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New Estimates for Exponentially Convex Functions via Conformable Fractional Operator

Saima Rashid [1,2,*], Muhammad Aslam Noor [2] and Khalida Inayat Noor [2]

1 Department of Mathematics, Govt. College University, Faisalabad 38000, Pakistan
2 Department of Mathematics, COMSATS University Islamabad, Islamabad 45550, Pakistan; noormaslam@gmail.com (M.A.N.); khalidan@gmail.com (K.I.N.)
* Correspondence: saimarashid@gcuf.edu.pk

Received: 28 March 2019; Accepted: 12 April 2019; Published: 15 April 2019

Abstract: In this paper, we derive a new Hermite–Hadamard inequality for exponentially convex functions via α-fractional integral. We also prove a new integral identity. Using this identity, we establish several Hermite–Hadamard type inequalities for exponentially convexity, which can be obtained from our results. Some special cases are also discussed.

Keywords: convex function; exponential convex function; Hermite–Hadamard inequality; α-fractional; integral inequalities

MSC: 26A15; 26E60; 41A55

1. Introduction

Convexity theory has played fundamental parts in the developments of various fields of pure and applied sciences. Due to its impermanence, convex functions and convex sets have been generalized and extended in different directions. It has been shown that a function is convex, if and only if, it satisfies an integral inequality, which his called the Hermite-Hadamard inequality. On other hand, the minimum of the differentiable convex functions can be characterized by variational inequalities. These two aspects of the convexity theory have far reaching applications and have provided powerful tools for studying difficult problems. In recent years, integral inequalities are being derived via fractional analysis, which has emerged as another interesting technique. Fractional analysis is an area that is constantly developing and trying to renew itself to produce solutions to the changing world and problems. Various types of fractional derivative and integral operators were studied. In fractional calculus, the fractional derivatives are defined via fractional integrals. The conformable fractional integral plays a major role in fractional calculus. There were several studies in the literature that include further properties such as expansion formulas, variational calculus applications, control theoretical applications, convexity and integral inequalities and Hermite-Hadamard type inequalities of this new operator and similar operators.

Exponentially convex functions have emerged an a significant new class of convex functions, which have important applications in technology, data science and statistics. The main motivation of this paper depends on a new identity that has been proved via α-fractional integrals (conformable fractional integral operators) and applied for exponentially convex functions. This identity offers new upper bounds and estimations of Hadamard type integral inequalities. Some special cases such for $\alpha = 1$ have been discussed, which can be deduced from these results. In derivation of these results, we have used integration techniques, some integral inequalities as power-mean inequality and Jensen inequality.

We now recall some well known concepts and basic results, which are needed in the derivation of our results.

Definition 1. *A set $K \subset \mathbb{R} \to \mathbb{R}$ is said to be a convex set, if*

$$tu + (1-t)v \in K, \ \ \forall u, v \in K, \ \ t \in [0,1].$$

Definition 2. *A function $f : K \subset \mathbb{R} \to \mathbb{R}$ is said to be convex function, if*

$$f\big(tu + (1-t)v\big) \leq tf(u) + (1-t)f(v), \ \ \forall u, v \in K, \ \ t \in [0,1].$$

For convex functions, many equalities or inequalities have been established by many authors; for example, Hardy type inequality [1], Ostrowski type inequality [2], Olsen type inequality [3] and Gagliardo–Nirenberg type inequality [4] but the most celebrated and significant inequality is the Hermite–Hadamard type inequality [5–12], which is defined as:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1}$$

A number of mathematicians in the field of applied and pure mathematics have dedicated their efforts to extend, generalize, counterpart, and refine the Hermite–Hadamard inequality (1) for different classes of convex functions and mappings. For more more recent results obtained on inequality (1), we refer the reader to References [5–12].

In recent years, the concept of convexity has been extended and generalized in various directions using novel and innovative ideas and techniques to study a a wide class of unrelated problems in a unified framework. Awan et al. [5] considered and studied a new class of exponentially convex functions. Antczak [13] explored the applications of the exponentially convex functions in the mathematical programming problems. Dragomir and Gomm [7] derived some integral inequalities for the exponentially convex functions.

We now recall the definition of exponentially convex function.

Definition 3. *(See [7,13]) Let $f : K \subset \mathbb{R} \longrightarrow \mathbb{R}$ is exponentially convex function, if f is positive, $\forall u, v \in K$ and $t \in [0,1]$, we have*

$$e^{f((1-t)u+tv)} \leq (1-t)e^{f(u)} + te^{f(v)}.$$

Exponentially convex functions are used to manipulate for statistical learning, sequential prediction and stochastic optimization, see [13–15] and the references therein. The class of exponentially convex functions was introduced by Antczak [13] and Dragomir et al. [7].

One can easily show that the minimum $u \in K$ is the minimum of the differentiable exponentially convex functions f, if and only if, $u \in K$ satisfies the inequality

$$\langle f'(u)e^{f(u)}, v-u\rangle \geq 0, \qquad \forall v \in K,$$

which is called the exponentially variational inequality and appears to be new one. It is an open problem to study the exponentially variational inequalities and their properties. For the applications, numerical methods and other aspects of variational inequalities, see Noor [16].

An important definition called Riemann–Liouville fractional integrals which is a milestone in the theory of fractional calculus:

Definition 4. *(See [17]) Let $f \in L_1[a,b]$. The Riemann–Liouville integrals $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of order $\alpha > 0$ are defined by*

$$J_{a+}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-x)^{\alpha-1}f(x)dx, \qquad t > a$$

and

$$J_{b-}^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (x-t)^{\alpha-1}f(x)dx, \qquad t < b$$

respectively where $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt$. Here $J_{a+}^{0}f(t) = J_{b-}^{0}f(t) = f(t)$.

In the case of $\alpha = 1$, the fractional integral reduces to classical integral. Several researchers have focused on new integral inequalities involving Riemann–Liouville fractional integrals in recent years, see the papers [3,10,17–23]. Recently Khalil et al. [22] gave a new definition that is called the "conformable fractional derivative" and its properties. The conformable fractional derivative attracts attention with conformity to the classical derivative. Khalil et al. [22] have introduced the conformable fractional derivative by the equation which has a limit form similar to the classical derivative. Khalil et al. [22] have proved that this definition provides multiplication and division rules. They also express the Roll's theorem and the mean value theorem for functions which are differentiable with conformable fractional order.

Now, we give the definition of the conformable fractional derivative with its important properties which are useful in order to obtain our main results see, [18–20,22].

In our study, we use the Katugampola derivative formulation of conformable derivative, which is explained in the following definition:

Definition 5. *([21]) Given a function $f : [0,\infty) \to \mathbb{R}$. Then, the conformable fractional derivative of f of order α of f at t is defined by*

$$D_{\alpha}(f)(t) = \lim_{\epsilon \to 0} \frac{f(t+\epsilon t^{1-\alpha}) - h(t)}{\epsilon}, \ \alpha \in (0,1), \ t > 0. \tag{2}$$

If f is α-differentiable in some $(0,\alpha)$, $\alpha > 0$, $\lim_{t\to 0^+} h^{(\alpha)}(t)$ exist, then define

$$f^{\alpha}(0) = \lim_{t\to 0^+} h^{(\alpha)}(t).$$

Note that, if f is differentiable, then

$$D_{\alpha}(f)(t) = t^{1-\alpha}f'(t), \ \textit{where} \quad f'(t) = \lim_{\epsilon\to 0}\frac{f(t+\epsilon)-f(t)}{\epsilon}. \tag{3}$$

We can write $f^{\alpha}(t)$ for $D_{\alpha}(f)(t)$ denotes the conformable fractional derivatives of f of order α at t. If the conformable fractional derivative of f of order α exists, then we simply say f is α-differentiable.

Khalil et al. [22] considered the following definition:

Definition 6. *([22]) (Conformable fractional integral) Let $\alpha \in (0,1]$, $0 \le a < b$. A function $f : [a,b] \longrightarrow \mathbb{R}$ is α-fractional integrable on $[a,b]$ if the integral*

$$\int_a^b f(x)d_{\alpha}x = \int_a^b f(x)x^{\alpha-1}dx \tag{4}$$

exists and is finite. All $\alpha-$fractional integrable functions on $[a,b]$ is indicated by $L_{\alpha}^{1}([a,b])$.

Remark 1. *(See [22])*

$$I_{\alpha}^{a}(f)\,(t) = I_{1}^{a}(t^{\alpha-1}f) = \int_{a}^{t} \frac{f(x)}{x^{1-\alpha}}dx$$

where the integral is the usual Riemann improper integral, and $\alpha \in (0,1]$.

The motivation of this article is to discuss some new fractional bounds involving the functions having exponential convexity property. In order to obtain main results of the article, we derive several new conformable fractional integral identities. We hope that the ideas and techniques of this article will inspire interested readers.

2. Results

Our main results depend on the following inequality:

Lemma 1. *Let* $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ *be an* α*-fractional differentiable mapping on* (a,b) *with* $0 \leq a < b$*. If* $D_{\alpha}(e^{f}) \in L_{\alpha}[a,b]$*, then*

$$\Upsilon_{f}(\alpha;a,b) = \sum_{i=1}^{4} \theta_i, \tag{5}$$

where

$$\begin{aligned}
\theta_1 &= \frac{b-a}{4}\int_0^1 \Bigg[\left(at+(1-t)\frac{3a+b}{4}\right)^{2\alpha-1} \\
&\quad -\left(\frac{3a+b}{4}\right)^{\alpha}\left(at+(1-t)\frac{3a+b}{4}\right)^{\alpha-1}\Bigg] \times D_{\alpha}(e^{f})f^{\alpha}\left(at+(1-t)\frac{3a+b}{4}\right)d_{\alpha}t, \\
\theta_2 &= \frac{b-a}{4}\int_0^1 \Bigg[\left(t\frac{3a+b}{4}+(1-t)\frac{a+b}{2}\right)^{2\alpha-1} \\
&\quad -\left(\frac{3a+b}{4}\right)^{\alpha}\left(t\frac{3a+b}{4}+(1-t)\frac{a+b}{2}\right)^{\alpha-1}\Bigg] \times D_{\alpha}(e^{f})f^{\alpha}\left(t\frac{3a+b}{4}+(1-t)\frac{a+b}{2}\right)d_{\alpha}t, \\
\theta_3 &= \frac{b-a}{4}\int_0^1 \Bigg[\left(t\frac{a+b}{2}+(1-t)\frac{a+3b}{4}\right)^{2\alpha-1} \\
&\quad -\left(\frac{3a+b}{4}\right)^{\alpha}\left(t\frac{a+b}{2}+(1-t)\frac{a+3b}{4}\right)^{\alpha-1}\Bigg] \times D_{\alpha}(e^{f})f^{\alpha}\left(t\frac{a+b}{2}+(1-t)\frac{a+3b}{4}\right)d_{\alpha}t, \\
\theta_4 &= \frac{b-a}{4}\int_0^1 \Bigg[\left(t\frac{a+3b}{4}+(1-t)b\right)^{2\alpha-1} \\
&\quad -\left(\frac{3a+b}{4}\right)^{\alpha}\left(t\frac{a+3b}{4}+(1-t)b\right)^{\alpha-1}\Bigg] \times D_{\alpha}(e^{f})f^{\alpha}\left(t\frac{a+3b}{4}+(1-t)b\right)d_{\alpha}t,
\end{aligned}$$

and

$$\begin{aligned}
\Upsilon_{f}(\alpha;a,b) &= \left[\left(\frac{3a+b}{4}\right)^{\alpha}-a^{\alpha}\right]e^{f(a)}+\left[b^{\alpha}-\left(\frac{a+3b}{4}\right)^{\alpha}\right]e^{f(v)} \\
&\quad +\left[\left(\frac{a+3b}{4}\right)^{\alpha}-\left(\frac{3a+b}{4}\right)^{\alpha}\right]e^{f\left(\frac{a+b}{2}\right)}-\alpha\int_a^b e^{f(x)}d_{\alpha}x.
\end{aligned}$$

Proof. Using integration by parts, we have

$$\begin{aligned}\theta_1 &= \frac{b-a}{4}\int_0^1 \left[\left(at+(1-t)\frac{3a+b}{4}\right)^{\alpha}-\left(\frac{3a+b}{4}\right)^{\alpha}\right]e^{f\left(at+(1-t)\frac{3a+b}{4}\right)}f'\left(at+(1-t)\frac{3a+b}{4}\right)dt\\ &= \left[\left(\frac{3a+b}{4}\right)^{\alpha}-a^{\alpha}\right]e^{f(a)}-\alpha\int_0^1\left(at+(1-t)\frac{3a+b}{4}\right)^{\alpha-1}e^{f\left(at+(1-t)\frac{3a+b}{4}\right)}dt.\end{aligned}$$

Using the change of the variable $x := at+(1-t)\frac{3a+b}{4},\ t\in[0,1]$ and the definition of conformable fractional integral (4), we obtain

$$\begin{aligned}\theta_1 &= \left[\left(\frac{3a+b}{4}\right)^{\alpha}-a^{\alpha}\right]e^{f(a)}-\alpha\int_a^{\frac{3a+b}{4}}x^{\alpha-1}e^{f(x)}dx\\ &= \left[\left(\frac{3a+b}{4}\right)^{\alpha}-a^{\alpha}\right]e^{f(a)}-\alpha\int_a^{\frac{3a+b}{4}}e^{f(x)}d_{\alpha}x.\end{aligned}$$

Similarly, we get

$$\begin{aligned}\theta_2 &= \left[\left(\frac{a+b}{2}\right)^{\alpha}-\left(\frac{3a+b}{4}\right)^{\alpha}\right]e^{f(\frac{a+b}{2})}-\alpha\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}e^{f(x)}d_{\alpha}x,\\ \theta_3 &= \left[\left(\frac{a+3b}{4}\right)^{\alpha}-\left(\frac{a+b}{2}\right)^{\alpha}\right]e^{f(\frac{a+b}{2})}-\alpha\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}e^{f(x)}d_{\alpha}x,\end{aligned}$$

and

$$\theta_4 = \left[b^{\alpha}-\left(\frac{a+3b}{4}\right)^{\alpha}\right]e^{f(b)}-\alpha\int_{\frac{a+3b}{4}}^{b}e^{f(x)}d_{\alpha}x.$$

Adding $\theta_1,\ \theta_2\ \theta_3$ and θ_4 together, we obtain the desired inequality (5). □

Remark 2. *If we set $\alpha = 1$, then under the assumption of Lemma 1, the identity (5) reduces to the the following new identity.*

$$\begin{aligned}\Upsilon_f(1;a,b) &= \int_0^1(1-t)e^{f\left(\frac{3a+b}{4}t+(1-t)\frac{a+b}{2}\right)}f'(\frac{3a+b}{4}t+(1-t)\frac{a+b}{2})dt\\ &-\int_0^1 te^{f\left(at+(1-t)\frac{3a+b}{2}\right)}f'(at+(1-t)\frac{3a+b}{4})dt\\ &+\int_0^1(1-t)e^{f\left(\frac{a+3b}{4}t+(1-t)b\right)}f'(\frac{a+3b}{4}t+(1-t)b)dt\\ &-\int_0^1 te^{f\left(\frac{a+b}{2}t+(1-t)\frac{a+3b}{4}\right)}f'(\frac{a+b}{2}t+(1-t)\frac{a+3b}{4})dt,\end{aligned}\tag{6}$$

where

$$\Upsilon_f(1;a,b) \qquad = \frac{1}{2}\left[\frac{e^{f(a)}+e^{f(b)}}{2}+e^{f\left(\frac{a+b}{2}\right)}\right]-\frac{4}{b-a}\int\limits_a^b e^{f(x)}dx.$$

Theorem 1. *Let $f:[a,b]\subseteq\mathbb{R}\to\mathbb{R}$ be an α-fractional exponentially differentiable mapping on (a,b) with $0\leq a<b$. If $D_\alpha(e^f)\in L_\alpha[a,b]$ and $|f'|$ is convex on $[a,b]$, then*

$$\begin{aligned}&\Upsilon_f(\alpha;a,b)\leq\frac{b-a}{240}\\&\times\Bigg[|e^{f(a)}f'(a)|H_1(\alpha)+|e^{f(b)}f'(b)|H_2(\alpha)+\big|e^{f\left(\frac{3a+b}{4}\right)}f'\big(\frac{3a+b}{4}\big)\big|H_3(\alpha)\\&+\big|e^{f\left(\frac{a+b}{2}\right)}f'\big(\frac{a+b}{2}\big)\big|H_4(\alpha)+\big|e^{f\left(\frac{a+3b}{4}\right)}f'\big(\frac{a+3b}{4}\big)\big|H_5(\alpha)+\Delta_1\big(a,\frac{3a+b}{4}\big)H_6(\alpha)\\&+\Delta_2\big(\frac{3a+b}{4},\frac{a+b}{2}\big)H_7(\alpha)+\Delta_3\big(\frac{a+b}{2},\frac{a+3b}{4}\big)H_8(\alpha)+\Delta_4+\big(\frac{a+3b}{4},b\big)H_9(\alpha)\Bigg],\end{aligned}$$

where

$$\begin{aligned}H_1(1)=&12a+3a\left(\frac{3a+b}{4}\right)^{\alpha-1}+3a^{\alpha-1}\left(\frac{3a+b}{4}\right)-18\left(\frac{3a+b}{4}\right)^{\alpha},\\H_2(\alpha)=&-18\left(\frac{a+3b}{4}\right)^{\alpha}+3b\left(\frac{a+3b}{4}\right)^{\alpha-1}+3b^{\alpha-1}\left(\frac{a+3b}{4}\right)+12b^{\alpha},\\H_3(\alpha)=&10\left(\frac{a+b}{2}\right)^{\alpha}+3b\left(\frac{a+3b}{4}\right)^{\alpha-1}+3b^{\alpha-1}\left(\frac{a+3b}{4}\right)+3\left(\frac{3a+b}{4}\right)^{\alpha-1}\frac{a+b}{2}\\&+3\left(\frac{a+b}{2}\right)^{\alpha-1}\frac{3a+b}{4}+5\left(\frac{a+b}{2}\right)^{\alpha}+2a^{\alpha},\\H_4(\alpha)=&10\left(\frac{a+b}{2}\right)^{\alpha}+3\left(\frac{3a+b}{4}\right)^{\alpha-1}\frac{a+b}{2}+3\left(\frac{3a+b}{4}\right)\left(\frac{a+b}{2}\right)^{\alpha-1}+5\left(\frac{a+3b}{4}\right)^{\alpha-1}\frac{a+b}{2}\\&+5\left(\frac{a+b}{2}\right)^{\alpha-1}\frac{a+3b}{4}-28\left(\frac{a+3b}{4}\right)^{\alpha},\\H_5(\alpha)=&10\left(\frac{a+b}{2}\right)^{\alpha}+3\left(\frac{a+3b}{4}\right)^{\alpha-1}\frac{a+b}{2}+3\left(\frac{a+3b}{4}\right)\left(\frac{a+b}{2}\right)^{\alpha-1}-16\left(\frac{a+3b}{4}\right)^{\alpha}\\&+3b\left(\frac{a+3b}{4}\right)^{\alpha-1}+3b^{\alpha-1}\left(\frac{a+3b}{4}\right)+2b^{\alpha},\\H_6(\alpha)=&3a^{\alpha}+2a\left(\frac{3a+b}{4}\right)^{\alpha-1}+3a^{\alpha-1}\left(\frac{3a+b}{4}\right)-7\left(\frac{3a+b}{4}\right)^{\alpha},\\H_7(\alpha)=&3\left(\frac{3a+b}{4}\right)^{\alpha}+2\left(\frac{3a+b}{4}\right)^{\alpha-1}\left(\frac{a+b}{2}\right)+2\left(\frac{3a+b}{4}\right)\left(\frac{a+b}{2}\right)^{\alpha-1}-2\left(\frac{3a+b}{4}\right)^{\alpha},\\H_8(\alpha)=&3\left(\frac{a+b}{2}\right)^{\alpha}+10\left(\frac{a+3b}{4}\right)^{\alpha-1}\frac{a+b}{2}+10\left(\frac{a+b}{2}\right)^{\alpha-1}\frac{a+3b}{4}-7\left(\frac{a+3b}{4}\right)^{\alpha-1},\\H_9(\alpha)=&3b^{\alpha}-7\left(\frac{a+3b}{4}\right)^{\alpha}+10b\left(\frac{a+3b}{4}\right)^{\alpha-1}+10b^{\alpha-1}\left(\frac{a+3b}{4}\right).\end{aligned}$$

and

$$\Delta_1\big(a,\frac{3a+b}{4}\big)=|e^{f(a)}f'\Big(\frac{3a+b}{4}\Big)|+|e^{f\left(\frac{3a+b}{4}\right)}f'(a)|,$$
$$\Delta_2\big(\frac{3a+b}{4},\frac{a+b}{2}\big)=|e^{f\left(\frac{a+b}{2}\right)}f'\Big(\frac{3a+b}{4}\Big)|+|e^{f\left(\frac{3a+b}{4}\right)}f'\big(\frac{a+b}{2}\big)|,$$
$$\Delta_3\big(\frac{a+b}{2},\frac{a+3b}{4}\big)=|e^{f\left(\frac{a+b}{2}\right)}f'\Big(\frac{a+3b}{4}\Big)|+|e^{f\left(\frac{a+3b}{4}\right)}f'\big(\frac{a+b}{2}\big)|,$$
$$\Delta_4\big(\frac{a+3b}{4},b\big)=|e^{f\left(\frac{a+3b}{4}\right)}f'(b)|+|e^{f(b)}f'\big(\frac{a+3b}{4}\big)|.$$

Proof. Using Lemma 1 and the convexity of $|f'|$, we find

$$\begin{aligned}
&\Upsilon_f(\alpha;a,b)\\
&=\frac{b-a}{4}\Bigg\{\int_0^1\Bigg[\Bigg(at+(1-t)\Big(\frac{3a+b}{4}\Big)^{\alpha}-\Big(\frac{3a+b}{4}\Big)^{\alpha}\Bigg]e^{f\left(at+(1-t)\frac{3a+b}{4}\right)}f'\big(at+(1-t)\frac{3a+b}{4}\big)dt\\
&-\int_0^1\Bigg[\Bigg(\frac{3a+b}{4}t+(1-t)\frac{a+b}{2}\Bigg)^{\alpha}-\Big(\frac{3a+b}{4}\Big)^{\alpha}\Bigg]e^{f\left(\frac{3a+b}{4}t+(1-t)\frac{a+b}{2}\right)}f'\big(\frac{3a+b}{4}t+(1-t)\frac{a+b}{2}\big)dt\\
&+\int_0^1\Bigg[\Bigg(\frac{a+b}{2}t+(1-t)\frac{a+3b}{4}\Bigg)^{\alpha}-\Big(\frac{a+3b}{4}\Big)^{\alpha}\Bigg]e^{f\left(\frac{a+b}{2}t+(1-t)\frac{a+3b}{4}\right)}f'\big(\frac{a+b}{2}t+(1-t)\frac{a+3b}{4}\big)dt\\
&-\int_0^1\Bigg[\Bigg(\frac{a+3b}{4}t+(1-t)b\Bigg)^{\alpha}-\Big(\frac{a+3b}{4}\Big)^{\alpha}\Bigg]e^{f\left(\frac{a+3b}{4}t+(1-t)b\right)}f'\big(\frac{a+3b}{4}t+(1-t)b\big)dt\Bigg\}.
\end{aligned}\tag{7}$$

By using the convexity of $x^{\alpha-1}$ for $x>0,\ \alpha\in(0,1]$, we have

$$\begin{aligned}
&\Bigg(at+(1-t)\frac{3a+b}{4}\Bigg)^{\alpha}-\Big(\frac{3a+b}{4}\Big)^{\alpha}=\Bigg(at+(1-t)\frac{3a+b}{4}\Bigg)^{\alpha-1+1}-\Big(\frac{3a+b}{4}\Big)^{\alpha}\\
&=\Bigg(at+(1-t)\frac{3a+b}{4}\Bigg)^{\alpha}\Bigg(at+(1-t)\frac{3a+b}{4}\Bigg)-\Big(\frac{3a+b}{4}\Big)^{\alpha}\\
&\le\Bigg[a^{\alpha-1}t+(1-t)\Big(\frac{3a+b}{4}\Big)^{\alpha-1}\Bigg]\Bigg(at+(1-t)\frac{3a+b}{4}\Bigg)-\Big(\frac{3a+b}{4}\Big)^{\alpha},
\end{aligned}\tag{8}$$

$$\begin{aligned}
&\Bigg(\frac{3a+b}{4}t+(1-t)\frac{a+b}{2}\Bigg)^{\alpha}-\Big(\frac{3a+b}{4}\Big)^{\alpha}\\
&\le\Bigg[t\Big(\frac{3a+b}{4}\Big)^{\alpha-1}+(1-t)\Big(\frac{a+b}{2}\Big)^{\alpha-1}\Bigg]\Bigg(\frac{3a+b}{4}t+(1-t)\frac{a+b}{2}\Bigg)-\Big(\frac{3a+b}{4}\Big)^{\alpha},
\end{aligned}\tag{9}$$

$$\begin{aligned}
&\Bigg(\frac{a+b}{2}t+(1-t)\frac{a+3b}{4}\Bigg)^{\alpha}-\Big(\frac{a+3b}{4}\Big)^{\alpha}\\
&\le\Bigg[t\Big(\frac{a+b}{2}\Big)^{\alpha-1}+(1-t)\Big(\frac{a+3b}{4}\Big)^{\alpha-1}\Bigg]\Bigg(\frac{a+b}{2}t+(1-t)\frac{a+3b}{4}\Bigg)-\Big(\frac{a+3b}{4}\Big)^{\alpha},
\end{aligned}\tag{10}$$

and

$$\begin{aligned}
&\Bigg(\frac{a+3b}{4}t+(1-t)b\Bigg)^{\alpha}-\Big(\frac{a+3b}{4}\Big)^{\alpha}\\
&\le\Bigg[t\Big(\frac{a+3b}{4}\Big)^{\alpha-1}+(1-t)b^{\alpha-1}\Bigg]\Bigg(\frac{a+3b}{4}t+(1-t)b\Bigg)-\Big(\frac{a+3b}{4}\Big)^{\alpha}.
\end{aligned}\tag{11}$$

Using (8), (9), (10) and (11) in (7) and the properties of modulus, we get

$$\begin{aligned}
&\Upsilon_f(\alpha;a,b) \leq \frac{b-a}{4} \\
&\Bigg\{ \int_0^1 \Bigg[\bigg(\Big(a^{\alpha-1} t + (1-t) \Big(\frac{3a+b}{4}\Big)^{\alpha-1} \Big) \Big(at + (1-t)\big(\frac{3a+b}{4}\big) \Big) - \Big(\frac{3a+b}{4}\Big)^{\alpha} \Bigg] \\
&\times \left| e^{f\left(at+(1-t)\left(\frac{3a+b}{4}\right)\right)} f'\Big(at + (1-t)\big(\frac{3a+b}{4}\big) \Big) \right| dt \\
&+ \int_0^1 \Bigg[\bigg(\Big(\frac{3a+b}{4}\Big)^{\alpha-1} t + (1-t) \Big(\frac{a+b}{2}\Big)^{\alpha-1} \bigg) \Big(\frac{3a+b}{4} t + (1-t)\big(\frac{a+b}{2}\big) \Big) - \Big(\frac{3a+b}{4}\Big)^{\alpha} \Bigg] \\
&\times \left| e^{f\left(\frac{3a+b}{4}t+(1-t)\left(\frac{a+b}{2}\right)\right)} f'\Big(\frac{3a+b}{4} t + (1-t)\big(\frac{a+b}{2}\big) \Big) \right| dt \qquad (12) \\
&+ \int_0^1 \Bigg[\bigg(\Big(\frac{a+b}{2}\Big)^{\alpha-1} t + (1-t) \Big(\frac{a+3b}{4}\Big)^{\alpha-1} \bigg) \Big(\frac{a+b}{2} t + (1-t)\big(\frac{a+3b}{4}\big) \Big) - \Big(\frac{a+3b}{4}\Big)^{\alpha} \Bigg] \\
&\times \left| e^{f\left(\frac{a+b}{2}t+(1-t)\left(\frac{a+3b}{4}\right)\right)} f'\Big(\frac{a+b}{2} t + (1-t)\big(\frac{a+3b}{4}\big) \Big) \right| dt \\
&+ \int_0^1 \Bigg[\bigg(\Big(\frac{a+3b}{4}\Big)^{\alpha-1} t + (1-t) b^{\alpha-1} \bigg) \Big(\frac{a+3b}{4} t + (1-t)b \Big) - \Big(\frac{a+3b}{4}\Big)^{\alpha} \Bigg] \\
&\times \left| e^{f\left(\frac{a+3b}{4}t+(1-t)b\right)} f'\Big(\frac{a+3b}{4} t + (1-t)b \Big) \right| dt \Bigg\}.
\end{aligned}$$

Since $|f'|$ is exponentially convex on $[a,b]$ for any $t \in [0,1]$, so

$$\begin{aligned}
&\left| e^{f\left(ta+(1-t)\left(\frac{3a+b}{4}\right)\right)} f'\Big(ta + (1-t)\big(\frac{3a+b}{4}\big) \Big) \right| \\
&\leq \{ t|e^{f(a)}| + (1-t)|e^{f\left(\frac{3a+b}{4}\right)}| \} \{ t|f'(a)| + (1-t)|f'(\frac{3a+b}{4})| \} \qquad (13) \\
&= t^2|e^{f(a)}f'(a)| + (1-t)^2 |e^{f\left(\frac{3a+b}{4}\right)} f'(\frac{3a+b}{4})| + t(1-t) \left| e^{f(a)} f'(\frac{3a+b}{4}) + e^{f\left(\frac{3a+b}{4}\right)} f'(a) \right| \\
&= t^2|e^{f(a)}f'(a)| + (1-t)^2 |e^{f\left(\frac{3a+b}{4}\right)} f'(\frac{3a+b}{4})| + t(1-t)\Delta_1(a, \frac{3a+b}{4}).
\end{aligned}$$

Thus (12) and (13) become

$$
\begin{aligned}
&\Upsilon_f(\alpha;a,b)\\
&\le \frac{b-a}{4}\times\Bigg\{\int_0^1\Bigg[\Bigg(\Big(a^{\alpha-1}t+(1-t)\Big(\frac{3a+b}{4}\Big)^{\alpha-1}\Big)\Big(at+(1-t)(\frac{3a+b}{4})\Big)-\Big(\frac{3a+b}{4}\Big)^{\alpha}\Bigg]\\
&\times\Bigg[t^2|e^{f(a)}f'(a)|+(1-t)^2|e^{f\left(\frac{3a+b}{4}\right)}f'(\frac{3a+b}{4})|+t(1-t)\Delta_1(a,\frac{3a+b}{4})\Bigg]\\
&+\int_0^1\Bigg[\Bigg(\Big(\frac{3a+b}{4}\Big)^{\alpha-1}t+(1-t)\Big(\frac{a+b}{2}\Big)^{\alpha-1}\Bigg)\Bigg(\frac{3a+b}{4}t+(1-t)(\frac{a+b}{2})\Bigg)-\Big(\frac{3a+b}{4}\Big)^{\alpha}\Bigg]\\
&\times\Bigg[t^2|e^{f(\frac{3a+b}{4})}f'(\frac{3a+b}{4})|+(1-t)^2|e^{f\left(\frac{a+b}{2}\right)}f'(\frac{a+b}{2})|+t(1-t)\Delta_2(\frac{3a+b}{4},\frac{a+b}{2})\Bigg]\\
&+\int_0^1\Bigg[\Bigg(\Big(\frac{a+b}{2}\Big)^{\alpha-1}t+(1-t)\Big(\frac{a+3b}{4}\Big)^{\alpha-1}\Bigg)\Bigg(\frac{a+b}{2}t+(1-t)(\frac{a+3b}{4})\Bigg)-\Big(\frac{a+3b}{4}\Big)^{\alpha}\Bigg]\\
&\times\Bigg[t^2|e^{f(\frac{a+b}{2})}f'(\frac{a+b}{2})|+(1-t)^2|e^{f\left(\frac{a+3b}{4}\right)}f'(\frac{a+3b}{4})|+t(1-t)\Delta_3(\frac{a+b}{2},\frac{a+3b}{4})\Bigg]\\
&+\int_0^1\Bigg[\Bigg(\Big(\frac{a+3b}{4}\Big)^{\alpha-1}t+(1-t)b^{\alpha-1}\Bigg)\Bigg(\frac{a+3b}{4}t+(1-t)b\Bigg)-\Big(\frac{a+3b}{4}\Big)^{\alpha}\Bigg]\\
&\times\Bigg[t^2|e^{f\left(\frac{a+3b}{4}\right)}f'(\frac{a+3b}{4})|+(1-t)^2|e^{f(b)}f'(b)|+t(1-t)\Delta_4(\frac{a+3b}{4},b)\Bigg]\Bigg\}.
\end{aligned}
$$

Simple calculations yield

$$
\begin{aligned}
&\Upsilon_f(\alpha;a,b)\\
&\le \frac{b-a}{240}\Bigg[\Big|e^{f(a)}f'(a)\Big|H_1(\alpha)+|e^{f(b)}f'(b)|H_2(\alpha)+|e^{f\left(\frac{3a+b}{4}\right)}f'(\frac{3a+b}{4})|H_3(\alpha)\\
&\quad+|e^{f\left(\frac{a+b}{2}\right)}f'(\frac{a+b}{2})|H_4(\alpha)+|e^{f\left(\frac{a+3b}{4}\right)}f'(\frac{a+3b}{4})|H_5(\alpha)+\Delta_1(a,\frac{3a+b}{4})H_6(\alpha)\\
&\quad\Delta_2(\frac{3a+b}{4},\frac{a+b}{2})H_7(\alpha)+\Delta_3(\frac{a+b}{2},\frac{a+3b}{4})H_8(\alpha)+\Delta_4+(\frac{a+3b}{4},b)H_9(\alpha)\Bigg],
\end{aligned}
$$

which is the required result. □

Theorem 2. *Let $f:[a,b]\subseteq\mathbb{R}\to\mathbb{R}$ be an α-fractional exponentially differentiable mapping on (a,b) with $0\le a<b$. If $D_\alpha(e^f)\in L_\alpha[a,b]$ and $|f'|^q$ is concave on $[a,b]$, then*

$$
\begin{aligned}
&\Upsilon_f(\alpha;a,b)\le\frac{b-a}{4}\\
&\times\Bigg[\big(A_1(\alpha)\big)^{1-\frac{1}{q}}\Bigg[A_2(\alpha)|e^{f(a)}f'(a)|^q+A_3(\alpha)\Big|e^{f\left(\frac{a+3b}{4}\right)}f'(\frac{a+3b}{4})\Big|^q+A_4(\alpha)\Delta_5(a,\frac{3a+b}{4})\Bigg]^{\frac{1}{q}}\\
&+\big(B_1(\alpha)\big)^{1-\frac{1}{q}}\Bigg[B_2(\alpha)|e^{f\left(\frac{3a+b}{4}\right)}f'(\frac{3a+b}{4})|^q+B_3(\alpha)\Big|e^{f\left(\frac{a+b}{2}\right)}f'(\frac{a+b}{2})\Big|^q+B_4(\alpha)\Delta_5(\frac{3a+b}{4},\frac{a+b}{2})\Bigg]^{\frac{1}{q}}\\
&+\big(C_1(\alpha)\big)^{1-\frac{1}{q}}\Bigg[C_2(\alpha)|e^{f\left(\frac{a+b}{2}\right)}f'(\frac{a+b}{2})|^q+C_3(\alpha)\Big|e^{f\left(\frac{a+3b}{4}\right)}f'(\frac{a+3b}{4})\Big|^q+C_4(\alpha)\Delta_5(\frac{a+b}{2},\frac{a+3b}{4})\Bigg]^{\frac{1}{q}}\\
&+\big(D_1(\alpha)\big)^{1-\frac{1}{q}}\Bigg[D_2(\alpha)|e^{f\left(\frac{a+3b}{4}\right)}f'(\frac{a+3b}{4})|^q+D_3(\alpha)\Big|e^{f(b)}f'(b)\Big|^q+D_4(\alpha)\Delta_5(\frac{a+3b}{4},b)\Bigg]^{\frac{1}{q}}\Bigg],
\end{aligned}
$$

where

$$
\begin{aligned}
A_1(\alpha) &= \frac{(3a+b)^{\alpha+1} - (4a)^{\alpha+1}}{4^{\alpha}(b-a)(\alpha+1)} - \frac{(3a+b)^{\alpha}}{4^{\alpha}}, \\
A_2(\alpha) &= -\frac{1}{3}\frac{(3a+b)^{\alpha}}{4^{\alpha}} + \frac{(4a)^{\alpha+1}(\alpha+2)(\alpha+3)(a-b)^2}{4^{\alpha}(\alpha+1)(\alpha+2)(\alpha+3)(b-a)^3} \\
&+2\frac{\left\{(4a)^{\alpha+3} - (3a+b)^{\alpha+3}\right\} - (a-b)(\alpha+3)(3a+b)^{\alpha+2}}{4^{\alpha}(\alpha+1)(\alpha+2)(\alpha+3)(b-a)^3}, \\
A_3(\alpha) &= -\frac{1}{3}\frac{(3a+b)^{\alpha}}{4^{\alpha}} + \frac{(a-b)^2(\alpha+2)(\alpha+3)(3a+b)^{\alpha+1}}{4^{\alpha}(\alpha+1)(\alpha+2)(\alpha+3)(b-a)^3} \\
&- \frac{2(a-b)(\alpha+3)(3a+b)^{\alpha+2} - 2\left((3a+b)^{\alpha+3} - (4a)^{\alpha+3}\right)}{4^{\alpha}(\alpha+1)(\alpha+2)(\alpha+3)(b-a)^3}, \\
A_4(\alpha) &= -\frac{1}{6}\frac{(3a+b)^{\alpha}}{4^{\alpha}} + \frac{(a-b)(\alpha+3)\left[(4a)^{(\alpha+2)} - (3a+b)^{\alpha+2}\right]}{4^{\alpha}(\alpha+1)(\alpha+2)(\alpha+3)(b-a)^3} \\
&- \frac{2\left[(4a)^{(\alpha+3)} - (3a+b)^{\alpha+3}\right]}{4^{\alpha}(\alpha+1)(\alpha+2)(\alpha+3)(b-a)^3}
\end{aligned}
$$

$$
\begin{aligned}
B_1(\alpha) &= \frac{\left[(3a+b)^{\alpha+1} - (2a+2b)^{\alpha+1}\right]}{4^{\alpha}(\alpha+1)(a-b)} - \left(\frac{3a+b}{4}\right)^{\alpha}, \\
B_2(\alpha) &= \frac{-1}{3}\left(\frac{3a+b}{4}\right)^{\alpha} + \frac{2\left[(3a+b)^{\alpha+3} - (2a+2b)^{\alpha+3}\right]}{4^{\alpha}(\alpha+1)(\alpha+2)(\alpha+3)(b-a)^3} \\
&- \frac{2(a-b)(\alpha+3)(3a+b)^{\alpha+2} - (a-b)^2(\alpha+2)(\alpha+3)(3a+b)^{\alpha+1}]}{4^{\alpha}(\alpha+1)(\alpha+2)(\alpha+3)(b-a)^3}, \\
B_3(\alpha) &= \frac{-1}{3}\left(\frac{3a+b}{4}\right)^{\alpha} + \frac{2\left[(3a+b)^{\alpha+3} - (2a+2b)^{\alpha+3}\right]}{4^{\alpha}(\alpha+1)(\alpha+2)(\alpha+3)(b-a)^3} \\
&- \frac{2(a-b)(\alpha+3)(2a+2b)^{\alpha+2} + (\alpha+2)(\alpha+3)(a-b)^2(2a+2b)^{\alpha+1}}{4^{\alpha}(\alpha+1)(\alpha+2)(\alpha+3)(b-a)^3}, \\
B_4(\alpha) &= \frac{-1}{6}\left(\frac{3a+b}{4}\right)^{\alpha} + \frac{(\alpha+3)(a-b)\left[(3a+b)^{\alpha+2} + (2a+2b)^{\alpha+2}\right]}{4^{\alpha}(\alpha+1)(\alpha+2)(\alpha+3)(b-a)^3} \\
&- \frac{2\left[(3a+b)^{\alpha+3} + (2a+2b)^{\alpha+3}\right]}{4^{\alpha}(\alpha+1)(\alpha+2)(\alpha+3)(b-a)^3}.
\end{aligned}
$$

$$
\begin{aligned}
C_1(\alpha) &= \frac{(2a+2b)^{\alpha+1} - (a+3b)^{\alpha+1}}{4^{\alpha}(\alpha+1)(a-b)} - \left(\frac{a+3b}{4}\right)^{\alpha}, \\
C_2(\alpha) &= \frac{-1}{3}\left(\frac{a+3b}{4}\right)^{\alpha} + \frac{2\left[(2a+2b)^{\alpha+3} - (a+3b)^{\alpha+3}\right]}{4^{\alpha}(\alpha+1)(\alpha+2)(\alpha+3)(b-a)^3} + \\
&- \frac{2(2a+2b)^{\alpha+2}(\alpha+3)(a-b) - (2a+2b)^{\alpha+1}(\alpha+2)(\alpha+3)(a-b)^2]}{4^{\alpha}(\alpha+1)(\alpha+2)(\alpha+3)(b-a)^3},
\end{aligned}
$$

$$
\begin{aligned}
C_3(\alpha) &= \frac{-1}{3}\left(\frac{a+3b}{4}\right)^{\alpha} + \frac{2\left[(2a+2b)^{\alpha+3} - (a+3b)^{\alpha+3}\right]}{4^{\alpha}(\alpha+1)(\alpha+2)(\alpha+3)(b-a)^3} \\
&- \frac{(a-b)(\alpha+3)(a+3b)^{\alpha+2} + (a+3b)^{\alpha+1}(\alpha+2)(\alpha+3)(a-b)^2}{4^{\alpha}(\alpha+1)(\alpha+2)(\alpha+3)(b-a)^3}, \\
C_4(\alpha) &= \frac{-1}{6}\left(\frac{a+3b}{4}\right)^{\alpha} + \frac{\left[(2a+2b)^{\alpha+2} + (a+3b)^{\alpha+2}\right](\alpha+3)(a-b)}{4^{\alpha}(\alpha+1)(\alpha+2)(\alpha+3)(b-a)^3} \\
&- \frac{2\left[(2a+2b)^{\alpha+3} - (a+3b)^{\alpha+3}\right]}{4^{\alpha}(\alpha+1)(\alpha+2)(\alpha+3)(b-a)^3},
\end{aligned}
$$

$$
\begin{aligned}
D_1(\alpha) &= \frac{(a+3b)^{\alpha+1}-(4b)^{\alpha+1}}{4^\alpha(\alpha+1)(a-b)} - \left(\frac{a+3b}{4}\right)^\alpha, \\
D_2(\alpha) &= \frac{-1}{3}\left(\frac{a+3b}{4}\right)^\alpha + \frac{(\alpha+2)(\alpha+3)(a-b)^2(a+3b)^{\alpha+1}}{4^\alpha(\alpha+1)(\alpha+2)(\alpha+3)(b-a)^3} \\
&\quad - \frac{2(\alpha+3)(a-b)(a+3b)^{\alpha+2}+2\left[(a+3b)^{\alpha+3}-(4b)^{\alpha+3}\right]}{4^\alpha(\alpha+1)(\alpha+2)(\alpha+3)(b-a)^3}, \\
D_3(\alpha) &= \frac{-1}{3}\left(\frac{a+3b}{4}\right)^\alpha + \frac{(4b)^{\alpha+1}(\alpha+2)(\alpha+3)(a-b)^2}{4^\alpha(\alpha+1)(\alpha+2)(\alpha+3)(b-a)^3} \\
&\quad - \frac{2(4b)^{\alpha+2}(\alpha+3)(a-b)-2\left[(a+3b)^{\alpha+3}-(4b)^{\alpha+3}\right]}{4^\alpha(\alpha+1)(\alpha+2)(\alpha+3)(b-a)^3}, \\
D_4(\alpha) &= \frac{-1}{6}\left(\frac{a+3b}{4}\right)^\alpha \\
&\quad + \frac{\left[(a+3b)^{\alpha+2}+(4b)^{\alpha+2}\right]+2\left[(a+3b)^{\alpha+3}-(4b)^{\alpha+3}\right]}{4^\alpha(\alpha+1)(\alpha+2)(\alpha+3)(b-a)^3},
\end{aligned}
$$

and

$$
\begin{aligned}
\Delta_5\Big(a,\frac{3a+b}{4}\Big) &= \Big|e^{f(a)}f'\left(\frac{3a+b}{4}\right)\Big|^q + \Big|e^{f\left(\frac{3a+b}{4}\right)}f'(a)\Big|^q, \\
\Delta_6\Big(\frac{3a+b}{4},\frac{a+b}{2}\Big) &= \Big|e^{f\left(\frac{a+b}{2}\right)}f'\left(\frac{3a+b}{4}\right)\Big|^q + \Big|e^{f\left(\frac{3a+b}{4}\right)}f'\Big(\frac{a+b}{2}\Big)\Big|^q, \\
\Delta_7\Big(\frac{a+b}{2},\frac{a+3b}{4}\Big) &= \Big|e^{f\left(\frac{a+b}{2}\right)}f'\left(\frac{a+3b}{4}\right)\Big|^q + \Big|e^{f\left(\frac{a+3b}{4}\right)}f'\Big(\frac{a+b}{2}\Big)\Big|^q, \\
\Delta_8\Big(\frac{a+3b}{4},b\Big) &= \Big|e^{f\left(\frac{a+3b}{4}\right)}f'(b)\Big|^q + \Big|e^{f(b)}f'\Big(\frac{a+3b}{4}\Big)\Big|^q.
\end{aligned}
$$

Proof. Using Lemma 1 and the concavity of $|f'|^q$, we find

$$
\begin{aligned}
&\Upsilon_f(\alpha;a,b) \\
&= \frac{b-a}{4}\Bigg\{\int_0^1 \left[\left(at+(1-t)\left(\frac{3a+b}{4}\right)^\alpha-\left(\frac{3a+b}{4}\right)^\alpha\right]e^{f\left(at+(1-t)\frac{3a+b}{4}\right)}f'\Big(at+(1-t)\frac{3a+b}{4}\Big)dt \\
&-\int_0^1\left[\left(\frac{3a+b}{4}t+(1-t)\frac{a+b}{2}\right)^\alpha-\left(\frac{3a+b}{4}\right)^\alpha\right]e^{f\left(\frac{3a+b}{4}t+(1-t)\frac{a+b}{2}\right)}f'\Big(\frac{3a+b}{4}t+(1-t)\frac{a+b}{2}\Big)dt \\
&+\int_0^1\left[\left(\frac{a+b}{2}t+(1-t)\frac{a+3b}{4}\right)^\alpha-\left(\frac{a+3b}{4}\right)^\alpha\right]e^{f\left(\frac{a+b}{2}t+(1-t)\frac{a+3b}{4}\right)}f'\Big(\frac{a+b}{2}t+(1-t)\frac{a+3b}{4}\Big)dt \\
&-\int_0^1\left[\left(\frac{a+3b}{4}t+(1-t)b\right)^\alpha-\left(\frac{a+3b}{4}\right)^\alpha\right]e^{f\left(\frac{a+3b}{4}t+(1-t)b\right)}f'\Big(\frac{a+3b}{4}t+(1-t)b\Big)dt\Bigg\} \\
&:= \frac{b-a}{4}(\eta_1+\eta_2+\eta_3+\eta_4).
\end{aligned} \tag{14}
$$

It follows from the power-mean inequality that

$$
\begin{aligned}
\eta_1 &\le \left(\int_0^1\left[at+(1-t)\left(\frac{3a+b}{4}\right)^\alpha-\left(\frac{3a+b}{4}\right)^\alpha\right]dt\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1\left[\left(at+(1-t)\left(\frac{3a+b}{4}\right)^\alpha-\left(\frac{3a+b}{4}\right)^\alpha\right]\left|e^{f\left(at+(1-t)\frac{3a+b}{4}\right)}f'\Big(at+(1-t)\frac{3a+b}{4}\Big)\right|dt\right)^{\frac{1}{q}}
\end{aligned}
$$

Since $|f'|^q$ is exponentially concave on $[a,b]$ for any $t\in[0,1]$, we obtain

$$
\begin{aligned}
\eta_1 &\leq \left(\int_0^1 \left[at+(1-t)\left(\frac{3a+b}{4}\right)^{\alpha}-\left(\frac{3a+b}{4}\right)^{\alpha}\right]dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 \left[\left(at+(1-t)\left(\frac{3a+b}{4}\right)^{\alpha}-\left(\frac{3a+b}{4}\right)^{\alpha}\right]\left[\left(t^2|e^{f(a)}f'(a)|^q\right.\right.\right. \\
&+(1-t)^2\left|e^{f\left(\frac{a+3b}{4}\right)}f'(\frac{a+3b}{4})\right|^q+t(1-t)\{|e^{f(a)}f'(\frac{a+3b}{4})|^q+|e^{f\left(\frac{a+3b}{4}\right)}f'(a)|^q\}\Big)dt\Big]^{\frac{1}{q}} \\
&=(A_1(\alpha))^{1-\frac{1}{q}}\left[A_2(\alpha)|e^{f(a)}f'(a)|^q+A_3(\alpha)\left|e^{f\left(\frac{a+3b}{4}\right)}f'(\frac{a+3b}{4})\right|^q+A_4(\alpha)\Delta_5(a,\frac{3a+b}{4})\right]^{\frac{1}{q}},
\end{aligned}
$$

where we have used the fact that

$$
\begin{aligned}
A_1(\alpha) := \int_0^1 \left[\left(at+(1-t)\frac{3a+b}{4}\right)^{\alpha}-\left(\frac{3a+b}{4}\right)^{\alpha}\right]dt \\
= \frac{(3a+b)^{\alpha+1}-(4a)^{\alpha+1}}{4^{\alpha}(b-a)(\alpha+1)}-\frac{(3a+b)^{\alpha}}{4^{\alpha}},
\end{aligned}
$$

$$
\begin{aligned}
A_2(\alpha) &:= \int_0^1 t^2\left[\left(at+(1-t)\frac{3a+b}{4}\right)^{\alpha}-\left(\frac{3a+b}{4}\right)^{\alpha}\right]dt = -\frac{1}{3}\frac{(3a+b)^{\alpha}}{4^{\alpha}} \\
&+\frac{(4a)^{(\alpha+1)}(\alpha+2)(\alpha+3)(a-b)^2-2(4a)^{\alpha+1}(\alpha+3)(a-b)+2\left((4a)^{\alpha+3}-(3a+b)^{(\alpha+3)}\right)}{4^{\alpha}(\alpha+1)(\alpha+2)(\alpha+3)(b-a)^3},
\end{aligned}
$$

$$
\begin{aligned}
A_3(\alpha) &:= \int_0^1 (1-t)^2\left[\left(at+(1-t)\frac{3a+b}{4}\right)^{\alpha}-\left(\frac{3a+b}{4}\right)^{\alpha}\right]dt = -\frac{1}{3}\frac{(3a+b)^{\alpha}}{4^{\alpha}} \\
&+\frac{(a-b)^2(\alpha+2)(\alpha+3)(3a+b)^{\alpha+1}-2(a-b)(\alpha+3)(3a+b)^{\alpha+2}+2\left((3a+b)^{\alpha+3}-(4a)^{\alpha+3}\right)}{4^{\alpha}(\alpha+1)(\alpha+2)(\alpha+3)(b-a)^3},
\end{aligned}
$$

and

$$
\begin{aligned}
A_4(\alpha) &:= \int_0^1 t(1-t)\left[\left(at+(1-t)\frac{3a+b}{4}\right)^{\alpha}-\left(\frac{3a+b}{4}\right)^{\alpha}\right]dt = -\frac{1}{6}\frac{(3a+b)^{\alpha}}{4^{\alpha}} \\
&+\frac{(a-b)(\alpha+3)\left[(4a)^{(\alpha+2)}-(3a+b)^{\alpha+2}\right]-2\left[(4a)^{(\alpha+3)}-(3a+b)^{\alpha+3}\right]}{4^{\alpha}(\alpha+1)(\alpha+2)(\alpha+3)(b-a)^3}.
\end{aligned}
$$

Analogously:

$$
\eta_2 \leq (B_1(\alpha))^{1-\frac{1}{q}}\left[B_2(\alpha)|e^{f\left(\frac{3a+b}{4}\right)}f'(\frac{3a+b}{4})|^q+B_3(\alpha)\left|e^{f\left(\frac{a+b}{2}\right)}f'(\frac{a+b}{2})\right|^q+B_4(\alpha)\Delta_6(\frac{3a+b}{4},\frac{a+b}{2})\right]^{\frac{1}{q}},
$$

$$
\eta_3 \leq (C_1(\alpha))^{1-\frac{1}{q}}\left[C_2(\alpha)|e^{f\left(\frac{a+b}{2}\right)}f'(\frac{a+b}{2})|^q+C_3(\alpha)\left|e^{f\left(\frac{a+3b}{4}\right)}f'(\frac{a+3b}{4})\right|^q+C_4(\alpha)\Delta_7(\frac{a+b}{2},\frac{a+3b}{4})\right]^{\frac{1}{q}},
$$

$$
\eta_4 \leq (D_1(\alpha))^{1-\frac{1}{q}}\left[D_2(\alpha)|e^{f\left(\frac{a+3b}{4}\right)}f'(\frac{a+3b}{4})|^q+D_3(\alpha)\left|e^{f(b)}f'(b)\right|^q+D_4(\alpha)\Delta_8(\frac{a+3b}{4},b)\right]^{\frac{1}{q}}.
$$

Using η_1, η_2, η_3 and η_4 in (14), we obtain the desired inequality. □

Theorem 3. *Let $f : [a,b] \subseteq \mathbb{R} \to \mathbb{R}$ be an α-fractional exponentially differentiable mapping on (a,b) with $0 \leq a < b$. If $D_\alpha(e^f) \in L_\alpha[a,b]$ and $|f'|^q$ is convex on $[a,b]$, then*

$$\begin{aligned}\Upsilon_f(\alpha;a,b) \quad & \leq \frac{b-a}{4}\Big[A_1(\alpha)e^{f\left(\frac{Q_1(\alpha)}{A_1(\alpha)}\right)} f'(\frac{Q_1(\alpha)}{A_1(\alpha)}) + B_1(\alpha)e^{f\left(\frac{Q_2(\alpha)}{B_1(\alpha)}\right)} f'(\frac{Q_2(\alpha)}{B_1(\alpha)}) \\ & + C_1(\alpha)e^{f\left(\frac{Q_3(\alpha)}{C_1(\alpha)}\right)} f'(\frac{Q_3(\alpha)}{C_1(\alpha)}) + D_1(\alpha)e^{f\left(\frac{Q_4(\alpha)}{D_1(\alpha)}\right)} f'(\frac{Q_4(\alpha)}{D_1(\alpha)})\Big],\end{aligned} \tag{15}$$

where $A_1(\alpha)$, $B_1(\alpha)$, $C_1(\alpha)$ and $D_1(\alpha)$ are given in Theorem 2 and

$$\begin{aligned}Q_1(\alpha) \quad & = \int_0^1 [at+(1-t)(\frac{3a+b}{4})^\alpha - (\frac{3a+b}{4})^\alpha](at+(1-t)\frac{3a+b}{4})dt \\ & = \frac{2\left[(4a)^{\alpha+2}-(3a+b)^{\alpha+2}\right]-(\alpha+2)(3a+b)^\alpha(7a^2-6ab-b^2)}{(\alpha+2)(a-b)2^{2\alpha+3}},\end{aligned}$$

$$\begin{aligned}Q_2(\alpha) = \quad & \int_0^1 \left[\left(\left(\frac{3a+b}{4}\right)t+(1-t)(\frac{a+b}{2})\right)^\alpha - (\frac{3a+b}{4})^\alpha\right]\left[\left(\frac{3a+b}{4}\right)t+(1-t)(\frac{a+b}{2})\right]dt \\ & = \frac{2\left[(3a+b)^{\alpha+2}-(2(a+b))^{\alpha+2}\right]-(\alpha+2)(3a+b)^\alpha(5a^2-2ab-3b^2)}{(\alpha+2)(a-b)2^{2\alpha+3}},\end{aligned}$$

$$\begin{aligned}Q_3(\alpha) = \quad & \int_0^1 \left[\left(\left(\frac{a+b}{2}\right)t+(1-t)(\frac{a+3b}{4})\right)^\alpha - (\frac{a+3b}{4})^\alpha\right]\left[\left(\frac{a+b}{2}\right)t+(1-t)(\frac{a+3b}{4})\right]dt \\ & = \frac{2\left[(3a+b)^{\alpha+2}-(2(a+b))^{\alpha+2}\right]-(\alpha+2)(3a+b)^\alpha(5a^2-2ab-3b^2)}{(\alpha+2)(a-b)2^{2\alpha+3}},\end{aligned}$$

and

$$\begin{aligned}Q_4(\alpha) = \quad & \int_0^1 \left[\left(\left(\frac{a+3b}{4}\right)t+(1-t)b\right)^\alpha - (\frac{a+3b}{4})^\alpha\right]\left[\left(\frac{a+3b}{4}\right)t+(1-t)b\right]dt \\ & = \frac{2\left[(a+3b)^{\alpha+2}-(4b)^{\alpha+2}\right]-(\alpha+2)(a+3b)^\alpha(a^2+6ab-7b^2)}{(\alpha+2)(a-b)2^{2\alpha+3}}.\end{aligned}$$

Proof. By using the power-mean inequality and the concavity of $|f'|^q$ for any $t \in [0,1]$, we have

$$\begin{aligned}& \left|e^{f\left(at+(1-t)\frac{3a+b}{4}\right)} f'\left(at+(1-t)\frac{3a+b}{4}\right)\right|^q \\ & \geq \left(t|e^{f(a)}|+(1-t)|e^f(\frac{3a+b}{4})|\right)^q \left(t|f'(a)|+(1-t)|f'(\frac{3a+b}{4})|\right)^q.\end{aligned}$$

This implies that

$$\begin{aligned}& \left|e^{f\left(at+(1-t)\frac{3a+b}{4}\right)} f'\left(at+(1-t)\frac{3a+b}{4}\right)\right| \\ & \geq \left(t|e^{f(a)}|+(1-t)|e^f(\frac{3a+b}{4})|\right)\left(t|f'(a)|+(1-t)|f'(\frac{3a+b}{4})|\right).\end{aligned} \tag{16}$$

This shows that $|f'|$ is also concave. Using inequality (16) and (14) and applying the Jensen's integral inequality, we get

$$\rho_1 \leq \left(\int_0^1 \left[\left(at+(1-t)\left(\frac{3a+b}{4}\right)\right)^\alpha - \left(\frac{3a+b}{4}\right)^\alpha\right) dt\right) e^{f\left(\frac{\int_0^1 \left[\left(at+(1-t)\left(\frac{3a+b}{4}\right)\right)^\alpha - \left(\frac{3a+b}{4}\right)^\alpha\right]\left(at+(1-t)\frac{3a+b}{4}\right)dt}{\int_0^1 \left[\left(at+(1-t)\left(\frac{3a+b}{4}\right)\right)^\alpha - \left(\frac{3a+b}{4}\right)^\alpha\right]dt}\right)}$$
$$\times f'\left(\frac{\int_0^1 \left[\left(at+(1-t)\left(\frac{3a+b}{4}\right)\right)^\alpha - \left(\frac{3a+b}{4}\right)^\alpha\right]\left(at+(1-t)\frac{3a+b}{4}\right)dt}{\int_0^1 \left[\left(at+(1-t)\left(\frac{3a+b}{4}\right)\right)^\alpha - \left(\frac{3a+b}{4}\right)^\alpha\right]dt}\right) \tag{17}$$
$$= A_1(\alpha)e^{f\left(\frac{Q_1(\alpha)}{A_1(\alpha)}\right)} f'\left(\frac{Q_1(\alpha)}{A_1(\alpha)}\right),$$

where we have used the facts that

$$A_1(\alpha) = \int_0^1 \left[\left(at+(1-t)\left(\frac{3a+b}{4}\right)\right)^\alpha - \left(\frac{3a+b}{4}\right)^\alpha\right) dt$$
$$= \frac{(3a+b)^{\alpha+1}-(4a)^{\alpha+1}}{4^\alpha(b-a)(\alpha+1)} - \frac{(3a+b)^\alpha}{4^\alpha},$$
$$Q_1(\alpha) = \int_0^1 \left[at+(1-t)\left(\frac{3a+b}{4}\right)^\alpha - \left(\frac{3a+b}{4}\right)^\alpha\right]\left(at+(1-t)\frac{3a+b}{4}\right)dt$$
$$= \frac{2\left[(4a)^{\alpha+2}-(3a+b)^{\alpha+2}\right]-(\alpha+2)(3a+b)^\alpha(7a^2-6ab-b^2)}{(\alpha+2)(a-b)2^{2\alpha+3}},$$

$$Q_2(\alpha) = \int_0^1 \left[\left(\left(\frac{3a+b}{4}\right)t+(1-t)\left(\frac{a+b}{2}\right)\right)^\alpha - \left(\frac{3a+b}{4}\right)^\alpha\right]\left[\left(\frac{3a+b}{4}\right)t+(1-t)\left(\frac{a+b}{2}\right)\right]dt$$
$$= \frac{2\left[(3a+b)^{\alpha+2}-(2(a+b))^{\alpha+2}\right]-(\alpha+2)(3a+b)^\alpha(5a^2-2ab-3b^2)}{(\alpha+2)(a-b)2^{2\alpha+3}},$$

$$Q_3(\alpha) = \int_0^1 \left[\left(\left(\frac{a+b}{2}\right)t+(1-t)\left(\frac{a+3b}{4}\right)\right)^\alpha - \left(\frac{a+3b}{4}\right)^\alpha\right]\left[\left(\frac{a+b}{2}\right)t+(1-t)\left(\frac{a+3b}{4}\right)\right]dt$$
$$= \frac{2\left[(3a+b)^{\alpha+2}-(2(a+b))^{\alpha+2}\right]-(\alpha+2)(3a+b)^\alpha(5a^2-2ab-3b^2)}{(\alpha+2)(a-b)2^{2\alpha+3}},$$

and

$$Q_4(\alpha) = \int_0^1 \left[\left(\left(\frac{a+3b}{4}\right)t+(1-t)b\right)^\alpha - \left(\frac{a+3b}{4}\right)^\alpha\right]\left[\left(\frac{a+3b}{4}\right)t+(1-t)b\right]dt$$
$$= \frac{2\left[(a+3b)^{\alpha+2}-(4b)^{\alpha+2}\right]-(\alpha+2)(a+3b)^\alpha(a^2+6ab-7b^2)}{(\alpha+2)(a-b)2^{2\alpha+3}}.$$

Similarly, we get $\rho_2 \leq B_1(\alpha)e^{f\left(\frac{Q_2(\alpha)}{B_1(\alpha)}\right)} f'\left(\frac{Q_2(\alpha)}{B_1(\alpha)}\right)$, $\rho_3 \leq C_1(\alpha)e^{f\left(\frac{Q_3(\alpha)}{C_1(\alpha)}\right)} f'\left(\frac{Q_3(\alpha)}{C_1(\alpha)}\right)$ and $\rho_4 \leq D_1(\alpha)e^{f\left(\frac{Q_4(\alpha)}{D_1(\alpha)}\right)} f'\left(\frac{Q_4(\alpha)}{D_1(\alpha)}\right)$. Using ρ_1, ρ_2, ρ_3 and ρ_4 in (14), we obtain the required inequality (15). This completes the proof. □

Corollary 1. *If we choose $\alpha = 1$, then, under the assumption of Theorem 3, we have a new result*

$$\begin{aligned} \Upsilon_f(1;a,b) \quad & \leq \frac{b-a}{4}\Big[A_1(1)e^{f\left(\frac{Q_1(1)}{A_1(1)}\right)}f'\Big(\frac{Q_1(1)}{A_1(1)}\Big)+B_1(1)e^{f\left(\frac{Q_2(1)}{B_1(1)}\right)}f'\Big(\frac{Q_2(1)}{B_1(1)}\Big) \\ & +C_1(1)e^{f\left(\frac{Q_3(1)}{C_1(1)}\right)}f'\Big(\frac{Q_3(1)}{C_1(1)}\Big)+D_1(1)e^{f\left(\frac{Q_4(1)}{D_1(1)}\right)}f'\Big(\frac{Q_4(1)}{D_1(1)}\Big)\Big], \end{aligned}$$

where

$$A_1(1) = \frac{b-a}{8}, \quad B_1(1) = \frac{b-a}{8}, \quad C_1(1) = \frac{b-a}{8}, \quad D_1(1) = \frac{b-a}{8},$$

and

$$\begin{aligned} Q_1(1) \quad & = \frac{2\left[(4a)^3-(3a+b)^3\right]-3(3a+b)(7a^2-6ab-b^2)}{96(a-b)}, \\ Q_2(1) \quad & = \frac{2\left[(3a+b)^3-(2(a+b))^3\right]-3(3a+b)(5a^2-2ab-3b^2)}{96(a-b)}, \\ Q_3(1) \quad & = \frac{2\left[(3a+b)^3-(2(a+b))^3\right]-3(3a+b)(5a^2-2ab-3b^2)}{96(a-b)}, \\ Q_4(1) \quad & = \frac{2\left[(a+3b)^3-(4b)^3\right]-3(a+3b)(a^2+6ab-7b^2)}{96(a-b)}. \end{aligned}$$

3. Conclusions

In this paper, we have established several new conformable fractional integral inequalities of Hermite-Hadamard type for exponentially convex functions. If $\alpha = 1$, then, one can obtain the classical integrals (as a special case) from the general definition of Conformable fractional integrals. Consequently, we have obtained some new inequalities of Hermite–Hadamard type for exponentially convex functions involving classical integrals. The ideas and techniques of this paper may stimulate further research in this dynamic field.

Author Contributions: Conceptualization, M.A.N. and K.I.N.; Methodology, S.R.; Software, S.R.; Validation, M.A.N., K.I.N. and S.R.; Formal Analysis, M.A.N.; Investigation, S.R.; Resources, M.A.N.; Data Curation, S.R.; Writing—Original Draft Preparation, S.R.; Writing—M.A.N., K.I.N. and S.R. & Editing, M.A.N.; Visualization, M.A.N., K.I.N., S.R.; Supervision, M.A.N.; Project Administration, K.I.N.

Funding: This research received no external funding.

Acknowledgments: Authors are pleased to thank the Rector, COMSATS University Islamabad, Islamabad, Pakistan for providing excellent research and academic environments. authors are grateful to the referees for their valuable and constructive suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

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