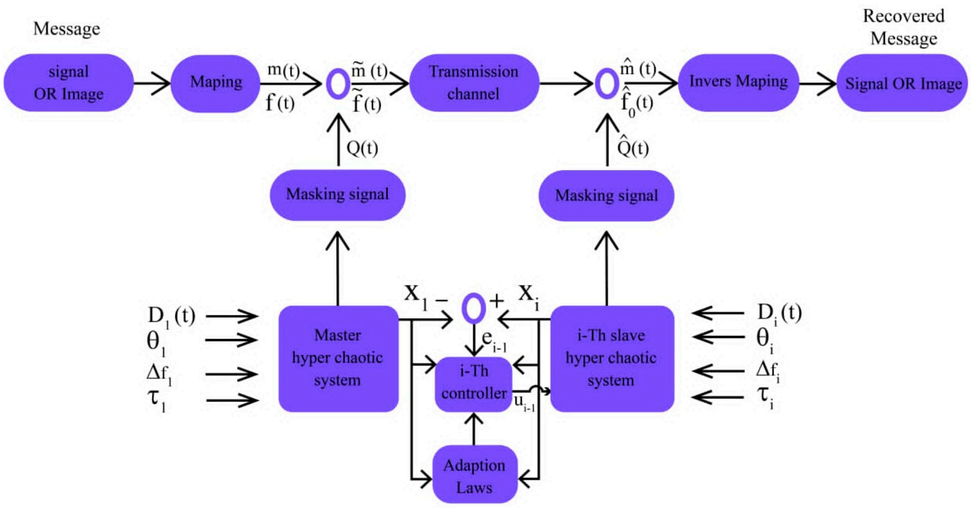

Figure 2. Block diagram of chaotic masking with multi-state synchronization for the distributed system.

4. Simulation and Results for the Duffing Distributed Fractional Order System

The two chaotic duffing systems of distribution fractional order are defined as master and slave [46]:

This section shows the results of synchronizing the duffing chaotic system for the distributed fractional order. The distribution master system is presented as follows:

$$\begin{cases} {}_{do}D^q x_1(t) = x_2(t) \\ {}_{do}D^q x_2(t) = x_1(t) - a_1 x_1(t)^2 x_2(t) + a_1 x_2(t) \\ a_2 \cos(w_0 t) + F(x(t-\tau_1)) \\ +\Delta f(x((t),t)) \\ +d_1(t) \end{cases} \quad (49)$$

The coefficients of the distribution fractional are as follows

$$M(q) = r_1\delta(q-q_1) + r_2\delta(q-q_2) \qquad F(x(t-\tau_{11}),) = x(t-\tau_{11})$$
$$q_1 = 0.2, q_2 = 0.4 \qquad w_0 = 2.467$$
$$r_1 = 0.2, r_1 = 0.9 \qquad \Delta f(x((t),t)) = 0.7 sin(5x_1(t) + 2x_2(t))$$

And the initial values of the variables, disturbance, and uncertainty are considered as follows:

$$x(0) = [0.1,\ 0.1], \qquad \tau_1 = 3 \qquad a_1 = 5, a_2 = 5$$

The chaotic duffing system for slave 1 is as follows

$$\begin{cases} {}_{do}D^q y_1(t) = y_2(t) + u_{11} \\ {}_{do}D^q y_2(t) = y_1(t) - a'_1 y_1(t)^2 y_2(t) + a'_1 y_2(t) \\ a''_1 \cos(w'_0 t) + G(y(t-\tau_{21})) \\ +\Delta g(y((t),t)) \\ +d_2(t) + u_{12} \end{cases} \quad (50)$$

The distribution fractional-order coefficients for the slave system (50) are as follows:

$$M(q) = r_1\delta(q-q_1) + r_2\delta(q-q_2) \qquad G(y(t-\tau_{11}),) = sin(y_1(t-\tau_{21}))y_2(t)$$
$$q_1 = 0.2, q_2 = 0.4 \qquad w'_0 = 2.450$$
$$r_1 = 0.2, r_1 = 0.9 \qquad \Delta g(y((t),t)) = 0.6 sin(5y_1(t) + 2y_2(t))$$

And the initial values of the variables, disturbance, and uncertainty for the slave system (50) are considered as follows:

$x(0) = [0.2,\ 0.3],\qquad d_1(t) = 0.5\,sin(5t) + (0.7\,cos(4t))^2\ \ \tau_2 = 3.5 a_1' = 4.8, a_2' = 5.1,$

To synchronize, first, it must be converted into a standard problem. For this purpose, by defining the variables z_i as $z_3 = x_2,\ \ z_1 = x_1$, the master system is transformed as a non-distributed fractional-order derivative:

$$\begin{array}{c} D^q z_1(t) = z_2(t)\ \ ,\qquad q = 0.2 \\ D^q z_2(t) = -\frac{r_1}{r_2} z_2(t) + \frac{1}{r_2} z_3(t) \\ D^q z_3(t) = z_4(t) \\ D^q z_4(t) = \frac{1}{r_2}(z_1(t) - a_1 z_1(t)^2 z_3(t) + a_1 z_2(t) \\ a_2\cos(w_0 t) + F(z(t-\tau_1)) \\ +\Delta f(z((t),t)) \\ +d_1(t)) - \frac{r_1}{r_2} z_4(t) \end{array} \tag{51}$$

The slave system will also be as follows by changing the variable ($v_3 = y_2, v_1 = y_1$):

$$\begin{array}{c} D^q v_1(t) = v_2(t)\ \ ,\qquad q = 0.2 \\ D^q v_2(t) = -\frac{r_1}{r_2} v_2(t) + \frac{1}{r_2} v_3(t) \\ D^q v_3(t) = v_4(t) + u_{11}(t) \\ D^q v_4(t) = \frac{1}{r_2}(v_1(t) - a_1' v_1(t)^2 v_3(t) + a_1' v_3(t) \\ a_2'\cos(w_0' t) + G(v(t-\tau_{21})) \\ +\Delta g(v((t),t)) \\ +d_2(t)) - \frac{r_1}{r_2} v_4(t) + u_{21}(t) \end{array} \tag{52}$$

The dynamic equations are as follows:

$$\begin{array}{c} D^q e_{11}(t) = e_{12}(t)\ \ ,\qquad q = 0.2 \\ D^q e_{12}(t) = -\frac{r_1}{r_2} e_{11}(t) + \frac{1}{r_2} e_{13}(t) \\ D^q e_{13}(t) = e_{14}(t) + u_{11}(t) \\ D^q e_{14}(t) = \frac{1}{r_2}(e_{11}(t) - a_1' v_1(t)^2 v_3(t) + a_1' v_3(t) \\ a_2'\cos(w_0' t) + G(v(t-\tau_{21})) + \Delta g(v((t),t)) \\ -(-a_1 z_1(t)^2 z_3(t) + a_1 z_2(t) + a_2\cos(w_0 t) + F(z(t-\tau_1)) \\ +\Delta f(z((t),t))) + d_2(t) - d_1(t)) - \frac{r_1}{r_2} e_{14}(t) + u_{21}(t) \end{array} \tag{53}$$

The control effort to synchronize the slave (1) system with the master system is as follows:

$$\begin{array}{c} u_{11}(t) = K_1^T E_1 \\ u_{12}(t) = -(v_1(t) - a_1' v_1(t)^2 v_4(t) + a_1' v_4(t) + a_2'\cos(w_0' t) + G(v(t-\hat{\tau}_2))) + \big(z_1(t) - a_1 z_1(t)^2 z_4(t) + \\ a_1 z_4(t) + a_2\cos(w_0 t) + F(z(t-\hat{\tau}_1))) - k_0 e_{14} - sign(e_{14})\big(\hat{\gamma}_1 + \hat{\gamma}_2 + \hat{\delta}_1 h_1(z) + \hat{\delta}_2 h_2(v)\big) \end{array} \tag{54}$$

Figure 3 shows the phase curve and Lyapunov exponents of the Duffing distribution master and slave system. It also shows both the master and slave systems exhibit chaotic behavior over time.

Figure 4 shows the disturbance and uncertainty in the master-slave distribution system.

The following is the control effort curve for the slave (1) distribution system for the two-time intervals.

Figure 5 shows that the control effort is initially increased and then tends to zero.

Figure 6 shows the error curve for tracking the state of slave and master system variables of number 1.

As can be seen from Figure 6, the tracking error is reduced to zero at the appropriate speed, indicating that our synchronization is appropriate and that the control effort to track the master system by the slave system is appropriate and the system remains stable.

Now, the slave distributed system (2) with different initial values for tracking the master distributed system is expressed as follows:

$$\left\{\begin{array}{c} {}_{do}D^q w_1(t) = w_2(t) + u_{21} \\ {}_{do}D^q w_2(t) = w_1(t) - a_1'' w_1(t)^2 w_2(t) + a_1'' w_2(t) \\ a_2''\cos(w_0'' t) + G(w(t-\tau_3)) \\ +\Delta g(w((t),t)) \\ +d_2(t) + u_{22} \end{array}\right. \tag{55}$$

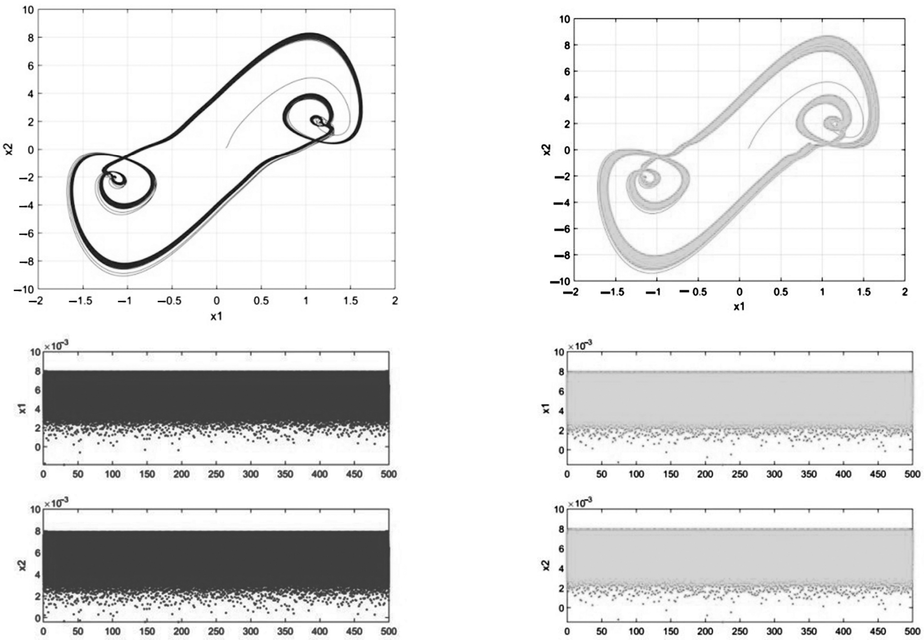

Figure 3. Phase curve and Lyapunov exponents of the master and slave system in distributed fractional order state.

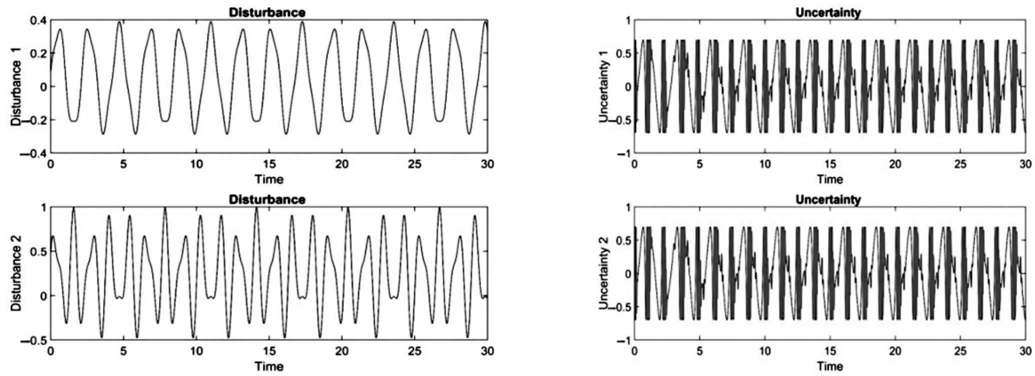

Figure 4. Error estimation of uncertainty and disturbance.

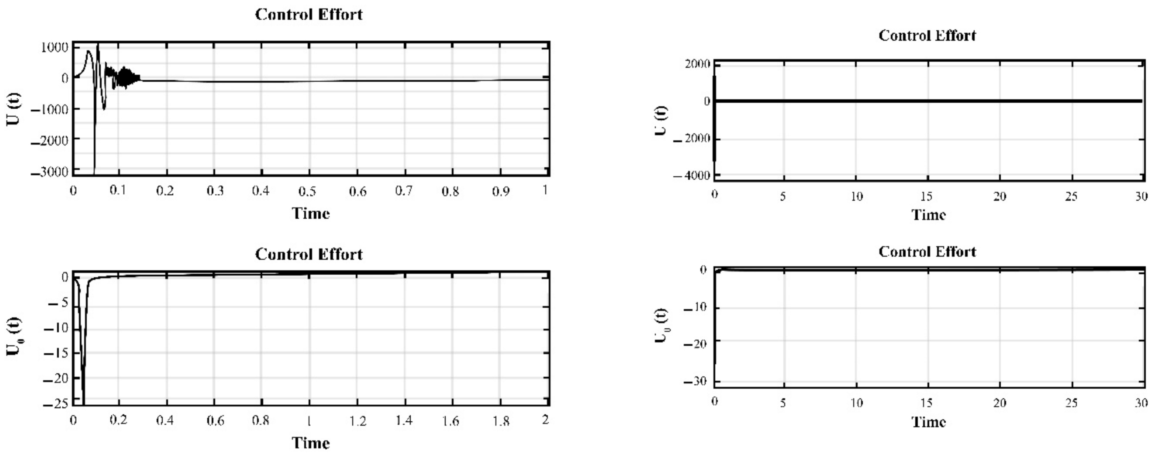

Figure 5. Control Effort of Slave system (1).

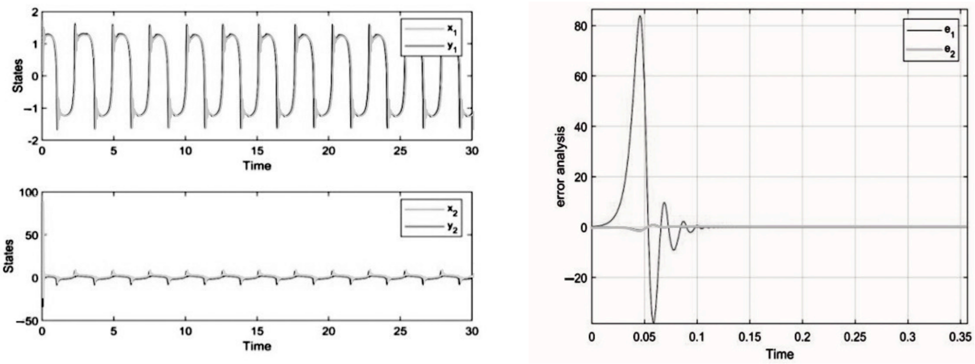

Figure 6. Tracking error curve of slave system 1 and master.

The initial values for the distribution slave(2) system are considered as follows:

$$\begin{array}{ccc} M(q) = r_1\delta(q-q_1) & & d_2(t) = (0.7\,sin(3t))^2 \\ +r_2\delta(q-q_2) & x(0) = [0.2,\ 0.3], & +0.7\,cos(4t) \\ q_1 = 0.2, q_2 = 2q_1 & \Delta g(y((t),t)) & a''_1 = 5.2, a''_2 = 5.2, \\ r_1 = 0.2, r_1 = 0.9 & = 0.6sin(7y_1(t) & w''_0 = 2.350 \\ G(w(t-\tau_3),) = sin(y_1(t-\tau_3)) & +6y_2(t)) & \end{array}$$

Now to convert the distributed slave system to a normal fraction order system (for simulation), we will have a variable ($\beta_3 = w_2, \beta_1 = w_1$):

$$\begin{array}{c} D^q\beta_1(t) = \beta_2(t)\ ,\ q = 0.2 \\ D^q\beta_2(t) = -\frac{r_1}{r_2}\beta_1(t) + \frac{1}{r_2}\beta_3(t) \\ D^q\beta_3(t) = \beta_4(t) + u_{21}(t) \\ D^q\beta_4(t) = \beta_1(t) - a''_1\beta_1(t)^2\beta_3(t) + a''_1\beta_3(t) \\ a''_2\cos(w''_0 t) + G(\beta(t-\tau_3)) \\ +\Delta g(\beta((t),t)) \\ +d_2(t) + u_{22}(t) \end{array} \tag{56}$$

Also, the dynamic equations of the error after changing the variable are as follows:

$$\begin{array}{c} D^q e_{21}(t) = e_{22}(t)\ ,\ q = 0.2 \\ D^q e_{22}(t) = -\frac{r_1}{r_2}e_{21}(t) + \frac{1}{r_2}e_{23}(t) \\ D^q e_{23}(t) = e_{24}(t) + u_{21}(t) \\ D^q e_{24}(t) = \frac{1}{r_2}(e_{21}(t) - a''_1\beta_1(t)^2\beta_3(t) + a''_1\beta_3(t) \\ a''_2\cos(w''_0 t) + G(\beta(t-\tau_{21})) + \Delta g(\beta((t),t)) \\ -(-a_1z_1(t)^2z_3(t) + a_1z_2(t) + a_2\cos(w_0t) + F(z(t-\tau_1)) \\ +\Delta f(z((t),t))) + d_2(t) - d_1(t)) - \frac{r_1}{r_2}e_{24}(t) + u_{21}(t) \end{array} \tag{57}$$

The control effort to track Slave System (2) with the Distribution Master system is designed as follows:

$$\begin{array}{c} u_{11}(t) = K_2^T E_2 \\ u_{21}(t) = -(\beta_1(t) - a''_1\beta_1(t)^2\beta_3(t) + a''_1\beta_3(t) + a''_1\cos(w''_0 t) + \\ G(\beta(t-\hat{\tau}_{21}))) + (z_1(t) - a_1z_1(t)^2z_4(t) + a_1z_4(t) + a_2\cos(wt) + \\ F(z(t-\hat{\tau}_1))) - k_0e_{24} - sign(e_{24})(\hat{\gamma}_1 + \hat{\gamma}_2 + \hat{\delta}_1h_1(z) + \hat{\delta}_2h_2(\beta)) \end{array} \tag{58}$$

In Figure 7, the tracking curve of the state variables of the slave system (2) is plotted with the master system of the distributed fractional order.

As Figure 7 shows, the tracking error of the slave system(2) with the master system of the duffing distribution fractional rate quickly converges to zero, indicating a proper synchronization of the proposed system. It can also be seen that the slave system followed the master system well and very quickly.

The following is a control effort at different time intervals for the slave system (2).

As can be seen from Figure 8, at first, the control effort was high for tracking, but after a while, the effort was reduced, which indicates the stability of the proposed design. Figure 9 shows that the estimation errors were high in the initial times but after 0.6 s reached zero.

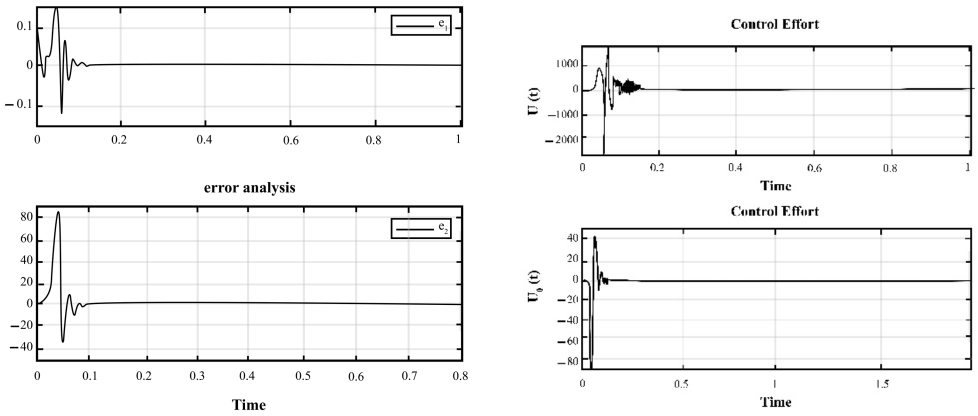

Figure 7. Error tracking state variables of master and slave system (2).

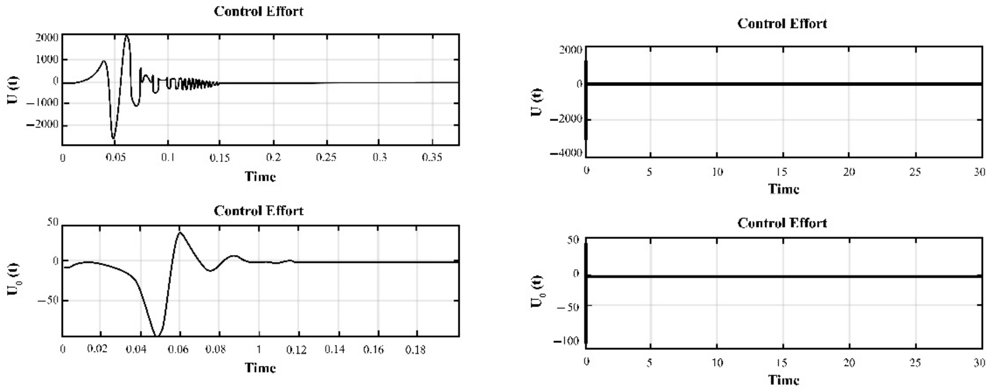

Figure 8. Control Effort of the slave system 2.

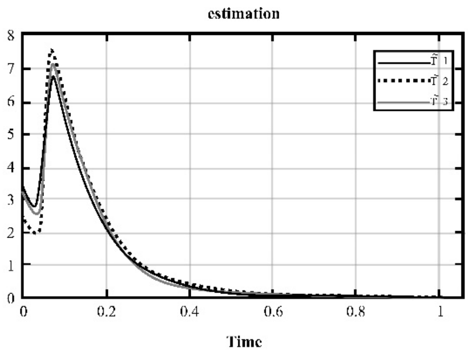

Figure 9. Time delay estimation error.

5. The Results of Chaotic Masking Experiments in Secure Communications

This section describes the synchronization results of distributed fractional order chaotic duffing systems for secure communications. First, the used signal is investigated to perform the experiments. In the following, the encrypted signals using the synchronization methods of the Duffing distribution fractional order chaotic systems are shown.

The used signal as the message signal is as follows:

$$\begin{aligned} y_1(t) &= 0.9\sin(2t) + 0.75\cos(\pi t - 1) + 1.05\sin(2\pi t - 0.5) + 0.65\cos(0.5\pi t - 0.3) \\ y_2(t) &= 0.9\sin(2t) + 0.75\cos(\pi t - 1) + 1.05\sin(2\pi t - 0.5) + 0.65\cos(0.5\pi t - 0.3) \\ y_3(t) &= 0.3\sin(2t) + 0.45\cos(\pi t - 1) + 0.35\sin(2\pi t - 0.45) + 0.35\cos(0.5\pi t - 0.63) \end{aligned} \quad (59)$$

The diagram of the chaotic masking error for the distribution system is shown in Figure 10.

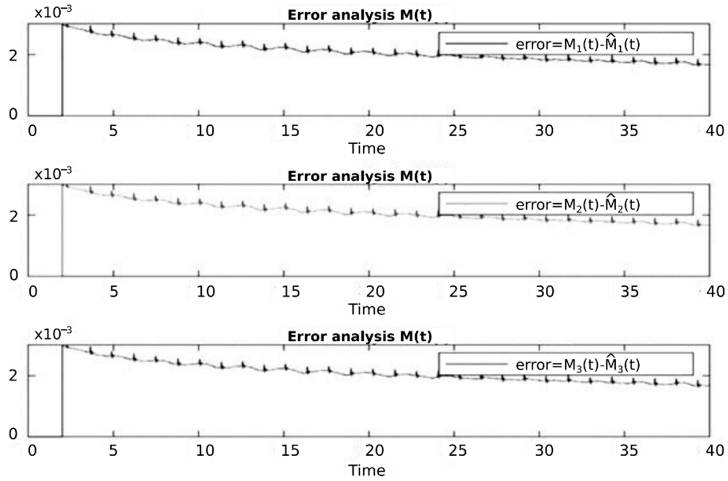

Figure 10. Error curve related to the chaotic masking method.

It can be seen from Figure 10 that the error related to the masking method in all three curves is very small in the range of two thousandths.

Figure 11 shows the message signal curve with the estimated signal.

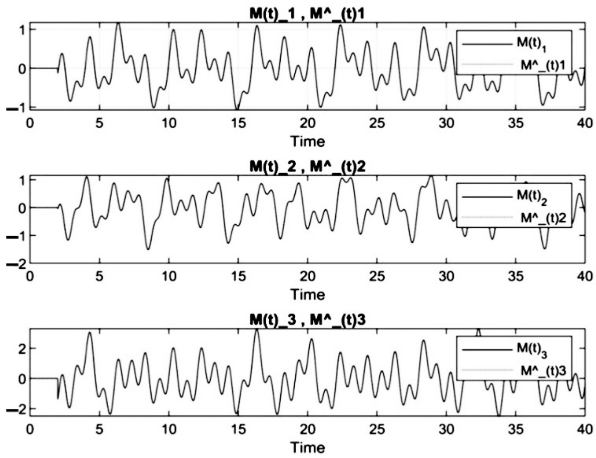

Figure 11. Message signal curve with the estimated message signal.

It can be seen from Figure 11 that the original message signal is well approximated by the synchronization method and the chaotic masking scheme. Figure 12 shows the x_1 signal curve with the message signal.

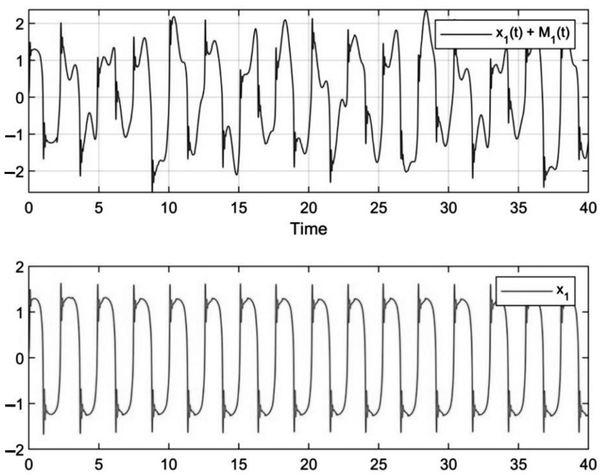

Figure 12. Curve $x_1(t) + M_1(t)$ per time.

It can be seen from Figure 12 that the signal $M_1(t)$ is well encoded by the chaotic signal $x_1(t)$. So that it is not recognizable.

Figure 13 shows that the M_2 message signal is encoded by the chaotic signal x_1. As can be seen, the M_2 message signal is not detectable.

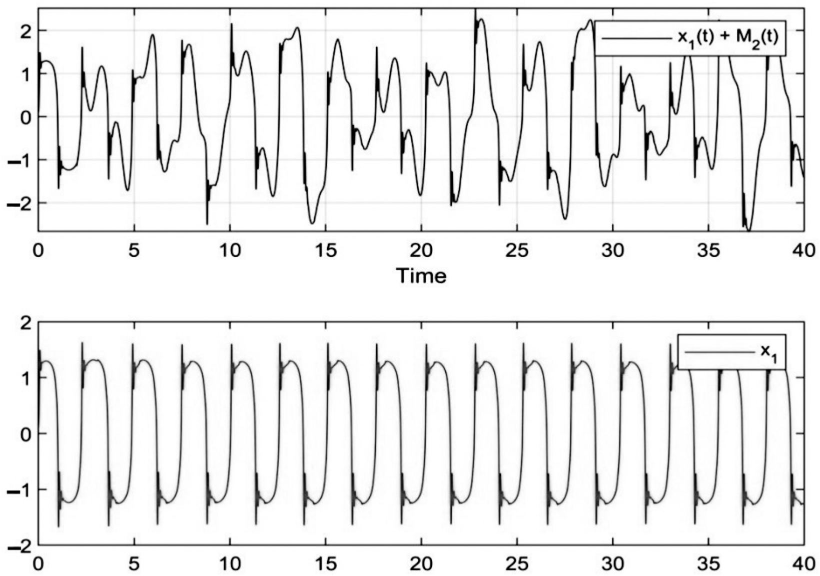

Figure 13. x_1 curve with x_1 signal added with M_2 message signal.

In Figure 14, the M_3 message signal is encoded by the chaotic signal x_1. It is impossible to detect the M_3 message signal.

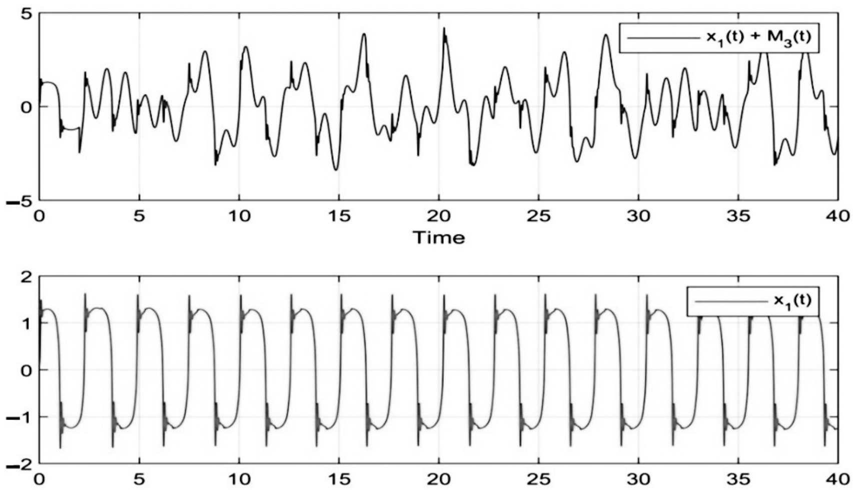

Figure 14. Signal curves x_1 and M_3 in terms of time.

6. Discussion

Using chaos in communications is among the most important ways to maintain information in communication systems. Recently, synchronization methods in secure communications have been the subject of intense research. According to research, synchronization is very effective in data encryption. This study presents secure communications according to multi-state chaotic synchronization of distributed fractional order based on robust adaptive control, which has not been considered in previous works. In this research, the synchronization is based on the multi-mode adaptive control method and the chaotic system of the Duffing fractional-order for signal encryption, followed by some experiments on the signals. In the next step, the secure communication method based on the synchronization of the distributed fractional-order chaotic system is applied to the standard signal. The unknown delay factor in systems complicates the synchronization problem. The technique of covering chaotic systems is used to encrypt signals. Then, simulations are performed on a distributed orderly Duffing chaotic system to test the proposed synchronization method's efficiency. Experimental results show the success of the Duffing distributed fraction synchronization method. Changing the derivative order makes the distribution system's behavior different. This issue is an essential point in cryptography that reduces the possibility of decoding. The result of controlling rules is clear as a continuous function.

Table 1 indicates the high efficiency of the proposed method compared to other studies. According to this table, it can be seen that our proposed method is different from other researches.

Table 1. Comparison of the proposed method with other related works.

Reference	Disturbance	Uncertainty	Type Order	Time Delay	Unknown Parameters
[45]	✓	Time varying-unknown	Integer-order	✓	✓
[47]	✓	×	Integer-order	×	×
[38]	×	×	Fractional-order	✓	✓
[48]	✓	Time varying-unknown	Fractional-order	✓	✓
[36]	×	×	Distributed Fractional Order	×	×
[19]	✓	×	Distributed Fractional Order	×	×
Proposed method	✓	unknown	Distributed Fractional Order	✓	✓

This study uses chaotic signals as a cover for message signals. In this mechanism, it was assumed that the system parameters and delay were unknown. In addition, parametric uncertainties and disturbance were considered in the analyses. As a result, a great deal of complexity was applied to synchronization, resulting in greater security in communications applications. The advantage of the proposed method is in using distributed fractional-order calculations instead of ordinary fractional-order calculations, which can provide more complexity in synchronization and security in sending communication signals. The results show that the proposed method is robust to disturbance and uncertainties. Also, the results were successful in coding and decoding the signal information. The main objective is to develop a multi-mode synchronization method for systems with distributed fractional order derivatives. Duffing's chaotic system may have been used in various cases of synchronization. Nevertheless, despite the unknown delay, it has not been used in the case where the derivative is of distributed fractional order.

For future work, using distributed fractional-order derivatives is recommended for other applications of synchronization of chaotic systems [49]. Also, using fuzzy methods types 1 and 2 is recommended regarding their intelligent nature. Sincefuzzy systems can provide better performance against uncertainties in the absence of fuzzy, the proposed method can also be used to transmit medical images, EEG, and ECG signals safely.

7. Conclusions

This paper investigates the multi-state synchronization of distributed fractional chaotic systems in the presence of uncertainty, disturbance, time delays, and unknown parameters. Parameter setting rules were obtained to converge the synchronization errors into zero with the presence of uncertainty. Rules were also set for estimating disturbance and uncertainty boundaries. In addition, the proposed method for synchronization ensures that the parameter estimation error, disturbance boundaries, and uncertainties converge to zero. The control rule was designed so that there was no problem with chattering. In addition, unknown delays were considered. Then adjustment rules were designed to estimate these delays. Next, a new chaotic masking scheme was proposed to encode information about message signals according to the synchronization method.

The results' accuracy was checked by performing some simulations to synchronize the chaotic system of the Duffing distributed fractional order. The results showed that synchronization was well achieved for both slave systems. Furthermore, the efficiency of the proposed encryption method was evaluated by applying the results to signal encryption. Afterward, simulations were performed in a secure communication application. Message signals were first masked and then retrieved by synchronization error on the receiver side. The results showed the successful use of cryptography on these signals. Overall, the proposed design was successful, according to the simulation results. Our higher efficiency results demonstrate the superiority of the proposed method.

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