Caputo Fractional Differential Equations with Non-Instantaneous Random Erlang Distributed Impulses

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Abstract: The p-moment exponential stability of non-instantaneous impulsive Caputo fractional differential equations is studied. The impulses occur at random moments and their action continues on finite time intervals with initially given lengths. The time between two consecutive moments of impulses is the Erlang distributed random variable. The study is based on Lyapunov functions. The fractional Dini derivatives are applied.

Keywords: impulsive fractional differential equations; random moments of impulses; non-instantaneous impulses; Erlang distribution; p-moment exponential stability

1. Introduction

Fractional differential equations are an adequate model of many physical phenomena that have memory and genetic characteristics [1–4]. At the same time, many evolution processes are characterized by the states' abrupt changes. These changes are adequately modeled by so-called impulses. There are two types of impulses: instantaneous impulses (their duration is negligible small) and non-instantaneous impulses (they start at some points and continue to be active on finite time intervals). Note the theory of instantaneous impulsive differential equation—known as impulsive differential equations—is well developed. At the same time, the theory of differential equations with non-instantaneous impulses is still at its initial stage (see, for example, Reference [5–11]). In many real world phenomena, the studied process has abrupt changes at random moments of time. In this case, the adequate model is, by differential equation, subject to random impulses. Impulsive differential equations with random impulsive moments differ from the study of stochastic differential equations with impulses (see, for example, Reference [12–16]). Some stability properties of differential equations with non-instantaneous impulses, starting at randomly distributed points, are studied in Reference [17,18]. Fractional differential equations with non-instantaneous impulses have recently been studied in Reference [19] but the meaning of the impulses are not random (as they are called in the paper) but arbitrary and the solutions are deterministic functions. Fractional equations with fixed, initially given non-instantaneous impulses, are studied over an interval with randomly varying trial lengths in Reference [20]. Fractional differential equations for random functions and with non-instantaneous impulses at fixed initially given intervals are studied in Reference [21]. Fractional differential equations with non-instantaneous impulses starting at randomly exponentially distributed times is studied in Reference [17].

In this paper, we study nonlinear fractional differential equations subject to non-instantaneous impulses starting abruptly at some random points. Note there are many types of fractional derivatives used and studied in the literature. Their application to differential equations leads to various types

of initial conditions (for example, see References [4,22,23]. Due to the natural initial value problem in the application of Caputo fractional derivative and its real practical meaning we will study this type of fractional derivative. The study of impulsive moments in random time is very limited. Caputo fractional differential equations with instantaneously acting impulses at random times were studied in References [24–27] but there are some inaccuracies there in the mixing properties of deterministic variables and random variables and inaccuracies in the convergence of a sequence of real numbers to a random variable. Also, the type of the random distribution of the random variables is not taken into account. The model with any other types of fractional derivatives such as Riemann-Liouville fractional derivative or Atangana-Baleanu fractional derivative with random impulses are not studied yet. We study the case when the time between two consecutive moments of impulses is Erlang distributed random variable. In this paper the *p*-moment exponential stability of the solution is studied by employing Lyapunov functions. This study combines the ones of the theory of fractional differential equations and the probability theory. Note that similar problem is studied in Reference [17] for exponentially distributed random moments of impulses. Since the exponential distribution is a partial case of Erlang distribution, the results in this paper are a generalization of those in Reference [17]. Note also that in this paper the stability is studied without the restriction about the constant length of the randomly occurred intervals of impulses.

Note obtained sufficient conditions could be very useful for adequate modeling of real world phenomena with impulses occurring at random moments and better described by fractional differential equations (for several models with ordinary derivatives and random impulses see Section 3.3 in Reference [28]).

2. Notes on Fractional Calculus

In this paper we will consider fractional derivatives of order $q \in (0,1)$. The main types of fractional derivatives are:

- Caputo fractional derivative (see, for example, Section 1.4.1.3 Reference [22])

$$
{}^{C}_{t_0}D^q m(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} m'(s) ds, \quad t \geq t_0,
$$

(1)

where $\Gamma(.)$ denotes the Gamma function.

- Grunwald-Letnikov fractional derivative (see, for example, Section 1.4.1.2, Reference [22])

$$
{}^{GL}_{t_0}D^q m(t) = \lim_{h \to 0} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r (qCr) m(t-rh), \quad t \geq t_0,
$$

and the Grunwald-Letnikov fractional Dini derivative by

$$
{}^{GL}_{t_0}D^q_{+} m(t) = \limsup_{h \to 0+} \frac{1}{h^q} \sum_{r=0}^{[\frac{t-t_0}{h}]} (-1)^r (qCr) m(t-rh), \quad t \geq t_0,
$$

where $qCr = \frac{q(q-1)\ldots(q-r+1)}{r!}$ where $r \geq 0$ is an integer and $[\frac{t-t_0}{h}]$ denotes the integer part of the fraction $\frac{t-t_0}{h}$.

According to Reference [23] the equality

$$
{}^{C}_{t_0}D^q_t m(t) = {}^{GL}_{t_0}D^q_t m(t) - m(t_0) \frac{(t-t_0)^{-q}}{\Gamma(1-q)},
$$

holds for $t \in (t_0, b]$ where $m \in C^1[t_0, b]$.

Definition 1. *([29]) We say $m \in C^q([t_0, T], \mathbb{R}^n)$ if $m(t)$ is differentiable (i.e., $m'(t)$ exists), the Caputo derivative ${}^c_{t_0}D^q m(t)$ exists and satisfies (1) for $t \in [t_0, T]$.*

Remark 1. *Definition 1 could be extended to any interval $I \subset \mathbb{R}_+$.*

3. Random Non-Instantaneous Impulses in Fractional Differential Equations

We will start with non-instantaneous fractional differential equations in the case when the time of these impulses is determinic one and given initially. Let two increasing sequences of nonnegative points $\{T_k\}_{k=1}^\infty$ and $\{s_i\}_{i=1}^\infty$ be given such that $0 < T_i \leq s_i < T_{i+1}$, $i = 1, 2, \ldots$ and $\lim_{k \to \infty} T_k = \infty$. Let $T_0 \in [0, T_1) \bigcup \cup_{k=1}^\infty [s_k, T_{k+1})$ be a given arbitrary point. Without loss of generality we will assume that $T_0 \in [0, T_1)$ and denote $s_0 = T_0$.

Consider the initial value problem (IVP) for the system of *non-instantaneous impulsive fractional differential equations* (NIFrDE) with fixed points of impulses

$$
\begin{aligned}
&{}^c_{T_0}D^q x = f(t, x(t)) \quad \text{for} \quad t \in (s_k, T_{k+1}], \; k = 0, 1, 2, \ldots, \\
&x(t) = I_i(t, x(T_i - 0)) \quad \text{for } t \in (T_i, s_i], \; k = 1, 2, \ldots, \\
&x(T_0) = x_0,
\end{aligned}
\tag{2}
$$

where $x, x_0 \in \mathbb{R}^n$, $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $f(t, 0) = 0$, $I_i : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $I_i(t, 0) = 0$, $(i = 1, 2, 3, \ldots)$.

In this paper, we will use Caputo fractional derivative with changeable lower limit at each point of jump. Note this approach is used for study various types of problems for deterministic non-instantaneous impulsive fractional differential equations in References [30–32].

Now we define *fractional differential equations with random non-instantaneous impulses*. Let $T_0 \geq 0$ $\{d_i\}_{i=1}^\infty$ be given. Denote $d_0 = 0$.

Consider the following condition:

H1. *The positive numbers $\{d_k\}_{k=1}^\infty$ are such that $\lim_{n \to \infty} \sum_{k=1}^n d_k = \infty$.*

H2. *The positive numbers $\{d_k\}_{k=1}^\infty$ are such that $\lim_{n \to \infty} \sum_{k=1}^n d_k = B < \infty$.*

Let the probability space $(\Omega, \mathcal{F}, P)$ be given. Let a sequence of independent Erlang distributed random variables $\{\tau_k\}_{k=1}^\infty$ defined on the sample space Ω.

Assume $\sum_{k=1}^\infty \tau_k = \infty$ with probability 1.

We will use the following condition:

H3. *The random variables $\{\tau_k\}_{k=1}^\infty$, $\tau_k \in Erlang(\alpha_k, \lambda)$ are independent where the positive integer parameter α_k is the "shape" and the positive real parameter λ is the "rate" with $\lim_{n \to \infty} \sum_{k=1}^n \alpha_k = \infty$.*

We will give some properties of the Erlang distribution:

(P1) If $X \in Erlang(\alpha_1, \lambda)$ and $Y \in Erlang(\alpha_2, \lambda)$ are independent random variables, then $X + Y \in Erlang(\alpha_1 + \alpha_2, \lambda)$;

(P2) The cumulative distribution function (CDF) of $Erlang(\alpha, \lambda)$ is

$$
F(x; \alpha, \lambda) = 1 - e^{-\lambda x} \sum_{j=1}^{\alpha-1} \frac{(\lambda x)^j}{j!}, \quad x \geq 0.
\tag{3}
$$

Define the increasing sequence of random variables $\{\xi_k\}_{k=0}^{\infty}$ by

$$\xi_k = T_0 + \sum_{i=1}^{k} \tau_i + \sum_{i=1}^{k-1} d_i, \quad k = 0, 1, 2, \ldots, \tag{4}$$

where $T_0 \geq 0$ is a fixed point.

Also, $\Xi_k = \sum_{i=1}^{n} \tau_i = \xi_k - T_0 - \sum_{i=1}^{k-1} d_i, k = 1, 2, \ldots$.

From properties (P1) and (P2) of Erlang distribution it follows the result:

Proposition 1. *Let condition (H3) be satisfied and $\Xi_n = \sum_{i=1}^{n} \tau_i$, n is a natural number. Then $\Xi_n \in Erlang(\sum_{i=1}^{n} \alpha_i, \lambda)$, that is, CDF of Ξ_n is*

$$F_{\Xi_n}(t) = P(\Xi_n < t) = 1 - e^{-\lambda t} \sum_{j=1}^{\sum_{i=1}^{n} \alpha_i - 1} \frac{(\lambda t)^j}{j!}, \quad t \geq 0.$$

The meaning of the above defined random variables and deterministic sequence is:

- the waiting time of the k-th impulse after the stop of action of $(k-1)$-st impulse is measured by random variable τ_k;
- the length of the time interval of acting the k-th impulse is given by d_k;
- the length of time until k impulses occur for $t \geq T_0$ is measured by the random variable ξ_k.

Let for any $k = 1, 2, \ldots$ the point t_k be an arbitrary value of the random variable τ_k. Define two increasing sequences of points $\{T_k\}_{k=1}^{\infty}$ and $\{s_k\}_{k=1}^{\infty}$ by

$$T_k = T_0 + \sum_{i=1}^{k} t_i + \sum_{i=1}^{k-1} d_i \tag{5}$$

and

$$s_0 = T_0, \qquad s_k = T_k + d_k = T_0 + \sum_{i=1}^{k} t_i + \sum_{i=1}^{k} d_i, \quad k = 1, 2, 3 \ldots. \tag{6}$$

Also, $T_{k+1} - s_k = t_{k+1}$, that is, it is a value of the random variable τ_{k+1}.

Note T_k and s_k are values of the random variables ξ_k and $\xi_k + d_k$, $k = 1, 2, \ldots$, respectively and $s_{k-1} \leq T_k \leq s_k$, $k = 1, 2, \ldots$.

Consider the IVP for NIFrDE (2). The solution of (2) depends not only on the initial condition (T_0, x_0) but on the moments of impulses T_k, $k = 1, 2, \ldots$, that is, the solution depends on the chosen arbitrary values t_k of the random variables τ_k, $k = 1, 2, \ldots$. We denote the solution of (2) by $x(t; T_0, x_0, \{T_k\})$. Assume $x(T_k; T_0, x_0, \{T_k\}) = \lim_{t \to T_k - 0} x(t; T_0, x_0, \{T_k\})$ for any $k = 1, 2, \ldots$. The set of all solutions $x(t; T_0, x_0, \{T_k\})$ of (2) generates a stochastic process with state space $\mathbb{R}^n$.

Consider the IVP for fractional differential equations with random moments of non-instantaneous impulses (RNIFrDE)

$$\begin{aligned}
&{}_{T_0}^{c} D^q x(t) = f(t, x(t)) \quad \text{for} \quad t \geq T_0, \; \xi_k + d_k < t < \xi_{k+1}, \; k = 0, 1, \ldots, \\
&x(t) = I_k(t, x(\xi_k)) \quad \text{for} \; \xi_k < t < \xi_k + d_k, \; k = 1, 2, \ldots, \\
&x(T_0) = x_0,
\end{aligned} \tag{7}$$

where $x_0 \in \mathbb{R}^n$, $d_0 = 0$, $f : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $I_k : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$, $(k = 1, 2, 3, \ldots)$.

Definition 2. *[33]. For any given values t_k of the random variables τ_k, $k = 1, 2, 3, \ldots$ respectively, the solution $x(t; T_0, x_0, \{T_k\})$ of the corresponding IVP for the NIFrDE (2) is called a sample path solution of the IVP for the RNIFrDE (7).*

Definition 3. *[33]. The stochastic process $x(t; T_0, x_0, \{\tau_k\})$ is said to be a solution of the IVP for RNIFrDE (7) if for any values t_k of the random variables τ_k, $k = 1, 2, \ldots$ the corresponding function $x(t; T_0, x_0, \{T_k\})$ is a sample path solution of the IVP for RNIFrDE (7).*

Remark 2. *Note that the Caputo fractional derivative in RNIFrDE (7) has a changeable lower limit on each interval without impulses. This lower limit is a value of the random variable ξ_k, $k = 1, 2, \ldots$.*

Any sample path solution $x(t; T_0, x_0, \{T_k\}) \in C^q((s_k, T_{k+1}], \mathbb{R}^n)$, $k = 0, 1, 2, \ldots$.

Definition 4. *We will say that the stochastic processes $y(t)$ and $u(t)$ satisfy the inequality $y(t) \leq u(t)$ for $t \in J \subset \mathbb{R}$ if the state space of the stochastic processes $v(t) = y(t) - v(t)$ is $(-\infty, 0]$.*

4. Preliminary Results for Erlang Distributed Moments of Impulses

We will give some results about Erlnag distributed moments of impulses studied in Reference [18] and applied to study ordinary differential equation with random non-instantaneous impulses.

For any $t \geq T_0$ consider the events

$$S_0(t) = \{\omega \in \Omega : 0 < t - T_0 < \tau_1(\omega)\},$$

$$S_k(t) = \{\omega \in \Omega : \xi_k(\omega) + d_k < t < \xi_{k+1}(\omega)\}, \quad k = 1, 2, \ldots,$$

and

$$W_k(t) = \{\omega \in \Omega : \xi_k(\omega) < t < \xi_k(\omega) + d_k\}, \quad k = 1, 2, \ldots,$$

where the random variables ξ_k, $k = 1, 2, \ldots$ are defined by (4).

Lemma 1. *[18]. For any $t \geq T_0$ the inequality*

$$P(S_0(t)) \leq e^{-\lambda(t - T_0)} \sum_{i=1}^{\alpha_1 - 1} \frac{(\lambda(t - T_0))^i}{(i)!}$$

holds.

Lemma 2. *(Upper bound of $S_k(t)$) [18]. Let condition (H3) and one of (H1) or (H2) be satisfied. Then for any natural number j we have*

$$
\begin{aligned}
P(S_j(t)) &\leq e^{-\lambda(t - T_0 - \sum_{i=1}^{j} d_i)} \sum_{i=\sum_{m=1}^{j} \alpha_m}^{\sum_{m=1}^{j+1} \alpha_m - 1} \frac{(\lambda(t - T_0 - \sum_{i=1}^{j} d_i))^i}{i!} \\
&\leq e^{-\lambda(t - T_0 - \sum_{i=1}^{j} d_i)} \sum_{i=1}^{\sum_{m=1}^{j+1} \alpha_m - 1} \frac{(\lambda(t - T_0 - \sum_{i=1}^{j} d_i))^i}{i!}
\end{aligned}
\tag{8}
$$

holds.

Lemma 3. *(Upper bound of $W_k(t)$) [18]. Let condition (H3) and one of (H1) or (H2) be satisfied. Then for any natural number j we have*

$$
\begin{aligned}
&P(W_j(t)) \\
&\leq e^{-\lambda(t - T_0 - \sum_{i=1}^{j} d_i)} \sum_{i=1}^{\sum_{m=1}^{j} \alpha_m - 1} \frac{(\lambda(t - T_0 - \sum_{i=1}^{j} d_i))^i - (\lambda(t - T_0 - \sum_{i=1}^{j-1} d_i))^i e^{-\lambda d_k}}{i!}
\end{aligned}
\tag{9}
$$

for $j \leq k$.

5. Linear Fractional Differential Equation with Random Impulses

Consider the scalar linear fractional differential equation with non-instantaneous random moments of impulses:

$$
\begin{aligned}
&{}_{T_0}^{c}D^q u = -a_k u \quad \text{for} \quad \xi_k + d_k < t < \xi_{k+1}, \ k = 0, 1, 2, \dots, \\
&u(t) = b_k u(\xi_k), \quad \text{for} \quad \xi_k < t < \xi_k + d_k, \ k = 1, 2, \dots, \\
&u(T_0) = u_0,
\end{aligned}
\tag{10}
$$

where $u_0 \in \mathbb{R}$, $a_k \geq 0, k = 0, 1, 2, \dots$, $b_k \neq 1$, $(k =, 2, \dots)$ and $d_k > 0$ are given real constants, $d_0 = 0$.

Lemma 4. *Let*

1. *Condition (H3) and one of (H1) and (H2) be satisfied.*
2. *Let there exist positive constants M_k, μ_k, $k = 0, 1, 2 \dots$ such that*

$$
E_q(-a_0(t - T_0)^q) \sum_{i=1}^{\alpha_1 - 1} \frac{(\lambda(t - T_0))^i}{(i)!} \leq M_0 e^{\mu_0(t - T_0)}, \quad t \geq T_0,
$$

and

$$
\frac{B(\lambda(t - T_0))^{\alpha_{k+1}} + 1}{\lambda(t - T_0 - \sum_{i=1}^{k} d_i)\Gamma(\sum_{m=1}^{k} \alpha_m)} \prod_{i=1}^{k} |b_i e^{\lambda d_i}(\lambda(t - T_0))^{\alpha_i}| \leq M_k e^{\mu_k(t - T_0)}, \quad t \geq T_0
\tag{11}
$$

with

$$
\sum_{k=0}^{\infty} M_k e^{\mu_k(t - T_0)} \leq M e^{\mu(t - T_0)},
$$

where $M, \mu > 0 : \ \mu < \lambda$.

Then the solution $u(t; T_0, x_0, \{\tau_k\})$ of the IVP for the linear RNIFrDE (10) is given by the formula

$$
u(t; T_0, u_0, \{\tau_k\}) =
\begin{cases}
u_0 \prod_{i=1}^{k} b_i \left(\prod_{i=0}^{k-1} E_q(-a_i(\tau_{i+1})^q) \right) \\
\qquad \text{for } \xi_k < t \leq \xi_k + d_k, \ k = 1, 2, \dots, \\
u_0 \prod_{i=1}^{k} b_i \left(\prod_{i=0}^{k-1} E_q(-a_i(\tau_{i+1})^q) \right) E_q(-a_k(t - \xi_k - d_k)^q) \\
\qquad \text{for } \xi_k + d_k < t \leq \xi_{k+1}, \ k = 0, 1, 2, \dots,
\end{cases}
\tag{12}
$$

where E_q is the Mittag-Leffler function (with one parameter) and the expected value of the solution satisfies the inequality

$$
E(|u(t; T_0, u_0, \{\tau_k\})|) \leq |u_0| e^{-(\lambda - \mu)(t - T_0)}, \quad t \geq T_0.
\tag{13}
$$

Proof. The sample path solution of (10) is given by (note the product is 1 if $k = 1$)

$$
u(t; T_0, x_0, \{T_k\}) =
\begin{cases}
u_0 \prod_{i=1}^{k} b_i \left(\prod_{i=0}^{k-1} E_q(-a_i(T_{i+1} - T_i - d_i)^q) \right) \\
\qquad \text{for } t \in (T_k, T_k + d_k], \ k = 1, 2, \dots, \\
u_0 \prod_{i=1}^{k} b_i \left(\prod_{i=0}^{k-1} E_q(-a_i(T_{i+1} - T_i - d_i)^q) \right) E_q(-a_k(t - T_k - d_k)^q) \\
\qquad \text{for } t \in (T_k + d_k, T_{k+1}], \ k = 0, 1, 2, \dots.
\end{cases}
$$

The equality (14) and Definition 3 establish (12) for the IVP for the linear RNIFrDE (10).

Then the expected value of the solution of the IVP for the scalar linear RNIFrDE (10) satisfies

$$
\begin{aligned}
E\Big(|u(t;T_0,u_0,\{\tau_k\})|\Big) &= E\Big(|u(t;T_0,u_0,\{\tau_k\})|\,\big|S_0(t)\Big)P(S_0(t)) \\
&\quad + \sum_{k=1}^{\infty} E\Big(|u(t;T_0,u_0,\{\tau_k\})|\,\big|S_k(t)\Big)P(S_k(t)) \\
&\quad + \sum_{k=1}^{\infty} E\Big(|u(t;T_0,u_0,\{\tau_k\})|\,\big|W_k(t)\Big)P(W_k(t)).
\end{aligned}
\tag{14}
$$

From Equation (12), the independence of the random variables τ_k, $k = 1, 2, \ldots$, the inequalities $0 < E_q(-A) \le 1$ for $A \ge 0$, $E(\eta) \le E(\xi)$ for the random variables $\eta, \xi : \ 0 \le \eta \le \xi$ we see

- for $\xi_0 < t \le \xi_1$

$$
\begin{aligned}
E\Big(|u(t;T_0,u_0,\{\tau_k\})|\,\big|S_0(t)\Big) &= E\Big(|u_0|E_q(-a_0(t-T_0)^q)\Big) \\
&= |u_0|E_q(-a_0(t-T_0)^q) \le |u_0|;
\end{aligned}
\tag{15}
$$

- for $\xi_k + d_k < t \le \xi_{k+1}$, k is a natural number we get $t - \xi_k - d_k < 0$, $E_q(-a_k(t - \xi_k - d_k)^q) \le 1$ and

$$
\begin{aligned}
&E\Big(|u(t;T_0,u_0,\{\tau_k\})|\,\big|S_k(t)\Big) \\
&= E\Big(|u_0|\prod_{i=1}^{k}|b_i|\Big(\prod_{i=0}^{k-1}E_q(-a_i(\tau_{i+1})^q)\Big)E_q(-a_k(t-\xi_k-d_k)^q)\Big) \\
&\le |u_0|\prod_{i=1}^{k}|b_i|\prod_{i=0}^{k-1}E\Big(E_q(-a_i(\tau_{i+1})^q)\Big) \le |u_0|\prod_{i=1}^{k}|b_i|;
\end{aligned}
\tag{16}
$$

- for $\xi_k < t \le \xi_k + d_k$, k is a natural number, we get $E_q(-a_i(\tau_{i+1})^q) \le 1$ and

$$
\begin{aligned}
&E\Big(|u(t;T_0,u_0,\{\tau_k\})|\,\big|W_k(t)\Big) = |u_0|\prod_{i=1}^{k}|b_i|\Big(E\prod_{i=0}^{k-1}E_q(-a_i(\tau_{i+1})^q)\Big) \\
&\le |u_0|\prod_{i=1}^{k}|b_i|.
\end{aligned}
\tag{17}
$$

Apply inequalities (15)–(17), Lemmas 1–3 to inequality (14) and obtain for the expected value

$$
\begin{aligned}
&E\Big(|u(t;T_0,u_0,\{\tau_k\})|\Big) \\
&\le |u_0|E_q(-a_0(t-T_0)^q)P(S_0(t)) + |u_0|\sum_{k=1}^{\infty}\prod_{i=1}^{k}|b_i|\Big(P(S_k(t)) + P(W_k(t))\Big) \\
&\le |u_0|e^{-\lambda(t-T_0)}\Big\{E_q(-a_0(t-T_0)^q)\sum_{i=1}^{\alpha_1-1}\frac{(\lambda(t-T_0))^i}{(i)!} \\
&\quad + \sum_{k=1}^{\infty}\prod_{i=1}^{k}|b_i|\Big[e^{\lambda\sum_{i=1}^{k}d_i}\Big(\sum_{i=1}^{\sum_{m=1}^{k+1}\alpha_m-1}\frac{(\lambda(t-T_0-\sum_{i=1}^{k}d_i))^i}{i!} \\
&\quad + \sum_{i=1}^{\sum_{m=1}^{k}\alpha_m-1}\frac{(\lambda(t-T_0-\sum_{i=1}^{k}d_i))^i - (\lambda(t-T_0-\sum_{i=1}^{k-1}d_i))^i e^{\lambda d_k}}{i!}\Big)\Big]\Big\}.
\end{aligned}
\tag{18}
$$

Now apply the inequalities $x^{a-1}e^{-x} < \Gamma(a,x) < Bx^{a-1}e^{-x}$ (see Equation (1.5) and (2.3) [1]) for $a > 1, B > 1, x > \dfrac{B}{B-1}(a-1)$,

$$\sum_{i=1}^{b} \frac{c^i}{i!} = e^c \frac{\Gamma(1+b,c)}{\Gamma(1+b)} - 1 < \frac{Bc^b}{\Gamma(1+b)} - 1, \quad \Gamma(a,x) \leq \frac{x^a e^{-x}}{x - (a-1)}, \; a \geq 1, x > 0,$$

for any $t > T_0 + \sum_{i=1}^{k} d_i$ we get $Bx - x > B(a-1)$, $B(x-a+1) > x$ or $B(c-b) > c$ or $\lambda(t - T_0 - \sum_{i=1}^{k} d_i) > \sum_{m=1}^{k+1} \alpha_m - 1$ and $B > \dfrac{\lambda(t - T_0 - \sum_{i=1}^{k} d_i)}{\lambda(t - T_0 - \sum_{i=1}^{k} d_i) - \sum_{m=1}^{k+1} \alpha_m + 1} > 1$

$$B > \frac{\lambda(t - T_0 - \sum_{i=1}^{k} d_i)}{\lambda(t - T_0 - \sum_{i=1}^{k} d_i) - \sum_{m=1}^{k+1} \alpha_m} \geq \frac{\lambda(t - T_0 - \sum_{i=1}^{k} d_i)}{\lambda(t - T_0 - \sum_{i=1}^{k} d_i) - \sum_{m=1}^{k+1} \alpha_m + 1} > 1,$$

$$\sum_{i=1}^{\sum_{m=1}^{k+1} \alpha_m - 1} \frac{(\lambda(t - T_0 - \sum_{i=1}^{k} d_i))^i}{i!}$$

$$= e^{(\lambda(t - T_0 - \sum_{i=1}^{k} d_i))} \frac{\Gamma(\sum_{m=1}^{k+1} \alpha_m, \lambda(t - T_0 - \sum_{i=1}^{k} d_i))}{\Gamma(\sum_{m=1}^{k+1} \alpha_m)} - 1$$

$$\leq \frac{B(\lambda(t - T_0 - \sum_{i=1}^{k} d_i))^{\sum_{m=1}^{k+1} \alpha_m - 1}}{\Gamma(\sum_{m=1}^{k+1} \alpha_m)} - 1,$$

and

$$\sum_{i=1}^{\sum_{m=1}^{k} \alpha_m - 1} \frac{(\lambda(t - T_0 - \sum_{i=1}^{k} d_i))^i - (\lambda(t - T_0 - \sum_{i=1}^{k-1} d_i))^i e^{\lambda d_k}}{i!}$$

$$< B \frac{(\lambda(t - T_0 - \sum_{i=1}^{k} d_i))^{\sum_{m=1}^{k} \alpha_m - 1}}{\Gamma(\sum_{m=1}^{k} \alpha_m)} - e^{\lambda d_k} \frac{(\lambda(t - T_0 - \sum_{i=1}^{k-1} d_i))^{\sum_{m=1}^{k} \alpha_m}}{\Gamma(\sum_{m=1}^{k} \alpha_m)}. \tag{19}$$

Then using that the function $\Gamma(u)$ is increasing for $u \geq 2$, that is, $\Gamma(\sum_{m=1}^{k+1} \alpha_m) \geq \Gamma(\sum_{m=1}^{k} \alpha_m)$ we get

$$E\left(|u(t; T_0, u_0, \{\tau_k\})|\right)$$

$$\leq |u_0| e^{-\lambda(t - T_0)} \left\{ E_q(-a_0(t - T_0)^q) \sum_{i=1}^{\alpha_1 - 1} \frac{(\lambda(t - T_0))^i}{(i)!} \right.$$

$$+ \sum_{k=1}^{\infty} \prod_{i=1}^{k} |b_i e^{\lambda d_i}| \left[\frac{B(\lambda(t - T_0 - \sum_{i=1}^{k} d_i))^{\sum_{m=1}^{k+1} \alpha_m - 1}}{\Gamma(\sum_{m=1}^{k+1} \alpha_m)} - 1 \right.$$

$$\left. \left. + B \frac{(\lambda(t - T_0 - \sum_{i=1}^{k} d_i))^{\sum_{m=1}^{k} \alpha_m - 1}}{\Gamma(\sum_{m=1}^{k} \alpha_m)} - e^{\lambda d_k} \frac{(\lambda(t - T_0 - \sum_{i=1}^{k-1} d_i))^{\sum_{m=1}^{k} \alpha_m}}{\Gamma(\sum_{m=1}^{k} \alpha_m)} \right] \right\}$$

$$\leq |u_0| e^{-\lambda(t - T_0)} \left\{ E_q(-a_0(t - T_0)^q) \sum_{i=1}^{\alpha_1 - 1} \frac{(\lambda(t - T_0))^i}{(i)!} \right.$$

$$\left. + \sum_{k=1}^{\infty} \prod_{i=1}^{k} |b_i e^{\lambda d_i}| \left[\frac{B((\lambda(t - T_0 - \sum_{i=1}^{k} d_i))^{\alpha_{k+1}} + 1)(\lambda(t - T_0 - \sum_{i=1}^{k} d_i))^{\sum_{m=1}^{k} \alpha_m}}{\lambda(t - T_0 - \sum_{i=1}^{k} d_i) \Gamma(\sum_{m=1}^{k} \alpha_m)} \right] \right\} \tag{20}$$

$$\leq |u_0| e^{-\lambda(t - T_0)} \left\{ E_q(-a_0(t - T_0)^q) \sum_{i=1}^{\alpha_1 - 1} \frac{(\lambda(t - T_0))^i}{(i)!} \right.$$

$$\left. + \sum_{k=1}^{\infty} \prod_{i=1}^{k} |b_i e^{\lambda d_i} (\lambda(t - T_0 - \sum_{j=1}^{k} d_j))^{\alpha_i}| \frac{B((\lambda(t - T_0 - \sum_{i=1}^{k} d_i))^{\alpha_{k+1}} + 1)}{\lambda(t - T_0 - \sum_{i=1}^{k} d_i) \Gamma(\sum_{m=1}^{k} \alpha_m)} \right\}$$

$$\leq |u_0| e^{-\lambda(t - T_0)} \left\{ E_q(-a_0(t - T_0)^q) \sum_{i=1}^{\alpha_1 - 1} \frac{(\lambda(t - T_0))^i}{(i)!} \right.$$

$$\left. + \sum_{k=1}^{\infty} \frac{B(\lambda(t - T_0))^{\alpha_{k+1}} + 1}{\lambda(t - T_0 - \sum_{i=1}^{k} d_i) \Gamma(\sum_{m=1}^{k} \alpha_m)} \prod_{i=1}^{k} |b_i e^{\lambda d_i} (\lambda(t - T_0))^{\alpha_i}| \right..$$

Then according to condition 2 we get

$$E\left(\left|u(t; T_0, u_0, \{\tau_k\})|\right|\right) \le |u_0| e^{-(\lambda - \mu)(t - T_0)}, \ \ t \ge T_0.$$

□

6. Lyapunov Functions

Definition 5. *Let $J \subset \mathbb{R}_+$ and $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$. The function $V(t, x) : J \times \Delta \to \mathbb{R}_+$, $V(t, 0) \equiv 0$ is from the class $\Lambda(J, \Delta)$ if it is continuous on $J \times \Delta$ and locally Lipschitzian with respect to its second argument.*

We will use fractional Dini derivatives of Lyapunov function $V \in \Lambda([T_0, \infty), \Delta)$ defined by:

$$^cD_+^q V(t, x; T_0) = \limsup_{h \to 0^+} \frac{1}{h^q} \left\{ V(t, x) - \sum_{r=1}^{[\frac{t - T_0}{h}]} (-1)^{r+1} qCr V(t - rh, x - h^q f(t, x)) \right\}, \tag{21}$$

for $t \ge T_0$,

where $qCr = \dfrac{q(q-1)(q-2)\ldots(q-r+1)}{r!}$.

7. *p*-Moment Exponential Stability for RNIFrDE

Definition 6. *Let $p > 0$. Then the trivial solution of the RNIFrDE (7) is*

- *p-moment exponentially stable if for any initial point $(T_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ there exist constants $\alpha, \mu > 0$ such that $E[||x(t; T_0, x_0, \{\tau_k\})||^p] < \alpha ||x_0||^p e^{-\mu(t - T_0)}$ for all $t > T_0$, where $x(t; T_0, x_0, \{\tau_k\})$ is the solution of the IVP for the RNIFrDE (7);*
- *eventually p-moment exponentially stable if for any initial point $(T_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ there exist constants $\Theta = \Theta(T_0), \alpha, \mu > 0$ such that $E[||x(t; T_0, x_0, \{\tau_k\})||^p] < \alpha ||x_0||^p e^{-\mu(t - T_0)}$ for all $t > T_0 + \Theta$.*

Theorem 1. *Let the following conditions be satisfied:*

1. *Conditions (H3) and one of (H1) and (H2) hold.*
2. *The function $V \in \Lambda([0, \infty), \mathbb{R}^n)$, for any $x \in \mathbb{R}^n$ the inequality $V(t, x)$ is nondecreasing in $t \ge 0$ and there exist positive constants a, b such that*

    (i) *$a||x||^p \le V(t, x) \le b||x||^p$ for $t \ge T_0$ $x \in \mathbb{R}^n$;*
    (ii) *there exists a constant $m \ge 0$ such that for any $T \in \mathbb{R}_+, x_0 \in \mathbb{R}^n$ the inequality*

    $$^cD_+^q V(t, x; T) \le -mV(t, x), \ \ \text{for } t \ge T, \ \ x \in \mathbb{R}^n$$

    holds;
    (iii) *for any $k = 1, 2, \ldots$ there exist functions $w_k \in C(\mathbb{R}_+, \mathbb{R}_+)$ and positive constants $C : C < e^{-\lambda d}$ and $C_k : w_k(t) \le C_k \le C < 1$ for $t \ge 0$ such that*

    $$V(t, I_k(t, x)) \le w_k(t) V(t, x) \ \ \ \text{for } t \ge 0, \ x \in \mathbb{R}^n. \tag{22}$$

3. *Let condition 2 of Lemma 4 is satisfied with $b_k = C_k$ and $a_k = m$.*

    Then the trivial solution of the RNIFrDE (7) is p-moment exponentially stable.

Proof. Choose an arbitrary initial data $(T_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ and let the stochastic process $x_\tau(t) = x(t; T_0, x_0, \{\tau_k\})$ be the solution of IVP for RNIFrDE (7).

Now consider the IVP for the scalar linear RNIFrDE

$$
\begin{aligned}
&{}^c_{T_0}D^q_+ u(t) = -m\,u(t) \quad \text{for} \quad \xi_k + d < t < \xi_{k+1}, \ k = 0,1,2,\ldots, \\
&u(t) = C_k u(\xi_k) \quad \text{for} \quad \xi_k < t < \xi_k + d, \ k = 1,2,\ldots, \\
&u(T_0) = V(T_0, x_0).
\end{aligned}
\tag{23}
$$

According to Lemma 4 the IVP for RNIFrDE (23) has a solution $u(t; T_0, V(T_0, x_0), \{\tau_k\})$ given by (12) with $a = m$, $b_k = C_k$ and $u_0 = V(T_0, x_0)$ and inequality (13) holds.

Choose arbitrary values t_k of the random variables τ_k, $k = 1, 2, \ldots$. Define $T_k = T_0 + \sum_{i=1}^{k} t_i + kd$, $k = 1, 2, \ldots$ and $s_k = T_k + d$. Note T_k are values of the random variables ξ_k and s_k are values of random variables $\xi_k + d$. The corresponding function $x(t; T_0, x_0, \{T_k\})$ is a sample path solution of the IVP for RNIFrDE (7) and the corresponding function $u(t) = u(t; T_0, V(T_0, x_0), \{T_k\})$ is a sample path solution of the IVP for RNIFrDE (23).

Define $v(t) = V(t, x(t; T_0, x_0, \{T_k\}))$ for $t \geq T_0$, $t \neq T_k$ and $v(T_k) = V(T_k, x(T_k - 0; T_0, x_0, \{T_k\}))$.

Let $t \in (T_k, s_k]$, $k = 0,1,2,\ldots$. From the continuity and monotonicity of the function $V(t, x)$ and condition 2(iii) we obtain

$$
\begin{aligned}
v(t) &= V(t, x(t; T_0, x_0, \{T_k\})) = V(t, I_k(t, x(T_k - 0; T_0, x_0, \{T_k\}))) \\
&\leq w_k(t) V(t, x(T_k - 0; T_0, x_0, \{T_k\})) \leq w_k(t) V(T_k, x(T_k - 0; T_0, x_0, \{T_k\})) \\
&= w_k(t) v(T_k) \leq C_k v(T_k), \quad t \in (s_k, T_k], \ k = 0,1,2,\ldots.
\end{aligned}
\tag{24}
$$

Now, consider any interval $(s_k, T_{k+1}]$. Then using $v(s_k) = V(s_k, x(s_k; T_0, x_0, \{T_k\}))$ with $x_k = x(s_k)$ and $x(t) = x(t; T_0, x_0, \{T_k\})$ we obtain

$$
\begin{aligned}
&v(t) - v(s_k) - \sum_{r=1}^{[\frac{t - s_k}{h}]} (-1)^{r+1} qCr \left[v(t - rh) - v(s_k) \right] \\
&= \left\{ V(t, x(t)) - V(s_k, x_k) \right. \\
&\quad - \sum_{r=1}^{[\frac{t - s_k}{h}]} (-1)^{r+1} qCr \left[V(t - rh, x(t) - h^q f(t, x(t))) - V(s_k, x_k) \right] \Bigg\} \\
&\quad + \sum_{r=1}^{[\frac{t - s_k}{h}]} (-1)^{r+1} qCr \left\{ V(t - rh, x(t) - h^q f(t, x(t))) - V(t - rh, x(t - rh)) \right\}.
\end{aligned}
\tag{25}
$$

Since $x(t) \in C^q((T_k, T_{k+1}], \mathbb{R}^n)$ from Remark 2 we have ${}^c_{s_k}D^q x(t) = {}^c_{s_k}D^q_+ x(t)$ and

$$
\begin{aligned}
{}^c_{s_k}D^q_+ x(t) &= \limsup_{h \to 0+} \frac{1}{h^q} \left[x(t) - x(s_k) - \sum_{r=1}^{[\frac{t - s_k}{h}]} (-1)^{r+1} (qCr) \left(x(t - rh) - x(s_k) \right) \right] \\
&= f(t, x(t))
\end{aligned}
$$

or

$$
x(t) - h^q f(t, x(t)) = S_k(x(t), h) + x(s_k) + \Lambda(h^q)
\tag{26}
$$

with $\dfrac{\Lambda(h^q)}{h^q} \to 0$ as $h \to 0$ where

$$S_k(x(t), h) = \sum_{r=1}^{[\frac{t-s_k}{h}]} (-1)^{r+1} qCr \left[x(t - rh) - x_k \right].$$

Therefore, we get

$$
\begin{aligned}
&\sum_{r=1}^{[\frac{t-s_k}{h}]} (-1)^{r+1} qCr \left\{ V(t - rh, x(t) - h^q f(t, x(t))) - V(t - rh, x(t - rh)) \right\} \\
&\leq L || \sum_{r=1}^{[\frac{t-s_k}{h}]} qCr \left(S_k(x(t), h) \, k + \Lambda(h^q) - (x(t - rh) - x_k) \right) || \\
&\leq L || \sum_{r=1}^{[\frac{t-s_k}{h}]} (-1)^{r+1} qCr \sum_{j=1}^{[\frac{t-s_k}{h}]} (-1)^{j+1} qCj \left(x(t - jh) - x_k \right) \\
&\quad - \sum_{r=1}^{[\frac{t-s_k}{h}]} (-1)^{r+1} qCr \left((x(t - rh) - x_k) \right) || + L|\Lambda(h^q)| \sum_{r=1}^{[\frac{t-s_k}{h}]} qCr \\
&= L || \left(\sum_{r=0}^{[\frac{t-s_k}{h}]} (-1)^{r+1} qCr \right) \left(\sum_{j=1}^{[\frac{t-s_k}{h}]} (-1)^{r+1} qCj \left(x(t - jh) - x_k \right) \right) || \\
&\quad + L \, |\Lambda(h^q)| \sum_{r=1}^{[\frac{t-s_k}{h}]} qCr.
\end{aligned}
\tag{27}
$$

Substitute inequality (27) in (25), divide both sides by h^q, take the limit as $h \to 0^+$, use $\sum_{r=0}^{\infty} qCr z^r = (1 + z)^q$ if $|z| \leq 1$ and condition 2(ii) and get

$$
\begin{aligned}
{}^c_{s_k} D^q_+ v(t) &\leq {}^c D^q_+ V(t, x; s_k) + L \lim_{h \to 0+} \frac{\Lambda(h^q)}{h^q} \lim_{h \to 0+} \sum_{r=1}^{[\frac{t-s_k}{h}]} qCr \\
&\quad + L \lim_{h \to 0^+} \sup \left\| \left(\sum_{r=0}^{[\frac{t-s_k}{h}]} (-1)^{r+1} qCr \right) \left(\frac{1}{h^q} \sum_{j=1}^{[\frac{t-s_k}{h}]} qCj \left(x(t - jh) - x_k \right) \right) \right\| \\
&= {}^c D^q_+ V(t, x; s_k) \leq -m V(t, x(t))) = -mv(t), \, t \in (s_k, T_{k+1}].
\end{aligned}
\tag{28}
$$

From (24) and (28) it follows the function $v(t)$ satisfies

$$
\begin{aligned}
&{}^c_{T_0} D^q_+ v(t) \leq -m \, v(t) \quad \text{for } s_k < t \leq T_{k+1}, \, k = 1, 2, \ldots, \\
&v(T_k+) \leq C_k v(T_k), \quad \text{for } T_k < t \leq s_k, \, k = 1, 2, \ldots, \\
&v(T_0) = V(T_0, x_0).
\end{aligned}
\tag{29}
$$

Consider the solution of the IVP for RNIFrDE corresponding to (29) (with equalities instead of inequalities) and with a solution $u(t; T_0, x_0, \{T_k\})$, $t \geq T_0$. Therefore, the inequality

$v(t) = V(t, x(t; T_0, x_0, \{T_k\})) \leq u(t) = u(t; T_0, V(T_0, x_0), \{T_k\})$ holds and $V(t, x(t; T_0, x_0, \{\tau_k\})) \leq u(t; T_0, V(T_0, x_0), \{\tau_k\})$.

From Inequality (13), condition 2(*i*) of Theorem 1 and Lemma 4 for the function $u(t; T_0, x_0, \{\tau_k\})$ we obtain for $t \geq T_0$ the inequalities

$$
\begin{aligned}
E(||x(t; T_0, x_0, \{\tau_k\})||^p) &= \frac{1}{a} E(a||x(t; T_0, x_0, \{\tau_k\})||^p) \\
&\leq \frac{1}{a} E(V(t, x(t; T_0, x_0, \{\tau_k\}))) \leq \frac{1}{a} E(u(t; T_0, V(T_0, x_0), \{\tau_k\})) \\
&\leq \frac{1}{a} V(T_0, x_0) e^{-(\lambda-\mu)(t-T_0)} \leq \frac{b}{a} ||x_0||^p e^{-(\lambda-\mu)(t-T_0)}.
\end{aligned}
\tag{30}
$$

Inequality (30) proves the *p*-moment exponential stability. □

Remark 3. *If the condition 2(ii) of Theorem 1 is satisfied for $t > T + \Theta$ where the constant $\Theta = \Theta(T) > 0$ then the zero solution is eventually p-moment exponentially stable.*

Example 1. *Let $d_0 = 0$, $d_k = \frac{1}{3^{k-1}}$, $k = 1, 2, 3, \ldots$ be a given numbers. Then the condition (H2) is satisfied with $B = 1.5$.*
Consider the IVP for RNIFrDE

$$
\begin{aligned}
{}^c_0 D^{0.5} x(t) &= -0.5 \frac{t}{\Gamma(0.5)} (x+y) \\
{}^c_0 D^{0.5} y(t) &= 0.5 \frac{t}{\Gamma(0.5)} (x-y) \quad \text{for} \quad t \geq 0, \ \xi_k + d_k < t < \xi_{k+1}, \\
x(t) &= 0.5 \sin(t) x(\xi_k - 0), \quad y(t) = \frac{0.5t}{t+1} y(\xi_k - 0) \quad \text{for} \ \xi_k < t < \xi_k + d_k, \ k = 1, 2, \ldots, \\
x(0) &= x_0, \quad y(0) = y_0,
\end{aligned}
\tag{31}
$$

where $x, y \in \mathbb{R}$, random variables ξ_k are defined by (4).

Let $V(x, y) = x^2 + y^2$. The condition 3(i) of Theorem 1 is satisfied for $p = 2$. The condition 3(iii) is reduced to

$$
V(t, I_k(t, x)) = (0.5\sin(t)x)^2 + (\frac{0.5t}{t+1} y)^2 \leq 0.25(x^2 + y^2) = w_k(t)V(t, x),
$$

where $w_k(t) \equiv 0.25$.

Let $T \in \mathbb{R}_+$ be an arbitrary. Then using (21) we get

$$
{}^c D_+^{0.5} V(t, x; T)
$$

$$
= \limsup_{h \to 0^+} \frac{1}{h^{0.5}} \left\{ (x^2 + y^2) - \sum_{r=1}^{[\frac{t-T}{h}]} (-1)^r 0.5 C_r \left[(x - h^{0.5} f(t, x, y))^2 + (y - h^{0.5} g(t, x, y))^2 \right] \right\}
$$

$$
= (x^2 + y^2) \limsup_{h \to 0^+} \frac{1}{h^{0.5}} \sum_{r=0}^{[\frac{t-T}{h}]} (-1)^r 0.5 C_r
$$

$$
+ \limsup_{h \to 0^+} \frac{1}{h^{0.5}} \sum_{r=1}^{[\frac{t-T}{h}]} (-1)^r 0.5 C_r \left[(x - h^{0.5} f(t, x, y))^2 + (y - h^{0.5} g(t, x, y))^2 - (x^2 + y^2) \right] \Bigg\}
$$

$$
= \frac{(x^2 + y^2)}{(t - T)^{0.5} \Gamma(0.5)} - \Big(2xf(t, x, y) + 2yg(t, x, y) \Big) \limsup_{h \to 0^+} \sum_{r=1}^{[\frac{t-T}{h}]} (-1)^r 0.5 C_r \tag{32}
$$

$$
+ \Big((f(t, x, y))^2 + (g(t, x, y))^2 \Big) \limsup_{h \to 0^+} h^{0.5} \sum_{r=1}^{[\frac{t-T}{h}]} (-1)^r 0.5 C_r
$$

$$
= \frac{(x^2 + y^2)}{(t - T)^{0.5} \Gamma(0.5)} + \Big(2xf(t, x, y) + 2yg(t, x, y) \Big)
$$

$$
= (x^2 + y^2) \Big(\frac{1}{(t - T)^{0.5} \Gamma(0.5)} - \frac{t}{\Gamma(0.5)} \Big)
$$

$$
= \frac{V(x, y)}{\Gamma(0.5)} \Big(\frac{1}{(t - T)^{0.5}} - t \Big).
$$

Note that for any $T > 0$ there exists $\Theta > 0$ such that $\dfrac{1}{(t - T)^{0.5}} - t < -1$ for $t \geq T + \Theta$. Therefore, the condition 2(iii) of Theorem 1 is satisfied with $m = \dfrac{1}{\Gamma(0.5)}$ for $t > T + \Theta$.

Also,

$$
E_{0.5}(-a_0(t - T_0)^{0.5})(\lambda(t - T_0)) \leq (\lambda(t - T_0)) \leq M_0 e^{\mu_0(t - T_0)}, \quad t \geq T_0,
$$

with $M_0 = 1$, $\mu = 0.5\lambda$ and

$$
\frac{B(\lambda(t - T_0))^{\alpha_{k+1}} + 1}{\lambda(t - T_0 - \sum_{i=1}^k d_i) \Gamma(\sum_{m=1}^k \alpha_m)} \prod_{i=1}^k |b_i e^{\lambda d_i} (\lambda(t - T_0))^{\alpha_i}|
$$

$$
= \frac{B(\lambda(t - T_0))^2 + 1}{\lambda(t - T_0 - \sum_{i=1}^k \frac{1}{3^{i-1}}) \Gamma(\sum_{m=1}^k \alpha_m)} \prod_{i=1}^k |b_i e^{\frac{\lambda}{3^{i-1}}} (\lambda(t - T_0))^2| \tag{33}
$$

$$
= \frac{B(\lambda(t - T_0))^2 + 1}{\lambda(t - T_0 - 1.5) \Gamma(2k)} (\lambda(t - T_0))^2 e^{1.5\lambda} (0.25)^k \Big(\frac{1}{\Gamma(0.5)} \Big)^k
$$

and there exist $M_k, \mu_k > 0$ such that the Inequality (11) holds for $t \geq T_0$. Therefore, the condition 3 of Theorem 1 is satisfied.

According to Remark 3 the zero solution is eventually mean square exponentially stable (eventually 2-moment exponentially stable).

8. Conclusions

The *p*-moment stability of the nonlinear Caputo fractional differential equations with impulses starting abruptly at some randomly distributed points and their action continue on intervals with

given finite lengths. The type of the distribution of the random variables determining the moments of the impulses is very important. We study the case of Erlang distributed random variables between two consecutive moments of impulses. The stability is studied without the restriction about the constant length of the randomly occurred intervals of impulses. This study combines the ones of the theory of fractional differential equations and the probability theory. The study by base on the employing Lyapunov functions. The fractional Dini derivatives are applied to obtain some sufficient conditions. Since the exponential distribution is a partial case of Erlang distribution, the results in this paper are a generalization of those in Reference [17].

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