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Some New Generalizations for Exponentially *s*-Convex Functions and Inequalities via Fractional Operators

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Abstract: The main objective of this paper is to obtain the Hermite–Hadamard-type inequalities for exponentially *s*-convex functions via the Katugampola fractional integral. The Katugampola fractional integral is a generalization of Riemann–Liouville fractional integral and Hadamard fractional integral. Some special cases and applications to special means are also discussed.

Keywords: convex function; exponentially convex function; Riemann–Liouville fractional integral; Hadamard fractional integral; Katugampola fractional integral

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1. Introduction

The fractional calculus has emerged as a nonlocal theory described with operators of a fractional nature [1]. Fractional calculus was born as a natural generalization of the traditional calculus (Leibniz, 1695; Euler, 1730; Fourier, 1822; Abel, 1823 [1,2]); however, until recently, this mathematical theory played an active role in disciplines such as physics and control theory [3]. In the last decade, several applications have emerged due to the fractional nature of the phenomena. For instance, in physics, fractional calculus has been applied to thermodynamics, materials, and waves [3,4]. In a fractional optimal control, either the performance index or the differential equations governing the dynamics of the system contains a term with a fractional derivative [5]. Recently, Agulilar and coauthors [6,7] and Barro et al. [8] provided a new fractional operator. Fractional calculus is a terminology that refers to the integration and differentiation of an arbitrary order [1,2,9]; in other words, the meaning of k-th derivative $d^k y/dx^k$ and k-th iterated integral $\int \dots \int dx$ are extended by considering a fractional $\alpha \in \mathbb{R}_+$ parameter instead of integer $k \in N$ parameter. Following this trend, some authors introduced new types of fractional derivatives and differences that allow the appearance of exponential function [10,11] or the Mittag-Leffler function [12,13] in the kernel of the operators that makes it difficult to solve certain complicated fractional systems in their frames. Nowadays, a variety of fractional integral operators are under discussion, and many generalized fractional integral operators also take a part in generalizing the theory of fractional calculus (see References [14–21]).

It is well known that convex functions are becoming increasingly important due to the variety of their nature; many generalizations for convexity can be found in the literature [22–31], and many remarkable inequalities have been established via convexity. Among these, the Hermite–Hadmard inequality [32,33] is one of the most important inequalities and can be stated as follows:

Let $K \subseteq \mathbb{R}$ be an interval and $f : K \longrightarrow \mathbb{R}$ be a convex function. Then, double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

holds for all $a, b \in K$ with $a \neq b$. If f is concave on interval K, then both inequalities in Inequality (1) hold in the reverse direction. Recently, many researchers have made extensions, generalizations, refinements, variations and applications [34–39] for Hermite–Hadmard Inequality (1). On other hand, the minimum of the differentiable convex functions can be characterized by variational inequalities. These two aspects of convexity theory have far-reaching applications and have provided powerful tools for studying difficult problems. In recent years, integral inequalities have been derived via fractional analysis, which has emerged as another interesting technique.

To the best of our knowledge, a comprehensive investigation of exponentially convex functions as Katugampola fractional integral in the present paper is new. The class of exponentially convex functions was introduced by Antczak [40] and Dragomir [41]. Motivated by these facts, Awan et al. [42] introduced and investigated another class of convex functions, namely, exponentially convex function, which is significantly different from the class introduced by References [39–41]. The growth of research on Big Data analysis and deep learning has recently increased interest in information theory involving exponentially convex functions. The smoothness of exponentially convex functions is exploited for statistical learning, sequential prediction, and stochastic optimization (see References [40,43,44] and the references therein).

It is known [41] that a function f is exponentially convex if, and only if, f satisfies inequality

$$e^{f\left(\frac{a+b}{2}\right)} \leq \frac{1}{b-a}\int_a^b e^{f(x)}dx \leq \frac{e^{f(a)}+e^{f(b)}}{2}, \tag{2}$$

Inequality (2) is called the Hermite–Hadamard inequality and provides the upper and lower estimates for the exponential integral.

In this paper, we introduce a new class of exponentially convex function, which is called the exponentially s-convex function. We derive some new inequalities using Katugampola fractional integral for exponentially s-convex functions. Some special cases are also discussed. We also give some applications to the special means of real numbers.

2. Preliminaries

Now, we recall and introduce some definitions for various convex functions.

Definition 1. *A set $K \subset \mathbb{R}$ is said to be convex, if*

$$tx + (1-t)y \in K, \quad \forall x, y \in K, t \in [0,1].$$

Definition 2. *A function $f : K \to \mathbb{R}$ is said to be a convex function, if and only if,*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y), \quad \forall x, y \in K, t \in [0,1],$$

function f is called concave if $-f$ is convex.

We now consider a class of exponentially convex function, which are mainly due to [40,41].

Definition 3 ([40,41])**.** *A positive real-valued function $f : K \subseteq \mathbb{R} \longrightarrow (0, \infty)$ is said to be exponentially convex on K, if*

$$e^{f(tx+(1-t)y)} \leq te^{f(x)} + (1-t)e^{f(y)}, \forall x, y \in K, \quad t \in [0,1].$$

Exponentially convex functions are used to manipulate for statistical learning, sequential prediction, and stochastic optimization (see References [40,43,44] and the references therein).

It is known that $x \in K$ is the minimum of the differentiable exponentially convex functions f if, and only if, $x \in K$ satisfies

$$\langle f'(x)e^{f(x)}, y - x\rangle \geq 0, \quad \forall y \in K. \tag{3}$$

Inequalities of the type (3) are known as exponentially variational inequalities and appear to be new. Using the idea and techniques of Noor [37], one can study some aspects of exponentially variational inequalities, which is itself an interesting problem for further research. For formulation, applications and other aspects of variational inequalities, see Noor [35–37].

We now give some examples of exponentially convex functions, (see Reference [39]).

1. $f(x) = c$ is exponentially convex on (R) for any $c \geq 0$.
2. $f(x) = e^{\alpha x}$ is exponentially convex on (R) for any $\alpha \in \mathbb{R}$.
3. $f(x) = x^{-\alpha}$ is exponentially convex on $(0, \infty)$ for any $\alpha > 0$.

We now introduce a new concept of exponentially s-convex functions.

Definition 4. *Let $s \in [0,1]$. A function $f : K \subset \mathbb{R} \to \mathbb{R}$ is said to be an exponential s-convex function in the first sense, if*

$$e^{f(tx+(1-t)y)} \leq t^s e^{f(x)} + (1-t)^s e^{f(y)}, \quad \forall t \in [0,1] \;\; x, y \in K. \tag{4}$$

For $t = \frac{1}{2}$, we have

$$e^{f\left(\frac{x+y}{2}\right)} \leq \frac{1}{2^s}[e^{f(x)} + e^{f(y)}], \quad x, y \in K. \tag{5}$$

Function f is called the exponentially Jensen-convex function.

We now recall a class of fractional integrals, which is mainly due to Katugampola [16].

Definition 5 ([16])**.** *Let $[a, b] \subset \mathbb{R}$ be a finite interval. Then, the left- and right-hand-side Katugampola fractional integrals of order $\alpha > 0$, of function $f \in J_c^p(a, b)$ are defined by*

$${}^{\rho}I^{\alpha}_{a+}f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x (x^{\rho} - t^{\rho})^{\alpha-1} t^{\rho-1} f(t)dt$$

and

$${}^{\rho}I^{\alpha}_{b-}f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x (t^{\rho} - x^{\rho})^{\alpha-1} t^{\rho-1} f(t)dt,$$

with $a < x < b$ and $\rho > 0$, where $J_c^p(a,b)$ $(c \in \mathbb{R},\ 1 < p < \infty)$ is the space of those complex valued Lebesgue measurable functions f on $[a,b]$ for which $\|f\|_{J_c^p(a,b)} < \infty$, where the norm is defined by

$$\|f\|_{J_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t} \right)^{\frac{1}{p}} < \infty$$

for $1 \le p < \infty,\ c \in \mathbb{R}$ and for the case $p = \infty$,

$$\|f\|_{J_c^\infty} = ess\ \ sup_{a \le t \le b} |t^c f(t)|,$$

where ess sup stands for essential supremum.

If $\rho = 1$, then, Definition 5 reduces to a Riemann–Liouville fractional integral.

Definition 6 ([19])**.** *Let $\alpha > 0$ with $n-1 < \alpha \le n,\ \ n \in \mathbb{N}$, and $1 < x < b$. The left- and right-hand-side Riemann–Liouville fractional integrals of order α of function f are given by*

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt,$$

where $\Gamma(\alpha)$ is the gamma function.

If $\rho = 0$, then, Definition 5 reduces to a Hadamard fractional integral.

Definition 7 ([20])**.** *Let $\alpha > 0$ with $n-1 < \alpha \le n,\ \ n \in \mathbb{N}$, and $1 < x < b$. The left- and right-hand-side Hadamard fractional integrals of order α of function f are given by*

$$H_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (\ln(\frac{x}{t}))^{\alpha-1} \frac{f(t)}{t} dt$$

and

$$H_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\ln(\frac{x}{t}))^{\alpha-1} \frac{f(t)}{t} dt.$$

Theorem 1 ([15])**.** *Let $\alpha > 0$ and $\rho > 0$. Then, for $x > a$,*

1. $lim_{\rho \to 1}^{\rho} I_{a+}^\alpha = J_{a+}^\alpha f(x)$,
2. $lim_{\rho \to 0^+}^{\rho} I_{a+}^\alpha = H_{a+}^\alpha f(x)$.

We recall the special functions that are known as Gamma function and Beta function, respectively.

$$\begin{aligned} \Gamma(x) &= \int_0^\infty e^{-t} t^{x-1} dt, \\ \mathbb{B}(x,y) &= \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad x,y > 0. \end{aligned}$$

For the appropriate and suitable choice of functions ρ, s, and α, one can obtain several new and known class of fractional convex functions as special cases. This shows that the concept of an exponentially s-convex function is quite general and unifying.

From now onward, we take $\mathbb{I} = [a, b]$, unless otherwise specified.

3. Main Results

In this section, we derive the Hermite–Hadamard-type inequalities for exponentially s-convex functions.

Theorem 2. *Let $\alpha > 0$ and $\rho > 0$. Let $f : \mathbb{I} = [a^\rho, b^\rho] \subset \mathbb{R}_+ \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $e^f \in L[a^\rho, b^\rho]$. If f is an exponential convex function on $[a^\rho, b^\rho]$, then*

$$\begin{aligned} \frac{2^s}{\rho\alpha} e^{f(\frac{a^\rho+b^\rho}{2})} &\leq \frac{\rho^{\alpha-1}\Gamma(\alpha)}{(b^\rho - a^\rho)^\alpha} [{}^\rho I^\alpha_{b_-} e^{f(a^\rho)} + {}^\rho I^\alpha_{a_+} e^{f(b^\rho)}] \\ &\leq \frac{1}{\rho}\Big\{\frac{1}{\alpha+s} + \mathbb{B}(\alpha, s+1)\Big\}[e^{f(a^\rho)} + e^{f(b^\rho)}], \end{aligned} \tag{6}$$

where $\mathbb{B}(x, y)$ denotes the beta as special function.

Proof. Let f be an exponentially s-convex function. Then, from Inequality (5),

$$e^{f(\frac{x^\rho+y^\rho}{2})} \leq \frac{e^{f(x^\rho)+f(y^\rho)}}{2^s}, \quad \forall x, y \in \mathbb{I}.$$

Taking $x = t^\rho a^\rho + (1 - t^\rho)b^\rho$ and $y = t^\rho b^\rho + (1 - t^\rho)a^\rho$, in the above inequality, we have

$$2^s e^{f(\frac{a^\rho+b^\rho}{2})} \leq e^{f(t^\rho a^\rho+(1-t^\rho)b^\rho)} + e^{f(t^\rho b^\rho+(1-t^\rho)a^\rho)},$$

Multiplying both sides of the above inequality by $t^{\alpha\rho-1}$, $\alpha > 0$, and then integrating with respect to t over $[0, 1]$, we have

$$\begin{aligned} \frac{2^s}{\rho\alpha} e^{f(\frac{a^\rho+b^\rho}{2})} &\leq \int_0^1 t^{\alpha\rho-1}[e^{f(t^\rho a^\rho+(1-t^\rho)b^\rho)}]dt + \int_0^1 t^{\alpha\rho-1} e^{f(t^\rho b^\rho+(1-t^\rho)a^\rho)}dt \\ &= \int_a^b \Big(\frac{b^\rho - x^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1} e^f(x^\rho)\frac{x^{\rho-1}}{a^\rho - b^\rho}dx + \int_a^b \Big(\frac{b^\rho - y^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1} e^f(y^\rho)\frac{y^{\rho-1}}{b^\rho - a^\rho}dx \\ &= \frac{\rho^{\alpha-1}\Gamma(\alpha)}{(b^\rho - a^\rho)^\alpha}[{}^\rho I^\alpha_{b_-} e^{f(a^\rho)} + {}^\rho I^\alpha_{a_+} e^{f(b^\rho)}] \\ &= \int_0^1 t^{\alpha\rho-1}\Big\{e^{f(t^\rho a^\rho+(1-t^\rho)b^\rho)} + e^{f(t^\rho b^\rho+(1-t^\rho)a^\rho)}\Big\}dt \\ &\leq \int_0^1 t^{\alpha\rho-1}\Big((t^\rho)^s e^{f(a^\rho)} + (1-t^\rho)^s e^{f(b^\rho)} + (t^\rho)^s e^{f(b^\rho)} + (1-t^\rho)^s e^{f(a^\rho)}\Big)dt \\ &= \int_0^1 t^{\alpha\rho-1}\Big([(t^\rho)^s + (1-(t^\rho)^s)][e^{f(a^\rho)} + e^{f(b^\rho)}]\Big)dt \\ &\leq \frac{1}{\rho}\Big(\frac{1}{\alpha+s} + \beta(\alpha, s+1)\Big)[e^{f(a^\rho)} + e^{f(b^\rho)}]. \end{aligned}$$

This completes the proof. □

We now discuss some new special cases of Theorem 2.

Corollary 1. *If $\rho = 1$, then, under the assumption of Theorem 2, we have*

$$2^s e^{f(\frac{a+b}{2})} \leq \frac{\Gamma(\alpha)}{(b-a)^\alpha}[J^\alpha_{b_-} e^{f(a)} + J^\alpha_{a_+} e^{f(b)}] \\ \leq \{\frac{1}{\alpha+s} + \mathbb{B}(\alpha, s+1)\}[e^{f(a)} + e^{f(b)}].$$

Corollary 2. *If $\rho = \alpha = 1$, then, under the assumption of Theorem 2, we have*

$$2^s e^{f(\frac{a+b}{2})} \leq \frac{1}{b-a}\int_a^b e^{f(x)}dx \leq \{\frac{1}{1+s} + \mathbb{B}(1, s+1)\}[e^{f(a)} + e^{f(b)}].$$

Corollary 3 ([18,42])**.** *If $\rho = \alpha = s = 1$, then, under the assumption of Theorem 2, we have*

$$2e^{f(\frac{a+b}{2})} \leq \frac{1}{b-a}\int_a^b e^{f(x)}dx \leq [e^{f(a)} + e^{f(b)}].$$

In order to prove the following result, we need the following lemma:

Lemma 1. *Let $\alpha > 0$ and $\rho > 0$. Let $f : \mathbb{I} = [a^\rho, b^\rho] \subset \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on interior $\mathbb{I}^\circ$ of $\mathbb{I}$. If $(e^f)' \in L[a^\rho, b^\rho]$ is s-convex function, then*

$$\frac{e^{f(a^\rho)} + e^{f(b^\rho)}}{2} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho - a^\rho)}[{}^\rho I^\alpha_{a_+} e^{f(b^\rho)} + {}^\rho I^\alpha_{b_-} e^{f(a^\rho)}] \\ = \frac{\rho(b^\rho - a^\rho)}{2}\int_0^1 [(1-t^\rho)^\alpha - (t^\rho)^\alpha]t^{\rho-1}e^{f(t^\rho a^\rho + (1-t^\rho)b^\rho)} f'(t^\rho a^\rho + (1-t^\rho)b^\rho)dt.$$

Proof. Consider

$$I = \frac{\rho(b^\rho - a^\rho)}{2}\int_0^1 [(1-t^\rho)^\alpha - (t^\rho)^\alpha]t^{\rho-1}e^{f(t^\rho a^\rho + (1-t^\rho)b^\rho)} f'(t^\rho a^\rho + (1-t^\rho)b^\rho)dt \\ = I_1 + I_2.$$

Now,

$$I_1 = \int_0^1 (1-t^\rho)^\alpha t^{\rho-1} e^{f(t^\rho a^\rho + (1-t^\rho)b^\rho)} f'(t^\rho a^\rho + (1-t^\rho)b^\rho)dt \\ = \frac{(1-t^\rho)^\alpha e^{f(t^\rho a^\rho + (1-t^\rho)b^\rho)}}{\rho(a^\rho - b^\rho)}\Bigg|_0^1 + \frac{\alpha}{a^\rho - b^\rho}\int_0^1 (1-t^\rho)^{\alpha-1}t^{\rho-1}e^{f(t^\rho a^\rho + (1-t^\rho)b^\rho)}dt \\ = \frac{e^{f(b^\rho)}}{\rho(b^\rho - a^\rho)} - \frac{\alpha}{b^\rho - a^\rho}\int_a^b \left(\frac{x^\rho - a^\rho}{b^\rho - a^\rho}\right)^{\alpha-1} e^{f(x^\rho)}\frac{x^{\rho-1}}{\rho(a^\rho - b^\rho)}dx \\ = \frac{e^{f(b^\rho)}}{\rho(b^\rho - a^\rho)} - \frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{(b^\rho - a^\rho)^{\alpha+1}} {}^\rho I^\alpha_{b_-} e^{f(x^\rho)}\Bigg|_{x=a}.$$

Similarly,

$$I_2 = \int_0^1 t^{\rho\alpha}.t^{\rho-1} e^{f(t^{\rho a^{\rho}}+(1-t^{\rho})b^{\rho})} f'(t^{\rho}a^{\rho} + (1-t^{\rho})b^{\rho})dt$$
$$= \frac{-e^{f(a^{\rho})}}{\rho(b^{\rho}-a^{\rho})} - \frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{(b^{\rho}-a^{\rho})^{\alpha+1}} {}^{\rho}I^{\alpha}_{a_+} e^{f(x^{\rho})}\Big|_{x=b}.$$

Adding I_1 and I_2, we get

$$I = \frac{e^{f(a^{\rho})} + e^{f(b^{\rho})}}{2} - \frac{\rho^{\alpha}\Gamma(\alpha+1)}{2(b^{\rho}-a^{\rho})}[{}^{\rho}I^{\alpha}_{a_+} e^{f(b^{\rho})} + {}^{\rho}I^{\alpha}_{b_-} e^{f(a^{\rho})}],$$

which is the required result. □

Theorem 3. *Let $\alpha > 0$ and $\rho > 0$. Let $f : \mathbb{I} = [a^{\rho}, b^{\rho}] \subset \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on interior $\mathbb{I}^{\circ}$ of $\mathbb{I}$. If $|(e^f)'| \in L[a^{\rho}, b^{\rho}]$ is s-convex function, then*

$$\left|\frac{e^{f(a^{\rho})} + e^{f(b^{\rho})}}{2} - \frac{\rho^{\alpha}\Gamma(\alpha+1)}{2(b^{\rho}-a^{\rho})}[{}^{\rho}I^{\alpha}_{a_+} e^{f(b^{\rho})} + {}^{\rho}I^{\alpha}_{b_-} e^{f(a^{\rho})}]\right|$$
$$\leq \frac{(b^{\rho}-a^{\rho})}{2\rho}\left[\left\{\frac{1}{\alpha+2s+1} + \mathbb{B}(\alpha+1, 2s+1)\right\}\{|e^{f(a^{\rho})}f'(a^{\rho})| + |e^{f(b^{\rho})}f'(b^{\rho})|\}\right.$$
$$\left. + 2\Delta(a,b)\mathbb{B}(\alpha+s+2, s+1)\right],$$

where

$$\Delta(a,b) = \{|e^{f(a^{\rho})}f'(b^{\rho})| + |e^{f(b^{\rho})}f'(a^{\rho})|\}. \tag{7}$$

Proof. Using Lemma 1 and the power mean inequality, we have

$$\left|\frac{e^{f(a^{\rho})} + e^{f(b^{\rho})}}{2} - \frac{\rho^{\alpha}\Gamma(\alpha+1)}{2(b^{\rho}-a^{\rho})}[{}^{\rho}I^{\alpha}_{a_+} e^{f(b^{\rho})} + {}^{\rho}I^{\alpha}_{b_-} e^{f(a^{\rho})}]\right|$$
$$\leq \frac{(\rho(b^{\rho}-a^{\rho}))}{2}\left\{\int_0^1 t^{\rho(\alpha+1)-1}\{e^{f(t^{\rho}b^{\rho}+(1-t^{\rho})a^{\rho}))}f'(t^{\rho}b^{\rho}+(1-t^{\rho})a^{\rho})\}\right.$$
$$\left. - \{e^{f(t^{\rho}a^{\rho}+(1-t^{\rho})b^{\rho}))}f'(t^{\rho}a^{\rho}+(1-t^{\rho})b^{\rho})\}\right\}dt$$
$$\leq \frac{(\rho(b^{\rho}-a^{\rho}))}{2}\left\{\int_0^1 t^{\rho(\alpha+1)-1}|e^{f(t^{\rho}b^{\rho}+(1-t^{\rho})a^{\rho}))}f'(t^{\rho}b^{\rho}+(1-t^{\rho})a^{\rho})|\right.$$
$$\left. - |e^{f(t^{\rho}a^{\rho}+(1-t^{\rho})b^{\rho}))}f'(t^{\rho}a^{\rho}+(1-t^{\rho})b^{\rho})|\right\}dt$$
$$\leq \frac{(b^{\rho}-a^{\rho})}{2}\left[\int_0^1 t^{\rho(\alpha+1)-1}\left\{\left((t^{\rho})^s|e^{f(b^{\rho})}| + (1-t^{\rho})^s|e^{f(a^{\rho})}|\}\{(t^{\rho})^s|f'(b^{\rho})| + (1-t^{\rho})^s|\right.\right.\right.$$
$$\left.\left.\left. f'(a^{\rho})|\right) + \left((t^{\rho})^s|e^{f(a^{\rho})}| + (1-t^{\rho})^s|e^{f(b^{\rho})}|\}\{(t^{\rho})^s|f'(a^{\rho})| + (1-t^{\rho})^s|f'(b^{\rho})|\right)\right\}\right]dt$$
$$= \frac{(b^{\rho}-a^{\rho})}{2}\left[\int_0^1 t^{\rho(\alpha+1)-1}\left\{t^{2\rho s}\{|e^{f(a^{\rho})}f'(a^{\rho})| + |e^{f(b^{\rho})}f'(b^{\rho})|\}\right.\right.$$
$$+ (1-t^{\rho})^{2s}\{|e^{f(a^{\rho})}f'(a^{\rho})| + (t^{\rho})^s(1-t^{\rho})^s|e^{f(b^{\rho})}f'(b^{\rho})|\}$$
$$\left.\left. + 2\{|e^{f(a^{\rho})}f'(b^{\rho})| + |e^{f(b^{\rho})}f'(a^{\rho})|\}\right\}\right]$$

$$= \frac{(b^\rho - a^\rho)}{2}\Big\{\{|e^{f(a^\rho)}f'(a^\rho)| + |e^{f(b^\rho)}f'(b^\rho)|\}\Big[\int_0^1 t^{\rho(\alpha+2s+1)-1}dt + \int_0^1 t^{\rho(\alpha+1)-1}(1-t^\alpha)^{2s}dt\Big]$$
$$+ 2\Delta(a,b)\int_0^1 t^{\rho(\alpha+s+1)-1}(1-t^\alpha)^s dt\Big\}$$
$$= \frac{(b^\rho - a^\rho)}{2\rho}\Big[\Big\{\frac{1}{\alpha+2s+1} + \mathbb{B}(\alpha+1, 2s+1)\Big\}\{|e^{f(a^\rho)}f'(a^\rho)| + |e^{f(b^\rho)}f'(b^\rho)|\}$$
$$+ 2\Delta(a,b)\mathbb{B}(\alpha+s+2, s+1)\Big],$$

which is the required result. □

Corollary 4. *If $\rho = 1$, then, under the assumption of Theorem 3, we have a new result:*

$$\left|\frac{e^{f(a)} + e^{f(b)}}{2} - \frac{\alpha\Gamma(\alpha+1)}{2(b-a)}[J^\alpha_{a_+}e^{f(b)} + J^\alpha_{b_-}e^{f(a)}]\right|$$
$$\leq \frac{(b-a)}{2}\Big\{\Big\{\frac{1}{\alpha+2s+1} + \mathbb{B}(\alpha+1, 2s+1)\Big\}\{|e^{f(a)}f'(a)| + |e^{f(b)}f'(b)|\}$$
$$+ 2\Delta(a,b)\mathbb{B}(\alpha+s+2, s+1)\Big\}.$$

Corollary 5. *If $\rho = s = 1$, then, under the assumption of Theorem 3, we have a new result:*

$$\left|\frac{e^{f(a)} + e^{f(b)}}{2} - \frac{\alpha\Gamma(\alpha+1)}{2(b-a)}[J^\alpha_{a_+}e^{f(b)} + J^\alpha_{b_-}e^{f(a)}]\right|$$
$$\leq \frac{(b-a)}{2}\Big\{\Big\{\frac{1}{\alpha+3} + \mathbb{B}(\alpha+1, 3)\Big\}\{|e^{f(a)}f'(a)| + |e^{f(b)}f'(b)|\}$$
$$+ 2\Delta(a,b)\mathbb{B}(\alpha+3, 2)\Big\}.$$

Corollary 6. *If $\rho = s = \alpha = 1$, then, under the assumption of Theorem 3, we have a new result:*

$$\left|\frac{e^{f(a)} + e^{f(b)}}{2} - \frac{1}{2(b-a)}\int_a^b e^{f(x)}dx\right| \leq \frac{(b-a)}{6}\Big\{\frac{5}{4}\{|e^{f(a)}f'(a)| + |e^{f(b)}f'(b)|\} + \Delta(a,b)\Big\}.$$

Theorem 4. *Let $\alpha > 0$ and $\rho > 0$. Let $f : \mathbb{I} = [a^\rho, b^\rho] \subset \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on interior $\mathbb{I}^\circ$ of $\mathbb{I}$. If $|(e^f)'|^q \in L[a^\rho, b^\rho]$ is s-convex function for some $q \geq 1$, then*

$$\left|\frac{e^{f(a^\rho)} + e^{f(b^\rho)}}{2} - \frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho - a^\rho)}[{}^\rho I^\alpha_{a_+}e^{f(b^\rho)} + {}^\rho I^\alpha_{b_-}e^{f(a^\rho)}]\right|$$
$$\leq \frac{(b^\rho - a^\rho)}{2}\left(\frac{1}{\rho(\alpha+1)}\right)^{1-\frac{1}{q}}\Big\{\mathbb{B}(\alpha+1, 2s+1)\{|e^{f(a^\rho)f'(a^\rho)}|^q + |e^{f(b^\rho)f'(b^\rho)}|^q\}$$
$$+ \{\mathbb{B}(\alpha+s+1, s+1) + \mathbb{B}(\alpha+2, s+1)\}\Delta_1(a,b)\Big\},$$

where

$$\Delta_1(a,b) = \{|e^{f(b^\rho)f'(a^\rho)}|^q + |e^{f(a^\rho)f'(b^\rho)}|^q\}. \tag{8}$$

Proof. Using Lemma 1 and the power mean inequality, we have

$$\left|\frac{e^{f(a^\rho)}+e^{f(b^\rho)}}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)}[{}^\rho I^\alpha_{a_+}e^{f(b^\rho)}+{}^\rho I^\alpha_{b_-}e^{f(a^\rho)}]\right|$$
$$=\left|\frac{\rho(b^\rho-a^\rho)}{2}\int_0^1\{(1-t^\rho)^\alpha-(t^\rho)^\alpha\}t^{\rho-1}e^{f(t^\rho a^\rho+(1-t^\rho)b^\rho)}f'(t^\rho a^\rho+(1-t^\rho)b^\rho)\right|$$
$$\le\left|\frac{\rho(b^\rho-a^\rho)}{2}\right|\left(\int_0^1\{(1-t^\rho)^\alpha-(t^\rho)^\alpha\}t^{\rho-1}dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1\left|(1-t^\rho)^\alpha-(t^\rho)^\alpha\right|t^{\rho-1}\left|e^{f(t^\rho a^\rho+(1-t^\rho)b^\rho)}f'(t^\rho a^\rho+(1-t^\rho)b^\rho)\right|^q dt\right)^{\frac{1}{q}}$$

$$\le\left|\frac{\rho(b^\rho-a^\rho)}{2}\right|\left(\int_0^1\{(1-t^\rho)^\alpha-(t^\rho)^\alpha\}t^{\rho-1}dt\right)^{1-\frac{1}{q}}\times\left(\int_0^1\left|(1-t^\rho)^\alpha-(t^\rho)^\alpha\right|t^{\rho-1}\right.$$
$$\left.\left[(t^\rho)^s|e^{f(a^\rho)}|^q+(1-t^\rho)^s|e^{f(b^\rho)}|^q\right]\left[(t^\rho)^s|f(a^\rho)|^q+(1-t^\rho)^s|f(b^\rho)|^q\right]dt\right)^{\frac{1}{q}}$$
$$=\frac{\rho(b^\rho-a^\rho)}{2}\left(\frac{1}{\rho(\alpha+1)}\right)^{1-\frac{1}{q}}\left\{|e^{f(a^\rho)f'(a^\rho)}|^q\int_0^1(1-t^\rho)^\alpha t^{\rho(2s+1)-1}dt\right.$$
$$+|e^{f(b^\rho)f'(b^\rho)}|^q\int_0^1(1-t^\rho)^{2s}t^{\rho(\alpha+1)-1}dt+\{\int_0^1(1-t^\rho)^{\alpha+s}t^{\rho(s+1)-1}dt$$
$$\left.+\int_0^1(1-t^\rho)^s t^{\rho(\alpha+1)-1}dt\}\{|e^{f(b^\rho)f'(a^\rho)}|^q+|e^{f(a^\rho)f'(b^\rho)}|^q\}\right\}^{\frac{1}{q}}$$
$$=\frac{(b^\rho-a^\rho)}{2}\left(\frac{1}{\rho(\alpha+1)}\right)^{1-\frac{1}{q}}\left\{\mathbb{B}(\alpha+1,2s+1)\{|e^{f(a^\rho)f'(a^\rho)}|^q+|e^{f(b^\rho)f'(b^\rho)}|^q\}\right.$$
$$\left.+\{\mathbb{B}(\alpha+s+1,s+1)+\mathbb{B}(\alpha+2,s+1)\}\Delta_1(a,b)\right\}^{\frac{1}{q}},$$

which is the required result. □

Corollary 7. *If $q=1$, then, under the assumption of Theorem 4, we have a new result:*

$$\left|\frac{e^{f(a^\rho)}+e^{f(b^\rho)}}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)}[{}^\rho I^\alpha_{a_+}e^{f(b^\rho)}+{}^\rho I^\alpha_{b_-}e^{f(a^\rho)}]\right|$$
$$\le\frac{(b^\rho-a^\rho)}{2}\left\{\mathbb{B}(\alpha+1,2s+1)\{|e^{f(a^\rho)f'(a^\rho)}|+|e^{f(b^\rho)f'(b^\rho)}|\}\right.$$
$$\left.+\{\mathbb{B}(\alpha+s+1,s+1)+\mathbb{B}(\alpha+2,s+1)\}\Delta_1(a,b)\right\}.$$

Corollary 8. *If $\rho = 1$, then, under the assumption of Theorem 4, we have a new result:*

$$\left|\frac{e^{f(a)}+e^{f(b)}}{2}-\frac{\alpha\Gamma(\alpha+1)}{2(b-a)}[J^{\alpha}_{a_+}e^{f(b)}+J^{\alpha}_{b_-}e^{f(a)}]\right|$$
$$\leq \frac{(b-a)}{2}\left(\frac{1}{(\alpha+1)}\right)^{1-\frac{1}{q}}\Big\{\mathbb{B}(\alpha+1,2s+1)\{|e^{f(a)f'(a)}|^q+|e^{f(b)f'(b)}|^q\}$$
$$+\{\mathbb{B}(\alpha+s+1,s+1)+\mathbb{B}(\alpha+2,s+1)\}\Delta_1(a,b)\Big\}^{\frac{1}{q}}.$$

Corollary 9. *If $\rho = s = 1$, then, under the assumption of Theorem 4, we have a new result:*

$$\left|\frac{e^{f(a)}+e^{f(b)}}{2}-\frac{\alpha\Gamma(\alpha+1)}{2(b-a)}[J^{\alpha}_{a_+}e^{f(b)}+J^{\alpha}_{b_-}e^{f(a)}]\right|$$
$$\leq \frac{(b-a)}{2}\left(\frac{1}{(\alpha+1)}\right)^{1-\frac{1}{q}}\Big\{\mathbb{B}(\alpha+2,3)\{|e^{f(a)}f'(a)|^q+|e^{f(b)}f'(b)|^q\}$$
$$+2\mathbb{B}(\alpha+2,2)\Delta_1(a,b)\Big\}^{\frac{1}{q}}.$$

Corollary 10. *If $\rho = s = \alpha = 1$, then, under the assumption of Theorem 4, we have a new result:*

$$\left|\frac{e^{f(a)}+e^{f(b)}}{2}-\frac{1}{2(b-a)}\int_a^b e^{f(x)}dx\right| \leq \frac{(b-a)}{2^{2-\frac{1}{q}}}\left\{\frac{1}{30}\{|e^{f(a)}f'(a)|^q+|e^{f(b)}f'(b)|^q\}+\frac{1}{6}\Delta_1(a,b)\right\}^{\frac{1}{q}}.$$

Theorem 5. *Let $\alpha > 0$ and $\rho > 0$. Let $f : \mathbb{I} = [a^\rho, b^\rho] \subset \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on interior $\mathbb{I}^\circ$ of $\mathbb{I}$. If $|(e^f)'|^q \in L[a^\rho, b^\rho]$ is s-convex function for some $q \geq 1$, then*

$$\left|\frac{e^{f(a^\rho)}+e^{f(b^\rho)}}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)}[{}^\rho I^{\alpha}_{a_+}e^{f(b^\rho)}+{}^\rho I^{\alpha}_{b_-}e^{f(a^\rho)}]\right|$$
$$\leq \frac{b^\rho-a^\rho}{2}\Big\{|e^{f(a^\rho)}f'(a^\rho)|^q\{\mathbb{B}(\alpha+1,2s+\frac{1}{\rho})+\frac{1}{\alpha+2s+1}\}+\{\mathbb{B}(\alpha+2s+1,\frac{1}{\rho})$$
$$+\{\mathbb{B}(2s+1,\alpha+\frac{1}{\rho})\}|e^{f(b^\rho)}f'(b^\rho)|^q+\{\mathbb{B}(\alpha+s+1,s+\frac{1}{\rho})$$
$$+\mathbb{B}(s+1,\alpha+s+\frac{1}{\rho})\}\Delta_1(a,b)\Big\}^{\frac{1}{q}},$$

where $\Delta_1(a,b)$ is given in (8).

Proof. Using Lemma 1 and the power mean inequality, we have

$$\left|\frac{e^{f(a^\rho)}+e^{f(b^\rho)}}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)}[{}^\rho I^{\alpha}_{a_+}e^{f(b^\rho)}+{}^\rho I^{\alpha}_{b_-}e^{f(a^\rho)}]\right|$$
$$\leq \left|\frac{\rho(b^\rho-a^\rho)}{2}\int_0^1\{(1-t^\rho)^\alpha-(t^\rho)^\alpha\}t^{\rho-1}|e^{f(t^\rho a^\rho+(1-t^\rho)b^\rho)}f'(t^\rho a^\rho+(1-t^\rho)b^\rho)|\right|$$
$$\leq \frac{\rho(b^\rho-a^\rho)}{2}\left(\int_0^1 t^{\rho-1}dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1 \{(1-t^\rho)^\alpha-(t^\rho)^\alpha\}|e^{f(t^\rho a^\rho+(1-t^\rho)b^\rho)}f'(t^\rho a^\rho+(1-t^\rho)b^\rho)|^q dt\right)^{\frac{1}{q}}$$
$$=\frac{\rho^{\frac{1}{q}}(b^\rho-a^\rho)}{2}\left(\int_0^1 \{(1-t^\rho)^\alpha-(t^\rho)^\alpha\}\{(t^\rho)^s|e^{f(a^\rho)}f'(a^\rho)|^q+(1-t^\rho)^s|e^{f(b^\rho)}f'(b^\rho)|^q\}dt\right)^{\frac{1}{q}}$$
$$=\frac{\rho^{\frac{1}{q}}(b^\rho-a^\rho)}{2}\Bigg\{|e^{f(a^\rho)}f'(a^\rho)|^q\int_0^1[(1-t^\rho)^\alpha t^{2\rho s}]dt$$
$$+|e^{f(b^\rho)}f'(b^\rho)|^q\int_0^1[(1-t^\rho)^{\alpha+2s}+(1-t^\rho)^{2s}t^{\rho s}dt]+\{|e^{f(a^\rho)}f'(b^\rho)|^q$$
$$+|e^{f(b^\rho)}f'(a^\rho)|^q\}\int_0^1[(1-t^\rho)^{\alpha+s}]t^{\rho s}+(1-t^\rho)^s t^{\rho(\alpha+s)}dt\Bigg\}^{\frac{1}{q}}$$
$$=\frac{b^\rho-a^\rho}{2}\Bigg\{|e^{f(a^\rho)}f'(a^\rho)|^q\{\mathbb{B}(\alpha+1,2s+\frac{1}{\rho})+\frac{1}{\alpha+2s+1}\}+\{\mathbb{B}(\alpha+2s+1,\frac{1}{\rho})$$
$$+\{\mathbb{B}(2s+1,\alpha+\frac{1}{\rho})\}|e^{f(b^\rho)}f'(b^\rho)|^q+\{\mathbb{B}(\alpha+s+1,s+\frac{1}{\rho})$$
$$+\mathbb{B}(s+1,\alpha+s+\frac{1}{\rho})\}\Delta_1(a,b)\Bigg\}^{\frac{1}{q}}.$$

This completes the proof. □

Corollary 11. *If $q=1$, then, under the assumption of Theorem 5, we have a new result:*

$$\left|\frac{e^{f(a^\rho)}+e^{f(b^\rho)}}{2}-\frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho-a^\rho)}[{}^\rho I^\alpha_{a_+}e^{f(b^\rho)}+{}^\rho I^\alpha_{b_-}e^{f(a^\rho)}]\right|$$
$$\leq\frac{b^\rho-a^\rho}{2}\Bigg\{|e^{f(a^\rho)}f'(a^\rho)|\{\mathbb{B}(\alpha+1,2s+\frac{1}{\rho})+\frac{1}{\alpha+2s+1}\}+\{\mathbb{B}(\alpha+2s+1,\frac{1}{\rho})$$
$$+\{\mathbb{B}(2s+1,\alpha+\frac{1}{\rho})\}|e^{f(b^\rho)}f'(b^\rho)|+\{\mathbb{B}(\alpha+s+1,s+\frac{1}{\rho})+\mathbb{B}(s+1,\alpha+s+\frac{1}{\rho})\}\Delta_1(a,b)\Bigg\}.$$

Corollary 12. *If $\rho=1$, then, under the assumption of Theorem 5, we have a new result:*

$$\left|\frac{e^{f(a)}+e^{f(b)}}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)}[J^\alpha_{a_+}e^{f(b)}+J^\alpha_{b_-}e^{f(a)}]\right|$$
$$\leq\frac{b-a}{2}\Bigg\{|e^{f(a)}f'(a)|^q\{\mathbb{B}(\alpha+1,2s+1)+\frac{1}{\alpha+2s+1}\}+\{\mathbb{B}(\alpha+2s+1,1)$$
$$+\{\mathbb{B}(2s+1,\alpha+1\}|e^{f(b)}f'(b)|^q+\{\mathbb{B}(\alpha+s+1,s+1)+\mathbb{B}(s+1,\alpha+s+1)\}\Delta_1(a,b)\Bigg\}^{\frac{1}{q}}.$$

Corollary 13. *If $\rho=s=1$, then, under the assumption of Theorem 5, we have a new result:*

$$\left|\frac{e^{f(a)}+e^{f(b)}}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)}[J^\alpha_{a_+}e^{f(b)}+J^\alpha_{b_-}e^{f(a)}]\right|$$
$$\leq\frac{b-a}{2}\Bigg\{|e^{f(a)}f'(a)|^q\{\mathbb{B}(\alpha+1,3)+\frac{1}{\alpha+3}\}+\{\mathbb{B}(\alpha+3,1)$$
$$+\{\mathbb{B}(3,\alpha+1\}|e^{f(b)}f'(b)|^q+\{\mathbb{B}(\alpha+2,2)+\mathbb{B}(2,\alpha+2)\}\Delta_1(a,b)\Bigg\}^{\frac{1}{q}}.$$

Corollary 14. *If $\alpha = \rho = s = 1$, then, under the assumption of Theorem 5, we have a new result:*

$$\left|\frac{e^{f(a)} + e^{f(b)}}{2} - \frac{1}{2(b+a)}\int_a^b e^{f(x)}dx\right|$$
$$\leq \frac{b-a}{2}\left\{\frac{5}{12}|e^{f(a)}f'(a)|^q + \frac{5}{12}|e^{f(b)}f'(b)|^q + \frac{1}{6}\Delta_1(a,b)\right\}^{\frac{1}{q}}.$$

Theorem 6. *Let $\alpha > 0$ and $\rho > 0$. Let $f : \mathbb{I} = [a^\rho, b^\rho] \subset \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on interior $\mathbb{I}^\circ$ of $\mathbb{I}$. If $(e^f)' \in L[a^\rho, b^\rho]$ is s-convex function, then*

$$\begin{aligned}&\frac{\rho^\alpha\Gamma(\alpha+1)}{2(b^\rho - a^\rho)^\alpha}[{}^\rho I^\alpha_{a_+} e^{f(b^\rho)} + {}^\rho I^\alpha_{b_-} e^{f(a^\rho)}] - e^{f(\frac{a^\rho+b^\rho}{2})}\\ =&\frac{\rho(b^\rho - a^\rho)}{2}\left[\int_0^1 \psi(t)t^{\rho-1}e^{f(t^\rho a^\rho)+(1-t^\rho)b^\rho}f'(t^\rho a^\rho) + (1-t^\rho)b^\rho)\right],\end{aligned} \tag{9}$$

where

$$\psi(t) = \begin{cases} 1, & t \in [0, \frac{1}{2}), \\ -1, & t \in [\frac{1}{2}, 1]. \end{cases}$$

Proof. It suffices to note that

$$\begin{aligned}I =& \int_0^{\frac{1}{2}} t^{\rho-1}e^{f(t^\rho a^\rho)+(1-t^\rho)b^\rho}f'(t^\rho a^\rho) + (1-t^\rho)b^\rho)dt\\
&- \int_{\frac{1}{2}}^1 t^{\rho-1}e^{f(t^\rho a^\rho)+(1-t^\rho)b^\rho}f'(t^\rho a^\rho) + (1-t^\rho)b^\rho)dt\\
&- \int_0^1 [(1-t^\rho)^\alpha - (t^\rho)^\alpha]t^{\rho-1}e^{f(t^\rho a^\rho)+(1-t^\rho)b^\rho}f'(t^\rho a^\rho) + (1-t^\rho)b^\rho)dt\\
=& \left\{\int_0^{\frac{1}{2}} t^{\rho-1}e^{f(t^\rho a^\rho)+(1-t^\rho)b^\rho}f'(t^\rho a^\rho) + (1-t^\rho)b^\rho)dt\right\}\\
&+ \left\{-\int_{\frac{1}{2}}^1 t^{\rho-1}e^{f(t^\rho a^\rho)+(1-t^\rho)b^\rho}f'(t^\rho a^\rho) + (1-t^\rho)b^\rho)dt\right\}\\
&+ \left\{-\int_0^1 [(1-t^\rho)^\alpha]t^{\rho-1}e^{f(t^\rho a^\rho)+(1-t^\rho)b^\rho}f'(t^\rho a^\rho) + (1-t^\rho)b^\rho)dt\right\}\\
&+ \left\{\int_0^1 [(t^\rho)^\alpha]t^{\rho-1}e^{f(t^\rho a^\rho)+(1-t^\rho)b^\rho}f'(t^\rho a^\rho) + (1-t^\rho)b^\rho)dt\right\}\\
=& I_1 + I_2 + I_3\end{aligned}$$

$$I_1=\left\{\int_0^{\frac{1}{2}} t^{\rho-1}e^{f(t^\rho a^\rho)+(1-t^\rho)b^\rho}f'(t^\rho a^\rho)+(1-t^\rho)b^\rho)dt\right\}$$

$$+\left\{-\int_{\frac{1}{2}}^{1} t^{\rho-1}e^{f(t^\rho a^\rho)+(1-t^\rho)b^\rho}f'(t^\rho a^\rho)+(1-t^\rho)b^\rho)dt\right\}$$

$$=\frac{1}{\rho(a^\rho-b^\rho)}\left[2e^{f(\frac{a^\rho+b^\rho}{2})}-e^{f(a^\rho)}-e^{f(b^\rho)}\right]$$

$$I_2=-\int_0^1[(1-t^\rho)^\alpha]t^{\rho-1}e^{f(t^\rho a^\rho)+(1-t^\rho)b^\rho}f'(t^\rho a^\rho)+(1-t^\rho)b^\rho)dt$$

$$=\frac{e^{f(b^\rho)}}{\rho(a^\rho-b^\rho)}+\frac{\alpha}{(b^\rho-a^\rho)}\int_0^1(1-t^\rho)^{\alpha-1}t^{\alpha-1}e^{f(t^\rho a^\rho)+(1-t^\rho)b^\rho)}dt$$

$$=\frac{e^{f(b^\rho)}}{\rho(a^\rho-b^\rho)}+\frac{\alpha}{(b^\rho-a^\rho)}\int_a^b(\frac{x^\rho-a^\rho}{b^\rho-a^\rho})^{\alpha-1}\frac{x^{\rho-1}}{a^\rho-b^\rho}e^{f(x^\rho)}dx$$

$$=\frac{e^{f(b^\rho)}}{\rho(a^\rho-b^\rho)}+\frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{(b^\rho-a^\rho)^{\alpha+1}}{}^\rho I^\alpha_{b_-}e^{f(a^\rho)}$$

$$I_3=\frac{e^{f(a^\rho)}}{\rho(a^\rho-b^\rho)}+\frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{(b^\rho-a^\rho)^{\alpha+1}}{}^\rho I^\alpha_{a_+}e^{f(b^\rho)},$$

it follows that

$$I=-\frac{2}{\rho(b^\rho-a^\rho)}e^{f(\frac{a^\rho-b^\rho}{2})}+\frac{\rho^{\alpha-1}\Gamma(\alpha+1)}{(a^\rho-b^\rho)^{\alpha+1}}[{}^\rho I^\alpha_{a_+}e^{f(b^\rho)}+{}^\rho I^\alpha_{b_-}e^{f(a^\rho)}].$$

Thus, by multiplying both sides by $\frac{\rho(b^\rho-a^\rho)}{2}$, we have Conclusion (9). □

Remark 1.

(i) *If $\rho=1$, then under the assumption of Theorem 6, we have a new result:*

$$\frac{1}{2(b-a)^\alpha}[J^\alpha_{a_+}e^{f(b)}+J^\alpha_{b_-}e^{f(a)}]-e^{f(\frac{a+b}{2})}$$
$$=\frac{(b-a)}{2}\left[\int_0^1\psi(t)e^{f(ta+(1-t)b)}f'(ta+(1-t)b)dt\right],$$

where

$$\psi(t)=\begin{cases}1, & t\in[0,\frac{1}{2}),\\ -1, & t\in[\frac{1}{2},1].\end{cases}$$

(ii) *If $\rho=\alpha=1$, then, under the assumption of Theorem 6, we have a new result:*

$$\frac{1}{2(b-a)}\int_a^b e^{f(x)}dx-e^{f(\frac{a+b}{2})}$$
$$=\frac{(b-a)}{2}\left[\int_0^1\psi(t)e^{f(ta+(1-t)b}f'(ta+(1-t)b)dt\right],$$

where

$$\psi(t) = \begin{cases} 1, & t \in [0, \frac{1}{2}), \\ -1, & t \in [\frac{1}{2}, 1]. \end{cases}$$

4. Applications to Special Means

Consider the following special means (see Pearce and Pecaric [45]) for arbitrary real numbers $a, b, a \neq b$ as follows.

1. The arithmetic mean:

$$A(a,b) = \frac{a+b}{2}.$$

2. The generalized log mean:

$$L_n(a,b) = [\frac{b^{n+1} - a^{n+1}}{(n+1)(b-1)}]^{\frac{1}{n}}; \; a, b \in \mathbb{R}, \; n \in \mathbb{Z} \setminus \{-1, 0\}, \; a, \; b \neq 0.$$

Proposition 1. *Let* $a, b \in \mathbb{R}, \; a < b, \; 0 \notin [a,b]$ *and* $n \in \mathbb{Z}, \; |n| \geq 2$, *then*

$$\left| A(a^n, b^n) - \frac{1}{2} L_n^n(a,b) \right| \leq \frac{|n|(b-a)}{3} \{ \frac{1}{8} |a^{2n-1}| + A(a^{2n-1}, b^{2n-1}) + |(ab)^{n-1}| A(a,b) \}.$$

Proof. By taking $f(x) = \ln x^n$ in Corollary 6, we obtain the desired result. □

Proposition 2. *Let* $a, b \in \mathbb{R}, \; a < b, \; 0 \notin [a,b]$ *and* $n \in \mathbb{Z}, \; |n| \geq 2$. *Then, for* $q \geq 1$, *we have*

$$\left| A(a^n, b^n) - \frac{1}{2} L_n^n(a,b) \right| \leq \frac{|n|(b-a)}{2^{2-\frac{1}{q}}} \{ \frac{1}{30} |a^{2n-1}|^q + |b^{2n-1}|^q + \frac{(ab)^{q(n-1)}}{3} A(a^q, b^q) \}^{\frac{1}{q}}.$$

Proof. By taking $f(x) = \ln x^n$ in Corollary 10, we get the desired result. □

Proposition 3. *Let* $a, b \in \mathbb{R}, \; a < b, \; 0 \notin [a,b]$ *and* $n \in \mathbb{Z}, \; |n| \geq 2$. *Then, for* $q \geq 1$, *we have*

$$\left| A(a^n, b^n) - \frac{1}{2} L_n^n(a,b) \right| \leq \frac{|n|(b-a)}{2} \{ \frac{5}{6} A(a^{(2n-1)q}, b^{(2n-1)q}) + \frac{(ab)^{q(n-1)}}{3} A(a^q, b^q) \}^{\frac{1}{q}}.$$

Proof. By taking $f(x) = \ln x^n$ in Corollary 14, we obtain the desired result. □

5. Conclusions

In this article, some Hermite–Hadamard type inequalities for exponentially s-convex functions in a generalized fractional form were obtained. Some special cases were discussed. Some new results related to exponentially s-convex functions via Riemann–Liouville fractional integrals and via classical integrals were obtained. Applications to special means were considered. The idea may stimulate further research in this dynamic field.

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