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Existence Theorems for Mixed Riemann–Liouville and Caputo Fractional Differential Equations and Inclusions with Nonlocal Fractional Integro-Differential Boundary Conditions

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Abstract: In this paper, we discuss the existence and uniqueness of solutions for a new class of single and multi-valued boundary value problems involving both Riemann–Liouville and Caputo fractional derivatives, and nonlocal fractional integro-differential boundary conditions. Our results rely on modern tools of functional analysis. We also demonstrate the application of the obtained results with the aid of examples.

Keywords: fractional derivatives; fractional integral; boundary value problems; existence; uniqueness; fixed-point theorems

1. Introduction

Fractional differential equations have gained much importance due to their widespread applications in various disciplines of social and natural sciences, and engineering. In recent years, there has been a remarkable development in fractional calculus and fractional differential equations; for instance, see the monographs by Kilbas et al. [1], Lakshmikantham et al. [2], Miller and Ross [3], Podlubny [4], Samko et al. [5], Diethelm [6], Ahmad et al. [7] and the papers [8–16].

In the literature, one can find many works on boundary value problems containing mixed fractional derivatives of different types. In [17] the authors studied a new class of nonlinear differential equations with Caputo-type fractional derivatives of different orders, and Caputo-type integro-differential boundary conditions:

$$\begin{cases} D^{\alpha}[D^{\beta}x(t) - g(t, x(t))] = f(t, x(t)),\ t \in J := [0, T], \\ x(0) = 0,\ \ (D^{\gamma}x)(T) = \lambda(I^{\delta}x)(T), \end{cases} \tag{1}$$

where D^{χ} is Caputo fractional derivative of order $\chi \in \{\alpha, \beta, \gamma\}$, $0 < \alpha, \beta, \gamma < 1$, I^{δ} is the Riemann–Liouville fractional integral of order δ, $f, g : J \times \mathbb{R} \to \mathbb{R}$ are given functions and $\lambda \neq \dfrac{\Gamma(\beta+\delta+1)}{T^{\gamma+\delta}\Gamma(\beta-\gamma+1)}$.

In [18] the authors considered two Caputo–Hadamard type fractional derivatives in a neutral-type differential equation supplemented with Dirichlet boundary conditions:

$$\begin{cases} D^{\omega}[D^{\kappa}y(t) - h(t, y(t))] = f(t, y(t)), \ t \in J := [1, T], \ T > 1, \\ y(1) = 0, \ y(T) = 0, \end{cases} \tag{2}$$

where D^{ρ} denotes the Caputo–Hadamard fractional derivatives of order $\rho \in (0,1)$, $\rho = \omega, \kappa$ and $f, h : J \times \mathbb{R} \to \mathbb{R}$ are appropriate functions.

More recently, in [19], both Caputo–Hadamard and Hadamard–Caputo fractional derivatives were considered in the boundary values problems:

$$\begin{cases} {}^{C}D^{p}({}^{H}D^{q}x)(t) = f(t, x(t)), \qquad t \in (a, b), \\ \alpha_1 x(a) + \alpha_2({}^{H}D^{q}x)(a) = 0, \quad \beta_1 x(b) + \beta_2({}^{H}D^{q}x)(b) = 0, \end{cases} \tag{3}$$

and

$$\begin{cases} {}^{H}D^{q}({}^{C}D^{p}x)(t) = f(t, x(t)), \qquad t \in (a, b), \\ \alpha_1 x(a) + \alpha_2({}^{C}D^{p}x)(a) = 0, \quad \beta_1 x(b) + \beta_2({}^{C}D^{p}x)(b) = 0, \end{cases} \tag{4}$$

where ${}^{C}D^{p}$ and ${}^{H}D^{q}$ are the Caputo and Hadamard fractional derivatives of orders p and q, respectively, $0 < p, q \le 1$, $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $a > 0$ and $\alpha_i, \beta_i \in \mathbb{R}$, $i = 1, 2$.

Motivated by the above papers, we introduce and investigate a new boundary value problem involving both Riemann–Liouville and Caputo fractional derivatives, and nonlocal fractional integro-differential boundary conditions given by

$$\begin{cases} {}^{RL}D^{q}({}^{C}D^{r}x(t)) = f(t, x(t)), \ \ 0 < t < T, \\ x'(\xi) = \lambda \, {}^{C}D^{\nu}x(\eta), \quad x(T) = \mu \, I^{p}x(\zeta), \ \ \xi, \eta, \zeta \in (0, T), \end{cases} \tag{5}$$

where ${}^{RL}D^{q}$ denote the Riemann–Liouville fractional derivative of order q, $0 < q < 1$, ${}^{C}D^{r}$, ${}^{C}D^{\nu}$ denote the Caputo fractional derivatives of orders r and ν respectively, $0 < r < 1, 0 < \nu < q + r$, I^{p} is the Riemann–Liouville fractional integral of order $p > 0$, $f : J \times \mathbb{R} \to \mathbb{R}$ and $\lambda, \mu \in \mathbb{R}$.

The main results for the problem (5), based on Banach contraction mapping principle, Krasnoselskii fixed-point theorem and nonlinear alternative of Leray–Schauder type, are obtained in Section 3. In Section 4 we extend our study to the multi-valued analogue of the problem (5) given by

$$\begin{cases} {}^{RL}D^{q}({}^{C}D^{r}x(t)) \in F(t, x(t)), \ \ 0 < t < T, \\ x'(\xi) = \lambda \, {}^{C}D^{\nu}x(\eta), \quad x(T) = \mu \, I^{p}x(\zeta), \ \ \xi, \eta, \zeta \in (0, T), \end{cases} \tag{6}$$

where $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multi-valued map ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$) and all other constants are the same as defined in problem (5). We derive the existence results for the inclusion boundary value problem (6) with the aid of standard fixed-point theorems for multi-valued maps. In case of convex-valued right-hand side of the inclusion, we use Leray–Schauder nonlinear alternative for multi-valued maps. In the case of non-convex-valued right-hand side of the inclusion, we apply a fixed-point theorem for multi-valued contractions due to Covitz and Nadler.

Examples illustrating the obtained results are presented in Section 5, while we recall some basic concepts of fractional calculus, multi-valued analysis and fixed-point theory in Section 2. We also establish a preliminary result related to the linear variant of the problem (5) in this section. Section 6 contains concluding remarks and some interesting discussion for possible extensions.

2. Preliminaries

In this section, we outline some basic concepts of fractional calculus and multi-valued analysis, and state some fixed-point theorems related to our work.

2.1. Fractional Calculus

In this subsection, we recall some basic ideas of fractional calculus [1,4] and present known results needed in our forthcoming analysis.

Definition 1. *The Riemann–Liouville fractional derivative of order q for a function $f : (0,\infty) \to \mathbb{R}$ is defined by*

$$^{RL}D^q f(t) = \frac{1}{\Gamma(n-q)}\left(\frac{d}{dt}\right)^n \int_{0+}^{t} (t-s)^{n-q-1} f(s)ds, \quad q > 0, \;\; n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q, provided the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2. *The Riemann–Liouville fractional integral of order q for a function $f : (0,\infty) \to \mathbb{R}$ is defined by*

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_{0+}^{t} (t-s)^{q-1} f(s)ds, \qquad q > 0,$$

provided the right-hand side is pointwise defined on $(0,\infty)$.

Definition 3. *The Caputo derivative of fractional order q for a n-times differential function $f : (0,\infty) \to \mathbb{R}$ is defined as*

$$^{C}D^q f(t) = \frac{1}{\Gamma(n-q)} \int_{0+}^{t} (t-s)^{n-q-1}\left(\frac{d}{ds}\right)^n f(s)ds, \quad q > 0, \;\; n = [q]+1.$$

Lemma 1. *If $\alpha+\beta > 1$, then the equation $(I^\alpha I^\beta u)(t) = (I^{\alpha+\beta}u)(t), t \in J$ is satisfied for $u \in L^1(J,\mathbb{R})$.*

Lemma 2. *Let $\beta > \alpha$. Then the equation $(D^\alpha I^\beta u)(t) = (I^{\beta-\alpha}u)(t), t \in J$ is satisfied for $u \in C(J,\mathbb{R})$.*

Lemma 3. *Let $n = [\alpha]+1]$ if $\alpha \notin \mathbb{N}$ and $n = \alpha$ if $\alpha \in \mathbb{N}$. Then the following relations hold:*

(i) *for $k \in \{0,1,2,\ldots,n-1\}$, $D^\alpha t^k = 0$;*
(ii) *if $\beta > n$ then $D^\alpha t^{\beta-1} = \dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} t^{\beta-\alpha-1}$;*
(iii) $I^\alpha t^{\beta-1} = \dfrac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} t^{\beta+\alpha-1}$.

Lemma 4. *(see [1]) Let $q > 0$. Then, for $y \in C(0,T) \cap L(0,T)$, the following formula holds:*

$$^{RL}I^q \left(^{RL}D^q y\right)(t) = y(t) + c_1 t^{q-1} + c_2 t^{q-2} + \cdots + c_n t^{q-n},$$

where $c_i \in \mathbb{R}$, $i = 1,2,\ldots,n$ and $n-1 < q < n$.

Lemma 5. *(see [1]) Let $q > 0$. Then for $y \in C(0,T) \cap L(0,T)$ holds*

$$^{RL}I^q \left(^{C}D^q y\right)(t) = y(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$ and $n = [q]+1$.

Lemma 6. *Let*

$$\begin{aligned} \Lambda_1 &= \frac{\Gamma(q)}{\Gamma(q+r-1)}\xi^{q+r-2} - \lambda\frac{\Gamma(q)}{\Gamma(q+r-\nu)}\eta^{q+r-\nu-1} \neq 0, \\ \Lambda_2 &= \mu\frac{\Gamma(q)}{\Gamma(q+r+p)}\zeta^{q+r+p-1} - \frac{\Gamma(q)}{\Gamma(q+r)}T^{q+r-1}, \; \Lambda_3 = 1 - \mu\frac{1}{\Gamma(1+p)}\zeta^p \neq 0, \end{aligned} \tag{7}$$

and $y \in C([0,T],\mathbb{R})$. *Then the unique solution of the linear problem*

$$\begin{cases} {}^{RL}D^q({}^CD^r x(t)) = y(t), \ 0 < t < T, \\ x'(\xi) = \lambda\, {}^CD^\nu x(\eta), \quad x(T) = \mu\, I^p x(\zeta), \ \ \xi,\eta,\zeta \in (0,T), \end{cases} \tag{8}$$

is given by

$$\begin{aligned} x(t) \quad = \quad & I^{q+r}y(t) + \frac{1}{\Lambda_1}\Big[\lambda I^{q+r-\nu}y(\eta) - I^{q+r-1}y(\xi)\Big]\Big[\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} - \frac{\Lambda_2}{\Lambda_3}\Big] \\ & + \frac{1}{\Lambda_3}\Big[\mu I^{q+r+p}y(\zeta) - I^{q+r}y(T)\Big]. \end{aligned} \tag{9}$$

Proof. Applying the Riemann–Liouville fractional integral of order q to both sides of equation in (8), and using Lemma 5, we get

$$ {}^CD^r x(t) = I^q y(t) + c_1 t^{q-1}, \tag{10}$$

where $c_1 \in \mathbb{R}$. Next, applying Riemann–Liouville fractional integral of order r to both sides (10), we get

$$x(t) = I^{q+r}y(t) + c_1\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} + c_2, \tag{11}$$

where $c_2 \in \mathbb{R}$. From (11), we have

$$x'(t) = I^{q+r-1}y(t) + c_1\frac{\Gamma(q)}{\Gamma(q+r-1)}t^{q+r-2}, \tag{12}$$

$${}^CD^\nu x(t) = I^{q+r-\nu}y(t) + c_1\frac{\Gamma(q)}{\Gamma(q+r-\nu)}t^{q+r-\nu-1}, \tag{13}$$

$$I^p x(t) = I^{q+r+p}y(t) + c_1\frac{\Gamma(q)}{\Gamma(q+r+p)}t^{q+r+p-1} + c_2\frac{1}{\Gamma(1+p)}t^p. \tag{14}$$

Using (11)–(14) in the boundary conditions of (8), we obtain

$$c_1 = \frac{1}{\Lambda_1}[\lambda I^{q+r-\nu}y(\eta) - I^{q+r-1}y(\xi)],$$

$$c_2 = \frac{1}{\Lambda_3}\Big[\mu I^{q+r+p}y(\zeta) - I^{q+r}y(T) - \frac{\Lambda_2}{\Lambda_1}\{\lambda I^{q+r-\nu}y(\eta) - I^{q+r-1}y(\xi)\}\Big],$$

which, on substituting in (11), yields the solution (9). The converse follows by direct computation. The proof is completed. □

2.2. Multi-Valued Analysis

Let $C(J,\mathbb{R})$ denote the Banach space of continuous functions x from J into $\mathbb{R}$ with the norm $\|x\| = \sup\{|x(t)| : t \in J\}$. By $L^1(J,\mathbb{R})$ we denote the Banach space of Lebesgue integrable functions $y : J \longrightarrow \mathbb{R}$ endowed with the norm by $\|y\|_{L^1} = \int_a^b |y(t)|dt$.

Let $(X, \|\cdot\|)$ be a Banach space. A multi-valued map $F : X \to \mathcal{P}(X)$:

- (i) is convex (closed) valued if $F(x)$ is convex (closed) for all $x \in X$;
- (ii) is bounded on bounded sets if $F(B) = \bigcup_{x\in B} F(x)$ is bounded in X for all bounded set B of X, i.e., $\sup_{x\in B}\{\sup\{|y| : y \in F(x)\}\} < \infty$;
- (iii) is called upper semi-continuous (u.s.c. for short) on X if for each $x_0 \in X$ the set $F(x_0)$ is nonempty, closed subset of X, and for each open set $\mathcal{U}$ of X containing $F(x_0)$, there exists an open neighborhood $\mathcal{V}$ of x_0 such that $F(\mathcal{V}) \subseteq \mathcal{U}$;
- (iv) is said to be completely continuous if $F(B)$ is relatively compact for every bounded subset B of X;
- (v) has a fixed point if there exists $x \in X$ such that $x \in F(x)$.

For each $y \in C(J, \mathbb{R})$, the set of selections for the multi-valued map F is defined by

$$S_{F,y} = \{f \in L^1(J, \mathbb{R}) : f(t) \in F(t, y) \text{ for a.e. } t \in J\}.$$

In the following, by $\mathcal{P}_p$ we denote the set of all nonempty subsets of X which have the property "p"where "p"will be bounded (b), closed (cl), convex (c), compact (cp) etc. Thus $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ is bounded}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact}\}$, and $\mathcal{P}_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \text{ is compact and convex}\}$. Next, we define the graph of G to be the set $Gr(G) = \{(x, y) \in X \times Y, y \in G(x)\}$ and recall two useful results regarding closed graphs and upper-semicontinuity.

Lemma 7. *([20] Proposition 1.2) If $G : X \to \mathcal{P}_{cl}(Y)$ is u.s.c., then $Gr(G)$ is a closed subset of $X \times Y$, i.e., for every sequence $\{x_n\}_{n \in \mathbb{N}} \subset X$ and $\{y_n\}_{n \in \mathbb{N}} \subset Y$, if $x_n \to x_*$, $y_n \to y_*$ and $y_n \in G(x_n)$ when $n \to \infty$, then $y_* \in G(x_*)$. Conversely, if G is completely continuous and has a closed graph, then it is upper semi-continuous.*

Lemma 8. *([21]) Let X be a separable Banach space. Let $F : [0, T] \times \mathbb{R} \to \mathcal{P}_{cp,c}(X)$ be an L^1-Carathéodory multi-valued map and let Θ be a linear continuous mapping from $L^1(J, X)$ to $C(J, X)$. Then the operator*

$$\Theta \circ S_F : C(J, X) \to \mathcal{P}_{cp,c}(C(J, X)), \;\; x \mapsto (\Theta \circ S_F)(x) = \Theta(S_{F,x,y})$$

is a closed graph operator in $C(J, X) \times C(J, X)$.

For more details on multi-valued maps and the proof of the known results cited in this section, we refer the interested reader to the books by Deimling [20], Gorniewicz [22] and Hu and Papageorgiou [23].

2.3. Fixed-Point Theorems

In this subsection we collect the fixed-point theorems which are used in the proofs of our main results.

Lemma 9. (Banach fixed-point theorem) *[24] Let X be a Banach space, $D \subset X$ be closed and $F : D \to D$ is a strict contraction, i.e., $|Fx - Fy| \leq k|x - y|$ for some $k \in (0, 1)$ and all $x, y \in D$. Then F has a unique fixed point in D.*

Lemma 10. (Krasnoselskii fixed-point theorem) *[25]. Let M be a closed, bounded, convex, and nonempty subset of a Banach space X. Let A, B be the operators such that (a) $Ax + By \in M$ whenever $x, y \in M$; (b) A is compact and continuous; (c) B is a contraction mapping. Then there exists $z \in M$ such that z=Az+Bz.*

Lemma 11. (Nonlinear alternative for single-valued maps) *[26]. Let E be a Banach space, C be a closed, convex subset of E, U be an open subset of C and $0 \in U$. Suppose that $A : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) F has a fixed point in $\overline{U}$, or
(ii) there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0, 1)$ with $u = \lambda F(u)$.

Lemma 12. (Nonlinear alternative for Kakutani maps) *[26]. Let C be a closed convex subset of a Banach space E, and U be an open subset of C with $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{c,cv}(C)$ is an upper semi-continuous compact map. Then either*

(i) F has a fixed point in $\overline{U}$, or
(ii) there is a $u \in \partial U$ and $\mu \in (0, 1)$ with $u \in \mu F(u)$.

Lemma 13. (Covitz and Nadler fixed-point theorem) [27] *Let (X, d) be a complete metric space. If $N : X \to \mathcal{P}_{cl}(X)$ is a contraction, then $FixN \neq \emptyset$.*

3. Main Results for Single-Valued Problem (5)

Let $\mathcal{C} = C([0, T], \mathbb{R})$ denote the Banach space of all continuous functions from $[0, T]$ to $\mathbb{R}$ endowed with the norm defined by $\|x\| = \sup_{t\in[0,T]} |x(t)|$. By Lemma 6, we define an operator $\mathcal{A} : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned}(\mathcal{A}x)(t) &= I^{q+r} f(s, x(s))(t) \\ &+ \frac{1}{\Lambda_1}\Big[\lambda I^{q+r-\nu} f(s, x(s))(\eta) - I^{q+r-1} f(s, x(s))(\xi)\Big]\Big[\frac{\Gamma(q)}{\Gamma(q+r)} t^{q+r-1} - \frac{\Lambda_2}{\Lambda_3}\Big] \\ &+ \frac{1}{\Lambda_3}\Big[\mu I^{q+r+p} f(s, x(s))(\zeta) - I^{q+r} f(s, x(s))(T)\Big], \end{aligned} \tag{15}$$

with $\Lambda_1, \Lambda_3 \neq 0$. It should be noticed that problem (5) has solutions if and only if the operator $\mathcal{A}$ has fixed points.

For the sake of convenience, we put

$$\Phi_0 = \frac{\Gamma(q)}{\Gamma(q+r)} T^{q+r-1} + \frac{|\Lambda_2|}{|\Lambda_3|}, \tag{16}$$

$$\begin{aligned}\Phi &= \frac{T^{q+r}}{\Gamma(q+r+1)} + \frac{\Phi_0}{|\Lambda_1|}\left[|\lambda| \frac{\eta^{q+r-\nu}}{\Gamma(q+r-\nu+1)} + \frac{\xi^{q+r-1}}{\Gamma(q+r)}\right] \\ &+ \frac{1}{|\Lambda_3|}\left[|\mu| \frac{\zeta^{q+r+p}}{\Gamma(q+r+p+1)} + \frac{T^{q+r}}{\Gamma(q+r+1)}\right]. \end{aligned} \tag{17}$$

Our first result, dealing with the existence of a unique solution, is based on the Banach contraction mapping principle.

Theorem 1. *Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the Lipschitz condition:*

(H_1) *there exists a positive constant L such that*

$$|f(t, x) - f(t, y)| \leq L|x - y|, \; t \in [0, T], \;\; x, y \in \mathbb{R}.$$

If

$$L\Phi < 1, \tag{18}$$

then the boundary value problem (5) has a unique solution on $[0, T]$, where Φ is given by (17).

Proof. We transform the problem (5) into a fixed-point problem, $x = \mathcal{A}x$, where the operator $\mathcal{A}$ is defined by (15). By using the Banach's contraction mapping principle, we shall show that $\mathcal{A}$ has a fixed point which is the unique solution of problem (5).

We set $\sup_{t\in[0,T]} |f(t, 0)| = M < \infty$ and choose

$$r \geq \frac{M\Phi}{1 - L\Phi}.$$

Now, we show that $\mathcal{A}B_r \subset B_r$, where $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$. For any $x \in B_r$, we obtain by the assumption (H_1) that

$$\begin{aligned}
\|\mathcal{A}x\| &\leq \sup_{t\in[0,T]}\Bigg\{ I^{q+r}|f(s,x(s))|(t) + \frac{\Phi_0}{|\Lambda_1|}\Big[|\lambda| I^{q+r-\nu}|f(s,x(s))|(\eta) + I^{q+r-1}|f(s,x(s))|(\xi)\Big] \\
&\quad + \frac{1}{|\Lambda_3|}\Big[|\mu| I^{q+r+p}|f(s,x(s))|(\zeta) + I^{q+r}|f(s,x(s))|(T)\Big]\Bigg\} \\
&\leq \sup_{t\in[0,T]}\Bigg\{ I^{q+r}(|f(s,x(s)) - f(s,0)| + |f(s,0)|)(t) \\
&\quad + \frac{\Phi_0}{|\Lambda_1|}\Big[|\lambda| I^{q+r-\nu}(|f(s,x(s)) - f(s,0)| + |f(s,0)|)(\eta) \\
&\quad + I^{q+r-1}(|f(s,x(s)) - f(s,0)| + |f(s,0)|)(\xi)\Big] \\
&\quad + \frac{1}{|\Lambda_3|}\Big[|\mu| I^{q+r+p}(|f(s,x(s)) - f(s,0)| + |f(s,0)|)(\zeta) \\
&\quad + I^{q+r}(|f(s,x(s)) - f(s,0)| + |f(s,0)|)(T)\Big]\Bigg\} \\
&\leq (Lr+M)\Bigg\{\frac{T^{q+r}}{\Gamma(q+r+1)} + \frac{\Phi_0}{|\Lambda_1|}\left[|\lambda|\frac{\eta^{q+r-\nu}}{\Gamma(q+r-\nu+1)} + \frac{\xi^{q+r-1}}{\Gamma(q+r)}\right] \\
&\quad + \frac{1}{|\Lambda_3|}\left[|\mu|\frac{\zeta^{q+r+p}}{\Gamma(q+r+p+1)} + \frac{T^{q+r}}{\Gamma(q+r+1)}\right]\Bigg\} \\
&= (Lr+M)\Phi \\
&\leq r,
\end{aligned}$$

which implies that $\mathcal{A}B_r \subset B_r$. For $t \in [0,T]$ and $x, y \in \mathcal{C}$, we have

$$\begin{aligned}
|\mathcal{A}x(t) - \mathcal{A}y(t)| &\leq I^{q+r}|f(s,x(s)) - f(s,y(s))|(t) \\
&\quad + \frac{\Phi_0}{|\Lambda_1|}\Big[|\lambda| I^{q+r-\nu}|f(s,x(s))|(\eta) + I^{q+r-1}|f(s,x(s)) - f(s,y(s))|(\xi)\Big] \\
&\quad + \frac{1}{|\Lambda_3|}\Big[|\mu| I^{q+r+p}|f(s,x(s)) - f(s,y(s))|(\zeta) + I^{q+r}|f(s,x(s)) - f(s,y(s))|(T)\Big] \\
&\leq L\|x-y\|\Bigg\{\frac{T^{q+r}}{\Gamma(q+r+1)} + \frac{\Phi_0}{|\Lambda_1|}\left[|\lambda|\frac{\eta^{q+r-\nu}}{\Gamma(q+r-\nu+1)} + \frac{\xi^{q+r-1}}{\Gamma(q+r)}\right] \\
&\quad + \frac{1}{|\Lambda_3|}\left[|\mu|\frac{\zeta^{q+r+p}}{\Gamma(q+r+p+1)} + \frac{T^{q+r}}{\Gamma(q+r+1)}\right]\Bigg\} \\
&= L\Phi\|x-y\|,
\end{aligned}$$

which, on taking the norm for $t \in [0,T]$, yields $\|\mathcal{A}x - \mathcal{A}y\| \leq L\Phi\|x-y\|$. As $L\Phi < 1$, therefore $\mathcal{A}$ is a contraction. Hence, by the conclusion of Banach contraction mapping principle, the operator $\mathcal{A}$ has a unique fixed point which corresponds to a unique solution of the problem (5). The proof is completed. □

Next, we prove an existence result for the given problem by using Krasnoselskii fixed-point theorem.

Theorem 2. *Assume that $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the assumption (H_1). In addition we suppose that:*

(H_2) $|f(t,x)| \leq \delta(t), \forall (t,x) \in [0,T] \times \mathbb{R}$ *and* $\delta \in C([0,T],\mathbb{R}^+)$.

Then the boundary value problem (5) has at least one solution on $[0, T]$ *if*

$$L\Big(\Phi - \frac{T^{q+r}}{\Gamma(q+r+1)}\Big) < 1, \tag{19}$$

where Φ *is given by (17).*

Proof. We define $\sup_{t\in[0,T]} |\delta(t)| = \|\delta\|$ and choose a suitable constant $\bar{r}$ such that

$$\bar{r} \geq \|\delta\|\Phi.$$

Furthermore, we define operators $\mathcal{P}$ and $\mathcal{Q}$ on $B_{\bar{r}} = \{x \in \mathcal{C} : \|x\| \leq \bar{r}\}$ by

$$\begin{aligned}
(\mathcal{P}x)(t) &= I^{q+r} f(s, x(s))(t), \;\; t \in [0, T], \\
(\mathcal{Q}x)(t) &= \frac{1}{\Lambda_1}\Big[\lambda I^{q+r-\nu} f(s, x(s))(\eta) - I^{q+r-1} f(s, x(s))(\xi)\Big]\Big[\frac{\Gamma(q)}{\Gamma(q+r)} t^{q+r-1} - \frac{\Lambda_2}{\Lambda_3}\Big] \\
&\quad + \frac{1}{\Lambda_3}\Big[\mu I^{q+r+p} f(s, x(s))(\zeta) - I^{q+r} f(s, x(s))(T)\Big], \;\; t \in [0, T].
\end{aligned}$$

Observe that $\mathcal{A}x = \mathcal{P}x + \mathcal{Q}x$. For $x, y \in B_{\bar{r}}$, we have

$$\begin{aligned}
\|\mathcal{P}x + \mathcal{Q}y\| &\leq \|\delta\|\Bigg\{\frac{T^{q+r}}{\Gamma(q+r+1)} + \frac{\Phi_0}{|\Lambda_1|}\Bigg[|\lambda|\frac{\eta^{q+r-\nu}}{\Gamma(q+r-\nu+1)} + \frac{\xi^{q+r-1}}{\Gamma(q+r)}\Bigg] \\
&\quad + \frac{1}{|\Lambda_3|}\Bigg[|\mu|\frac{\zeta^{q+r+p}}{\Gamma(q+r+p+1)} + \frac{T^{q+r}}{\Gamma(q+r+1)}\Bigg]\Bigg\} \\
&= \|\delta\|\Phi \\
&\leq \bar{r}.
\end{aligned}$$

This shows that $\mathcal{P}x + \mathcal{Q}y \in B_{\bar{r}}$. It follows from the assumption (H_1) together with (19) that $\mathcal{Q}$ is a contraction. Since the function f is continuous, we have that the operator $\mathcal{P}$ is continuous. It is easy to verify that

$$\|\mathcal{P}x\| \leq \|\delta\|\frac{T^{q+r}}{\Gamma(q+r+1)}.$$

Therefore, $\mathcal{P}$ is uniformly bounded on $B_{\bar{r}}$. Next, we prove the compactness of the operator $\mathcal{P}$. Let us set $\sup_{(t,x)\in[0,T]\times B_{\bar{r}}} |f(t, x)| = \bar{f} < \infty$. Let $t_1, t_2 \in [0, T]$ with $t_1 < t_2$. Then we have

$$\begin{aligned}
&|(\mathcal{P}x)(t_2) - (\mathcal{P}x)(t_1)| \\
\leq\;& \frac{\bar{f}}{\Gamma(q+r)}\Big|\int_0^{t_1}[(t_2 - s)^{q+r-1} - (t_1 - s)^{q+r-1} ds\Big| + \frac{\bar{f}}{\Gamma(q+r)}\Big|\int_{t_1}^{t_2}(t_2 - s)^{q+r-1} ds\Big| \\
\leq\;& \frac{\bar{f}}{\Gamma(q+r+1)}[|t_2^{q+r} - t_1^{q+r}| + 2(t_2 - t_1)^{q+r}],
\end{aligned}$$

which is independent of x and tends to zero as $t_2 \to t_1$. Thus, $\mathcal{P}$ is equicontinuous. So $\mathcal{P}$ is relatively compact on $B_{\bar{r}}$. Hence, by the Arzelá-Ascoli theorem, $\mathcal{P}$ is compact on $B_{\bar{r}}$. Thus, all the assumptions of Lemma 10 are satisfied. Therefore, the boundary value problem (5) has at least one solution on $[0, T]$. The proof is completed. □

Remark 1. *In the above theorem, we can interchange the role of the operators $\mathcal{P}$ and $\mathcal{Q}$ to obtain a second result by replacing (19) with the following condition:*

$$L\frac{T^{q+r}}{\Gamma(q+r+1)}<1.$$

Now we prove our next existence result by means of Leray–Schauder nonlinear alternative.

Theorem 3. *Assume that $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is a continuous function. In addition, we suppose that:*

(H_3) *there exist a continuous nondecreasing functions $\psi:[0,\infty)\to(0,\infty)$ and a function $\phi\in C([0,T],\mathbb{R}^+)$ such that*

$$|f(t,x)|\le\phi(t)\psi(\|x\|)\ \ \text{for each } (t,x)\in[0,T]\times\mathbb{R};$$

(H_4) *there exists a constant $N>0$ such that*

$$\frac{N}{\Phi\|\phi\|\psi(N)}>1,$$

where Φ is given by (17).

Then the boundary value problem (5) has at least one solution on $[0,T]$.

Proof. Firstly, we shall show that the operator $\mathcal{A}$, defined by (15), *maps bounded sets (balls) into bounded sets in $\mathcal{C}$*. For a positive number R, let $B_R=\{x\in\mathcal{C}:\|x\|\le R\}$ be a bounded ball in $\mathcal{C}$. Then, for $t\in[0,T]$, we have

$$\begin{aligned}|\mathcal{A}x(t)| \le\;& I^{q+r}|f(s,x(s))|(t)\\ &+\frac{\Phi_0}{|\Lambda_1|}\Big[|\lambda|I^{q+r-\nu}|f(s,x(s))|(\eta)+I^{q+r-1}|f(s,x(s))|(\xi)\Big]\\ &+\frac{1}{|\Lambda_3|}\Big[|\mu|I^{q+r+p}|f(s,x(s))|(\zeta)+I^{q+r}|f(s,x(s))|(T)\Big]\\ \le\;& \phi(t)\psi(\|x\|)\left\{\frac{T^{q+r}}{\Gamma(q+r+1)}+\frac{\Phi_0}{|\Lambda_1|}\left[|\lambda|\frac{\eta^{q+r-\nu}}{\Gamma(q+r-\nu+1)}+\frac{\xi^{q+r-1}}{\Gamma(q+r)}\right]\right.\\ &\left.+\frac{1}{|\Lambda_3|}\left[|\mu|\frac{\zeta^{q+r+p}}{\Gamma(q+r+p+1)}+\frac{T^{q+r}}{\Gamma(q+r+1)}\right]\right\},\end{aligned}$$

which leads to $\|\mathcal{A}x\|\le\Phi\|\phi\|\psi(R)$.

Secondly, we show that $\mathcal{A}$ *maps bounded sets into equicontinuous sets of $\mathcal{C}$*. Let $\nu_1,\nu_2\in[0,T]$ with $\nu_1<\nu_2$ and $x\in B_R$. Then, as argued in the proof of Theorem 2, we have

$$\begin{aligned}&|(\mathcal{A}x)(\nu_2)-(\mathcal{A}x)(\nu_1)|\\ \le\;& I^{q+r}(|f(s,x(s))(\nu_2)-f(s,x(s))(\nu_1)|)\\ &+\frac{|\nu_2^{q+r-1}-\nu_1^{q+r-1}|}{|\Lambda_1|}\frac{\Gamma(q)}{\Gamma(q+r)}\left[|\lambda|\frac{\eta^{q+r-\nu}}{\Gamma(q+r-\nu+1)}+\frac{\xi^{q+r-1}}{\Gamma(q+r)}\right]\\ \le\;& \frac{\|\phi\|\psi(R)}{\Gamma(r+1)}[|t_2^r-t_2^r|+2(t_2-t_1)^r]\\ &+\frac{|\nu_2^{q+r-1}-\nu_1^{q+r-1}|}{|\Lambda_1|}\frac{\Gamma(q)}{\Gamma(q+r)}\left[|\lambda|\frac{\eta^{q+r-\nu}}{\Gamma(q+r-\nu+1)}+\frac{\xi^{q+r-1}}{\Gamma(q+r)}\right].\end{aligned}$$

Obviously the right-hand side of the above inequality tends to zero independently of $x\in B_R$ as $\nu_2\to\nu_1$. Therefore, it follows by the Arzelá-Ascoli theorem that $\mathcal{F}:\mathcal{C}\to\mathcal{C}$ is completely continuous.

The result will follow from the Leray–Schauder nonlinear alternative once it is shown that the set of all solutions to the equation $x = \theta\mathcal{A}x$ is bounded for $\theta \in [0,1]$. For that, let x be a solution of $x = \theta\mathcal{A}x$ for $\theta \in [0,1]$. Then, for $t \in [0,T]$, and following the similar computations as in the first step, we have

$$\|x\| \leq \Phi\|\phi\|\psi(\|x\|).$$

Consequently, we have

$$\frac{\|x\|}{\Phi\|\phi\|\psi(\|x\|)} \leq 1.$$

In view of (H_4), there exists N such that $\|x\| \neq N$. Let us set

$$U = \{x \in \mathcal{C} : \|x\| < N\}. \tag{20}$$

Please note that the operator $\mathcal{A} : \overline{U} \to \mathcal{C}$ is continuous and completely continuous. From the choice of U, there is no $x \in \partial U$ such that $x = \theta\mathcal{A}x$ for some $\theta \in (0,1)$. Consequently, by nonlinear alternative of Leray–Schauder type (Lemma 11), we deduce that $\mathcal{A}$ has a fixed point in $\overline{U}$, which is a solution of the boundary value problem (5). This completes the proof. □

4. Existence Results for Multi-Valued Problem (6)

Definition 4. *A function $x \in C^1(J,\mathbb{R})$ is said to be a solution of the problem (6) if there exists a function $v \in L^1(J,\mathbb{R})$ with $v(t) \in F(t,x)$ a.e. on J such that*

$$\begin{aligned} x(t) \quad = \quad & I^{q+r}v(s)(t) + \frac{1}{\Lambda_1}\Big[\lambda I^{q+r-\nu}v(s)(\eta) - I^{q+r-1}v(s)(\xi)\Big]\Big[\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} - \frac{\Lambda_2}{\Lambda_3}\Big] \\ & + \frac{1}{\Lambda_3}\Big[\mu I^{q+r+p}v(s)(\zeta) - I^{q+r}v(s)(T)\Big], \end{aligned}$$

and $x'(\xi) = \lambda{}^{C}D^{\nu}x(\eta),\ x(T) = \mu I^{p}x(\zeta)$.

4.1. The Upper Semi-Continuous Case

Our first result, dealing with the convex-valued F, is based on Leray–Schauder nonlinear alternative for multi-valued maps.

Definition 5. *A multi-valued map $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if (i) $t \longmapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$ and (ii) $x \longmapsto F(t,x)$ is upper semi-continuous for almost all $t \in J$. Further a Carathéodory function F is called L^1-Carathéodory if (iii) for each $\rho > 0$, there exists $\varphi_\rho \in L^1(J,\mathbb{R}^+)$ such that $\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq \varphi_\rho(t)$ for all $x \in \mathbb{R}$ with $\|x\| \leq \rho$ and for a.e. $t \in J$.*

Theorem 4. *Assume that:*

- (A_1) *$F : J \times \mathbb{R} \to \mathcal{P}_{cp,c}(\mathbb{R})$ is L^1-Carathéodory;*
- (A_2) *there exists a continuous nondecreasing function $Q : [0,\infty) \to (0,\infty)$ and a function $P \in C(J,\mathbb{R}^+)$ such that*
  $\|F(t,x)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t,x)\} \leq P(t)Q(\|x\|)$ for each $(t,x) \in J \times \mathbb{R}$;
- (A_3) *there exists a constant $M > 0$ such that*

$$\frac{M}{\Phi\|P\|Q(M)} > 1,$$

  where Φ is given by (17).

Then the boundary value problem (6) has at least one solution on J.

Proof. Firstly, we transform the problem (6) into a fixed-point problem. Consider the multi-valued map: $N : C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ defined by

$$N(x) = \left\{ \begin{array}{l} h \in C(J,\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} I^{q+r}v(s)(t) \\ +\dfrac{1}{\Lambda_1}\Big[\lambda I^{q+r-\nu}v(s)(\eta) - I^{q+r-1}v(s)(\xi)\Big]\Big[\dfrac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} - \dfrac{\Lambda_2}{\Lambda_3}\Big] \\ +\dfrac{1}{\Lambda_3}\Big[\mu I^{q+r+p}v(s)(\zeta) - I^{q+r}v(s)(T)\Big], \end{array} \right. \end{array} \right\}$$

for $v \in S_{F,x}$. Clearly the fixed points of N are solutions of problem (6). Now we proceed to show that the operator N satisfies all condition of Lemma 12. This is done in several steps.

Step 1. *$N(x)$ is convex for each $x \in C(J,\mathbb{R})$.*

Indeed, if h_1, h_2 belongs to $N(x)$, then there exist $v_1, v_2 \in S_{F,x}$ such that for each $t \in J$, we have

$$\begin{aligned} h_i(t) &= I^{q+r}v_i(s)(t) \\ &+ \frac{1}{\Lambda_1}\Big[\lambda I^{q+r-\nu}v_i(s)(\eta) - I^{q+r-1}v_i(s)(\xi)\Big]\Big[\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} - \frac{\Lambda_2}{\Lambda_3}\Big] \\ &+ \frac{1}{\Lambda_3}\Big[\mu I^{q+r+p}v_i(s)(\zeta) - I^{q+r}v_i(s)(T)\Big], \; i = 1,2. \end{aligned}$$

Let $0 \le \theta \le 1$. Then, for each $t \in J$, we have

$$\begin{aligned} & [\theta h_1 + (1-\theta)h_2](t) \\ = \; & I^{q+r}[\theta v_1(s) + (1-\theta)v_2(s)])(s)(t) \\ & + \frac{1}{\Lambda_1}\Big[\lambda I^{q+r-\nu}[\theta v_1(s) + (1-\theta)v_2(s)])(s)(\eta) \\ & - I^{q+r-1}[\theta v_1(s) + (1-\theta)v_2(s)])(s)(\xi)\Big]\Big[\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} - \frac{\Lambda_2}{\Lambda_3}\Big] \\ & + \frac{1}{\Lambda_3}\Big[\mu I^{q+r+p}[\theta v_1(s) + (1-\theta)v_2(s)])(s)(\zeta) - I^{q+r}[\theta v_1(s) + (1-\theta)v_2(s)])(s)(T)\Big]. \end{aligned}$$

Since F has convex values ($S_{F,x}$ is convex), therefore $\theta h_1 + (1-\theta)h_2 \in N(x)$.

Step 2. *$N(x)$ maps bounded sets (balls) into bounded sets in $C(J,\mathbb{R})$.*

For a positive number r, let $B_r = \{x \in C(J,\mathbb{R}) : \|x\| \le r\}$ be a bounded ball in $C(J,\mathbb{R})$. Then, for each $h \in N(x), x \in B_r$, there exists $v \in S_{F,x}$ such that

$$\begin{aligned} h(t) &= I^{q+r}v(s)(t) \\ &+ \frac{1}{\Lambda_1}\Big[\lambda I^{q+r-\nu}v(s)(\eta) - I^{q+r-1}v(s)(\xi)\Big]\Big[\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} - \frac{\Lambda_2}{\Lambda_3}\Big] \\ &+ \frac{1}{\Lambda_3}\Big[\mu I^{q+r+p}v(s)(\zeta) - I^{q+r}v(s)(T)\Big]. \end{aligned}$$

In view of (H_2), for each $t \in J$, we have

$$\begin{aligned}
|h(t)| &\leq I^{q+r}|v(s)|(t) \\
&\quad + \frac{\Phi_0}{|\Lambda_1|}\Big[|\lambda| I^{q+r-\nu}|v(s)|(\eta) + I^{q+r-1}|v(s)|(\xi)\Big] \\
&\quad + \frac{1}{|\Lambda_3|}\Big[|\mu| I^{q+r+p}v(s)(\zeta) + I^{q+r}v(s)(T)\Big] \\
&\leq P(t)Q(\|x\|)\left\{\frac{T^{q+r}}{\Gamma(q+r+1)} + \frac{\Phi_0}{|\Lambda_1|}\left[|\lambda|\frac{\eta^{q+r-\nu}}{\Gamma(q+r-\nu+1)} + \frac{\xi^{q+r-1}}{\Gamma(q+r)}\right]\right. \\
&\quad \left. + \frac{1}{|\Lambda_3|}\left[|\mu|\frac{\zeta^{q+r+p}}{\Gamma(q+r+p+1)} + \frac{T^{q+r}}{\Gamma(q+r+1)}\right]\right\},
\end{aligned}$$

which yields

$$\|h\| \leq \Phi\|P\|Q(r).$$

Step 3. *$N(x)$ maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$.*

Let x be any element in B_r and $h \in N(x)$. Then there exists a function $v \in S_{F,x}$ such that for each $t \in J$, we have

$$\begin{aligned}
h(t) &= I^{q+r}v(s)(t) \\
&\quad + \frac{1}{\Lambda_1}\Big[\lambda I^{q+r-\nu}v(s)(\eta) - I^{q+r-1}v(s)(\xi)\Big]\Big[\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} - \frac{\Lambda_2}{\Lambda_3}\Big] \\
&\quad + \frac{1}{\Lambda_3}\Big[\mu I^{q+r+p}v(s)(\zeta) - I^{q+r}v(s)(T)\Big].
\end{aligned}$$

Let $\tau_1,\ \tau_2 \in J,\ \tau_1 < \tau_2$. Then

$$\begin{aligned}
|h(\tau_2) - h(\tau_1)| &\leq I^{q+r}(|v(s)(\nu_2) - v(s)(\nu_1)|) \\
&\quad + \frac{|\nu_2^{q+r-1} - \nu_1^{q+r-1}|}{|\Lambda_1|}\frac{\Gamma(q)}{\Gamma(q+r)}\left[|\lambda|\frac{\eta^{q+r-\nu}}{\Gamma(q+r-\nu+1)} + \frac{\xi^{q+r-1}}{\Gamma(q+r)}\right] \\
&\leq \frac{\|P\|Q(r)}{\Gamma(r+1)}[|t_2^r - t_2^r| + 2(t_2 - t_1)^r] \\
&\quad + \frac{|\nu_2^{q+r-1} - \nu_1^{q+r-1}|}{|\Lambda_1|}\frac{\Gamma(q)}{\Gamma(q+r)}\left[|\lambda|\frac{\eta^{q+r-\nu}}{\Gamma(q+r-\nu+1)} + \frac{\xi^{q+r-1}}{\Gamma(q+r)}\right].
\end{aligned}$$

The right-hand side of the above inequality tends to zero independently of $x \in B_r$ as $\tau_1 \to \tau_2$.

As a consequence of Steps 1–3 together with Arzelá-Ascoli theorem, we conclude that $N : C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ is completely continuous.

Since N is completely continuous, it is enough to show that it has a closed graph in view of Lemma 7, which will imply that N is u.s.c. This is done in the following step.

Step 4. *N has a closed graph.*

Let $x_n \to x_*$, $h_n \in N(x_n)$ and $h_n \to h_*$. Then we need to show that $h_* \in N(x_*)$. Observe that $h_n \in N(x_n)$ implies that there exists $v_n \in S_{F,x_n}$ such that for each $t \in J$,

$$\begin{aligned}
h_n(t) &= I^{q+r}v_n(s)(t) \\
&\quad + \frac{1}{\Lambda_1}\Big[\lambda I^{q+r-\nu}v_n(s)(\eta) - I^{q+r-1}v_n(s)(\xi)\Big]\Big[\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} - \frac{\Lambda_2}{\Lambda_3}\Big] \\
&\quad + \frac{1}{\Lambda_3}\Big[\mu I^{q+r+p}v_n(s)(\zeta) - I^{q+r}v_n(s)(T)\Big].
\end{aligned}$$

Therefore, we must show that there exists $v_* \in S_{F,x_*}$ such that, for each $t \in J$,

$$\begin{aligned} h_*(t) &= I^{q+r}v_*(s)(t) \\ &+ \frac{1}{\Lambda_1}\Big[\lambda I^{q+r-\nu}v_*(s)(\eta) - I^{q+r-1}v_*(s)(\xi)\Big]\Big[\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} - \frac{\Lambda_2}{\Lambda_3}\Big] \\ &+ \frac{1}{\Lambda_3}\Big[\mu I^{q+r+p}v_*(s)(\zeta) - I^{q+r}v_*(s)(T)\Big]. \end{aligned}$$

Consider the continuous linear operator $\Theta : L^1(J,\mathbb{R}) \to C(J,\mathbb{R})$ defined by

$$\begin{aligned} v \to \Theta(v)(t) &= I^{q+r}v(s)(t) \\ &+ \frac{1}{\Lambda_1}\Big[\lambda I^{q+r-\nu}v(s)(\eta) - I^{q+r-1}v(s)(\xi)\Big]\Big[\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} - \frac{\Lambda_2}{\Lambda_3}\Big] \\ &+ \frac{1}{\Lambda_3}\Big[\mu I^{q+r+p}v(s)(\zeta) - I^{q+r}v(s)(T)\Big]. \end{aligned}$$

Obviously $\|h_n(t) - h_*(t)\| \to 0$ as $n \to \infty$. So it follows from Lemma 8, that $\Theta \circ S_{F,x}$ is a closed graph operator. Moreover, we have

$$h_n \in \Theta(S_{F,x_n}).$$

Since $x_n \to x_*$, Lemma 8 implies that

$$\begin{aligned} h_*(t) &= I^{q+r}v_*(s)(t) \\ &+ \frac{1}{\Lambda_1}\Big[\lambda I^{q+r-\nu}v_*(s)(\eta) - I^{q+r-1}v_*(s)(\xi)\Big]\Big[\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} - \frac{\Lambda_2}{\Lambda_3}\Big] \\ &+ \frac{1}{\Lambda_3}\Big[\mu I^{q+r+p}v_*(s)(\zeta) - I^{q+r}v_*(s)(T)\Big], \end{aligned}$$

for some $v_* \in S_{F,x_*}$.

Step 5. *We show there exists an open set $U \subseteq C(J,\mathbb{R})$ with $x \notin \theta N(x)$ for any $\theta \in (0,1)$ and all $x \in \partial U$.*

Let $\theta \in (0,1)$ and $x \in \theta N(x)$. Then there exists $v \in L^1(J,\mathbb{R})$ with $v \in S_{F,x}$ such that, for $t \in J$, we have

$$\begin{aligned} x(t) &= \theta I^{q+r}v(s)(t) \\ &+ \theta\frac{1}{\Lambda_1}\Big[\lambda I^{q+r-\nu}v(s)(\eta) - I^{q+r-1}v(s)(\xi)\Big]\Big[\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} - \frac{\Lambda_2}{\Lambda_3}\Big] \\ &+ \theta\frac{1}{\Lambda_3}\Big[\mu I^{q+r+p}v(s)(\zeta) - I^{q+r}v(s)(T)\Big]. \end{aligned}$$

Using the method of computation employed in Step 2, for each $t \in J$, we get

$$|x(t)| \le \Phi\|P\|Q(\|x\|),$$

which can alternatively be written as

$$\frac{\|x\|}{\Phi\|P\|Q(\|x\|)} \le 1.$$

In view of (A_3), there exists M such that $\|x\| \neq M$. Let us set

$$U = \{x \in C(J,\mathbb{R}) : \|x\| < M\}.$$

Note that the operator $N : \overline{U} \to \mathcal{P}(C(J, \mathbb{R}))$ is a compact multi-valued map, u.s.c. with convex closed values. From the choice of U, there is no $x \in \partial U$ such that $x \in \theta N(x)$ for some $\theta \in (0, 1)$. Consequently, by the nonlinear alternative of Leray–Schauder type (Lemma 12), we deduce that N has a fixed point $x \in \overline{U}$ which is a solution of the boundary value problem (6). This completes the proof. □

4.2. The Lipschitz Case

Now we prove the existence of solutions for the boundary value problem (6) with a non-convex-valued right-hand side by applying a fixed-point theorem for multi-valued map due to Covitz and Nadler [27].

Let (X, d) be a metric space induced from the normed space $(X; \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{R} \cup \{\infty\}$ given by

$$H_d(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b)\},$$

where $d(A, b) = \inf_{a \in A} d(a; b)$ and $d(a, B) = \inf_{b \in B} d(a; b)$. Then $(\mathcal{P}_{b,cl}(X), H_d)$ is a metric space (see [28]).

Definition 6. *A multi-valued operator $N : X \to \mathcal{P}_{cl}(X)$ is called*

(**a**) *$\gamma-$Lipschitz if and only if there exists $\gamma > 0$ such that*

$$H_d(N(x), N(y)) \le \gamma d(x, y) \text{ for each } x, y \in X;$$

(**b**) *a contraction if and only if it is $\gamma-$Lipschitz with $\gamma < 1$.*

Theorem 5. *Assume that the following conditions hold:*

(A_4) *$F : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$ is such that $F(\cdot, x) : J \to \mathcal{P}_{cp}(\mathbb{R})$ is measurable for each $x \in \mathbb{R}$;*
(A_5) *$H_d(F(t, x), F(t, \bar{x})) \le m(t)|x - \bar{x}|$ for almost all $t \in J$ and $x, \bar{x} \in \mathbb{R}$ with $m \in C(J, \mathbb{R}^+)$ and $d(0, F(t, 0)) \le m(t)$ for almost all $t \in J$.*

Then the boundary value problem (6) has at least one solution on J if

$$\Phi\|m\| < 1,$$

where Φ is given by (17).

Proof. We transform the boundary value problem (6) into a fixed-point problem by considering the operator $N : C(J, \mathbb{R}) \to \mathcal{P}(C(J, \mathbb{R}))$ defined in the beginning of the proof of Theorem 12. Actually, we show that the operator N satisfies the assumptions of Lemma 13. The proof will be given in two steps.

Step I. $N(x)$ is nonempty and closed for every $v \in S_{F,x}$.

Since the set-valued map $F(\cdot, x(\cdot))$ is measurable, by the measurable selection theorem (e.g., ([29] Theorem III.6)), it admits a measurable selection $v : J \to \mathbb{R}$. Moreover, by the assumption (A_5), we have

$$|v(t)| \le m(t) + m(t)|x(t)|,$$

i.e., $v \in L^1(J, \mathbb{R})$ and hence F is integrably bounded. Therefore, $S_{F,x} \neq \emptyset$.

Next we show that $N(x)$ is closed for each $x \in C(J, \mathbb{R})$. Let $\{u_n\}_{n \ge 0} \in N(x)$ be such that $u_n \to u$ $(n \to \infty)$ in $C(J, \mathbb{R})$. Then $u \in C(J, \mathbb{R})$ and there exists $v_n \in S_{F,x_n}$ such that, for each $t \in J$,

$$\begin{aligned} u_n(t) &= I^{q+r}v_n(s)(t) \\ &+ \frac{1}{\Lambda_1}\Big[\lambda I^{q+r-\nu}v_n(s)(\eta) - I^{q+r-1}v_n(s)(\xi)\Big]\Big[\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} - \frac{\Lambda_2}{\Lambda_3}\Big] \\ &+ \frac{1}{\Lambda_3}\Big[\mu I^{q+r+p}v_n(s)(\zeta) - I^{q+r}v_n(s)(T)\Big]. \end{aligned}$$

As F has compact values, we pass onto a subsequence (if necessary) to obtain that v_n converges to v in $L^1(J,\mathbb{R})$. Thus, $v \in S_{F,x}$ and for each $t \in J$, we have

$$\begin{aligned} u_n(t) \to v(t) &= I^{q+r}v(s)(t) \\ &+ \frac{1}{\Lambda_1}\Big[\lambda I^{q+r-\nu}v(s)(\eta) - I^{q+r-1}v(s)(\xi)\Big]\Big[\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} - \frac{\Lambda_2}{\Lambda_3}\Big] \\ &+ \frac{1}{\Lambda_3}\Big[\mu I^{q+r+p}v(s)(\zeta) - I^{q+r}v(s)(T)\Big]. \end{aligned}$$

Hence $u \in N(x)$.

Step II. Next we show that there exists $0 < \hat{\theta} < 1$ ($\hat{\theta} = \Phi\|m\|$) such that

$$H_d(N(x), N(\bar{x})) \le \hat{\theta}\|x - \bar{x}\| \ \text{ for each } \ x, \bar{x} \in C(J,\mathbb{R}).$$

Let $x, \bar{x} \in C(J,\mathbb{R})$ and $h_1 \in N(x)$. Then there exists $v_1(t) \in F(t, x(t))$ such that, for each $t \in J$,

$$\begin{aligned} h_1(t) &= I^{q+r}v_1(s)(t) \\ &+ \frac{1}{\Lambda_1}\Big[\lambda I^{q+r-\nu}v_1(s)(\eta) - I^{q+r-1}v_1(s)(\xi)\Big]\Big[\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} - \frac{\Lambda_2}{\Lambda_3}\Big] \\ &+ \frac{1}{\Lambda_3}\Big[\mu I^{q+r+p}v_1(s)(\zeta) - I^{q+r}v_1(s)(T)\Big]. \end{aligned}$$

By (A_5), we have

$$H_d(F(t,x), F(t,\bar{x})) \le m(t)|x(t) - \bar{x}(t)|.$$

Therefore, there exists $w(t) \in F(t, \bar{x}(t))$ such that

$$|v_1(t) - w| \le m(t)|x(t) - \bar{x}(t)|, \ \ t \in J.$$

Define $U : J \to \mathcal{P}(\mathbb{R})$ by

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \le m(t)|x(t) - \bar{x}(t)|\}.$$

Since the multi-valued operator $U(t) \cap F(t, \bar{x}(t))$ is measurable (Proposition III.4 [29]), there exists a function $v_2(t)$ which is a measurable selection for U. So $v_2(t) \in F(t, \bar{x}(t))$ and for each $t \in J$, we have $|v_1(t) - v_2(t)| \le m(t)|x(t) - \bar{x}(t)|$.

For each $t \in J$, let us define

$$\begin{aligned} h_2(t) &= I^{q+r}v_2(s)(t) \\ &+ \frac{1}{\Lambda_1}\Big[\lambda I^{q+r-\nu}v_2(s)(\eta) - I^{q+r-1}v_2(s)(\xi)\Big]\Big[\frac{\Gamma(q)}{\Gamma(q+r)}t^{q+r-1} - \frac{\Lambda_2}{\Lambda_3}\Big] \\ &+ \frac{1}{\Lambda_3}\Big[\mu I^{q+r+p}v_2(s)(\zeta) - I^{q+r}v_2(s)(T)\Big]. \end{aligned}$$

In consequence, we get

$$\begin{aligned}
& |h_1(t)-h_2(t)| \\
\leq\ & I^{q+r}|v_2(s)-v_1(s)|(t) \\
& +\frac{\Phi_0}{|\Lambda_1|}\Big[|\lambda| I^{q+r-\nu}|v_2(s)-v_1(s)|(\eta)+I^{q+r-1}|v_2(s)-v_1(s)|(\xi)\Big] \\
& +\frac{1}{|\Lambda_3|}\Big[|\mu| I^{q+r+p}|v_2(s)-v_1(s)|(\zeta)+I^{q+r}|v_2(s)-v_1(s)|(T)\Big] \\
\leq\ & \left\{\frac{T^{q+r}}{\Gamma(q+r+1)}+\frac{\Phi_0}{|\Lambda_1|}\left[|\lambda|\frac{\eta^{q+r-\nu}}{\Gamma(q+r-\nu+1)}+\frac{\xi^{q+r-1}}{\Gamma(q+r)}\right]\right. \\
& \left.+\frac{1}{|\Lambda_3|}\left[|\mu|\frac{\zeta^{q+r+p}}{\Gamma(q+r+p+1)}+\frac{T^{q+r}}{\Gamma(q+r+1)}\right]\right\}\|m\|\|x-\overline{x}\|.
\end{aligned}$$

Hence

$$\|h_1-h_2\| \leq \Phi\|m\|\|x-\overline{x}\|.$$

Analogously, interchanging the roles of x and $\overline{x}$, we obtain

$$H_d(N(x),N(\bar{x})) \leq \Phi\|m\|\|x-\overline{x}\|.$$

Since N is a contraction, it follows by Lemma 13 that N has a fixed point x which is a solution of (6). This completes the proof. □

5. Examples

Consider the following nonlinear Riemann–Liouville and Caputo-type fractional boundary value problem

$$\begin{cases} {}^{RL}D^{4/5}\left({}^{C}D^{1/2}x(t)\right)(t)=f(t,x(t)) \text{ or } \in F(t,x(t)), \quad 0<t<3, \\ x'(1/2)=1/12\,{}^{C}D^{1/2}x(1/3), \quad x(3)=2\,I^{2/3}x(1/2). \end{cases} \tag{21}$$

Here $q=4/5$, $r=1/2$, $\xi=1/2$, $\lambda=1/12$, $\nu=1/2$, $\eta=1/3$, $T=3$, $\mu=2$, $p=1/2$, $\zeta=1/2$. With these data we find $\Lambda_1 \approx 0.13659$, $\Lambda_2 \approx 2.371158$, $\Lambda_3 \approx -1.256322$, $\Phi_0 \approx 6.525634$, $\Phi \approx 49.056696$.

5.1. Single-Valued Case

(i). Let

$$f(t,x)=\frac{\cos^2(\pi t)}{t^2+250}\left(\frac{|x|}{|x|+1}\right)+\cos(2\pi t). \tag{22}$$

Please note that $|f(t,x)-f(t,y)| \leq (1/(250))|x-y|$ and thus (H_1) is satisfied with $L=1/(250)$. Since $L\Phi \approx 0.736559 < 1$ by Theorem 1, the boundary value problem (21), with f given by (22), has a unique solution on $[0,3]$.

(ii). With the function f given by (22), we remark that $f(t,x)| \leq \frac{1}{t^2+250}+1$ and $L\left(\Phi-\dfrac{T^{q+r}}{\Gamma(q+r+1)}\right) \approx 0.181925 < 1$.

Hence, by Theorem 2, the boundary value problem (21), with f given by (22), has at least one solution on $[0,3]$.

(iii). Next consider

$$f(t,x)=\frac{1}{t+350}(x(t)\cos x(t)+5). \tag{23}$$

It is easy to find that $|f(t,x)| \leq \frac{1}{t+50}(\|x\|+5)$. Then by condition (H_4) with $\psi(\|x\|) = \|x\|+5$ snf $\|\phi\| = 1/(350)$, we find that $N > N_1 \approx 0.815048$.

Hence, by Theorem 3, the boundary value problem (21), with f given by (23), has at least one solution on $[0,3]$.

5.2. Multi-Valued Case

(I). Consider the multi-valued map $F : [0,3] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ given by

$$x \to F(t,x) = \left[\frac{1}{t^2+200}\Big(\frac{x}{3}\Big(\frac{|x|}{|x|+1}+2\Big)+1\Big), \frac{e^{-t}}{19+t}\Big(\sin x + \frac{1}{100}\Big)\right]. \tag{24}$$

Clearly the multi-valued map F satisfies condition (A_1) and that

$$\|F(t,x)\|_{\mathcal{P}} \leq \frac{1}{t^2+200}(\|x\|+1) := P(t)Q(|x|),$$

which yields $\|P\| = 1/200$ and $Q(\|x\|) = \|x\|+1$. Therefore, the condition (A_2) is fulfilled. By direct computation, there exists a constant $M > M_1 \approx 0.3250$ satisfying condition (A_3). Hence all assumptions of Theorem 4 hold and hence the problem (21), with F given by (24), has at least one solution on $[0,3]$.

(II). Let the multi-valued map $F : [0,3] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be defined by

$$x \to F(t,x) = \left[0, \frac{1}{200+12t}\left(\frac{x^2+2|x|}{1+|x|}\right)+\frac{1}{400}\right]. \tag{25}$$

Choosing $m(t) = 1/(200+12t)$, we can show that $H_d(F(t,x),F(t,y)) \leq m(t)|x-y|$ and $d(0,F(t,0)) = 1/400 \leq m(t)$ for almost all $t \in [0,3]$. In addition, we get $\|m\| = 1/200$ which leads to $\Phi\|m\| \approx 0.122641 < 1$. By the conclusion of Theorem 5, the problem (21), with F given by (25), has at least one solution on $[0,3]$.

6. Discussion

Interchanging the position of Riemann–Liouville and Caputo fractional derivatives in problem (5), we get the following boundary value problem:

$$\begin{cases} {}^{C}D^{r}({}^{RL}D^{q}x(t)) = f(t,x(t)), \;\; 0 < t < T, \\ x(0) = 0, \quad x(T) = \mu\, I^{p}x(\zeta), \;\; \zeta \in (0,T). \end{cases} \tag{26}$$

In this case the condition $x(0) = 0$ is necessary for the well-posedness of the problem. The solution for the problem (26) is given by the integral equation

$$x(t) = I^{r+q}f(s,x(s))(t) + \frac{t^q}{\Lambda\Gamma(1+q)}[\mu I^{r+p+q}f(s,x(s))(\zeta) - I^{r+q}f(s,x(s))(T)], \tag{27}$$

where

$$\Lambda = \frac{T^q}{\gamma(1+q)} - \frac{\zeta^{q+p}}{\Gamma(q+p+1)} \neq 0.$$

Another more general boundary value problem consisting of Riemann–Liouville and Caputo fractional derivatives of neutral type is

$$\begin{cases} {}^{RL}D^{q}[{}^{C}D^{r}x(t) - g(t,x(t))] = f(t,x(t)), \;\; 0 < t < T, \\ x'(\xi) = \lambda\, {}^{C}D^{\nu}x(\eta), \quad x(T) = \mu\, I^{p}x(\zeta), \;\; \xi,\eta,\zeta \in (0,T), \end{cases} \tag{28}$$

where $g : J \times \mathbb{R} \to \mathbb{R}$ a continuous function while all other quantities are the same as defined in (5). Please note that the problem (5) is a special case of the problem (28) when $g = 0$.

The solution for the problem (28) is given by

$$\begin{aligned} x(t) &= I^r g(s,x(s))(t) + I^{q+r} f(s,x(s))(t) \\ &+ \frac{1}{\Lambda}\left[t^{q+r-1}\frac{\Gamma(q)}{\Gamma(q+r)} + \frac{D}{\Omega}\right]\Big[\lambda I^{r-\nu} g(s,x(s))(\eta) + \lambda I^{q+r-\nu} f(s,x(s))(\eta) \\ &- I^{r-1} g(s,x(s))(\xi) - I^{q+r-1} f(s,x(s))(\xi)\Big] + \frac{1}{\Omega}\Big[\mu I^{r+p} g(s,x(s))(\zeta) \\ &+ \mu I^{q+r+p} f(s,x(s))(\zeta) - I^r g(s,x(s))(T) - I^{q+r} f(s,x(s))(T)\Big], \end{aligned}$$

where

$$\begin{aligned} \Omega &= 1 - \frac{\mu \zeta^p}{\Gamma(1+p)} \neq 0, \\ D &= \mu \frac{\Gamma(q)}{\Gamma(q+r+p)} \zeta^{q+r+p-1} - \frac{\Gamma(q)}{\Gamma(q+r)} T^{q+r-1}, \\ \Lambda &= \frac{\Gamma(q)}{\Gamma(q+r-1)} \xi^{q+r-2} - \lambda \frac{\Gamma(q)}{\Gamma(q+r-\nu)} \eta^{q+r-\nu-1} \neq 0. \end{aligned}$$

We can obtain the existence results for the problem (28) by following the procedure used in the previous sections.

7. Conclusions

We have developed the existence theory for nonlinear fractional differential equations and inclusions involving both Riemann–Liouville and Caputo fractional derivatives, equipped with nonlocal fractional integro-differential boundary conditions. We applied the fixed-point theorems for single-valued and multi-valued maps to derive the desired results for the given problems. We also discussed the case obtained by interchanging the position of Riemann–Liouville and Caputo fractional derivatives in the original equation in (5), supplemented with nonlocal integral boundary conditions. Finally, we introduced a neutral-type fractional differential equation containing both Riemann–Liouville and Caputo fractional derivatives subject to the nonlocal fractional integro-differential boundary conditions and provided the outline for obtaining the existence results for this problem. It is imperative to note that the results obtained in this paper are similar to theoretically well-known propagation properties of fractional Schrödinger equation [30,31]. Moreover, our results are comparable to parity-time symmetry in a fractional Schrödinger equation [32] and propagation dynamics of light beam in a fractional Schrödinger equation [33]. In fact, the work established in the given configuration is new and contributes significantly to the literature on fractional order boundary value problems.

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