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Existence and Uniqueness Results for a Coupled System of Caputo-Hadamard Fractional Differential Equations with Nonlocal Hadamard Type Integral Boundary Conditions

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Abstract: In this paper, we study a coupled system of Caputo-Hadamard type sequential fractional differential equations supplemented with nonlocal boundary conditions involving Hadamard fractional integrals. The sufficient criteria ensuring the existence and uniqueness of solutions for the given problem are obtained. We make use of the Leray-Schauder alternative and contraction mapping principle to derive the desired results. Illustrative examples for the main results are also presented.

Keywords: Caputo-Hadamard fractional derivative; coupled system; Hadamard fractional integral; boundary conditions; existence

MSC: 34A08, 34B10, 34B15

1. Introduction

Fractional calculus has emerged as an important area of investigation in view of its extensive applications in mathematical modeling of many complex and nonlocal nonlinear systems. An important characteristic of fractional-order operators is their nonlocal nature that accounts for the hereditary properties of the underlying phenomena. The interactions among macromolecules in the damping phenomenon give rise to a macroscopic stress-strain relation in terms of fractional differential operators. For the fractional law dealing with the viscoelastic materials, see [1] and the references cited therein. In [2], transport processes influenced by the past and present histories are described by the Caputo power law. For the details on dynamic memory involved in the economic processes, see [3,4].

In 1892, Hadamard [5] suggested a concept of fractional integro-differentiation in terms of the fractional power of the type $(t\frac{d}{dt})^q$ in contrast to its Riemann-Liouville counterpart of the form $(\frac{d}{dt})^q$. The Hadamard fractional derivative contains a logarithmic function of an arbitrary exponent in the kernel of the integral appearing in its definition. For the details of Hadamard fractional calculus, we refer the reader to the works [6–9]. Fractional differential equations involving Hadamard derivative attracted significant attention in recent years, for instance, see [10–20] and the references cited therein.

More recently, Jarad et al. [21] introduced Caputo modification of Hadamard fractional derivative which is more suitable for physically interpretable initial conditions as in case of Caputo fractional differential equations. One can find some recent results on Caputo-Hadamard type fractional differential equations in [22–28] and the references cited therein.

In this paper, we introduce a new class of boundary value problems consisting of Caputo-Hadamard type fractional differential equations and Hadamard type fractional integral boundary conditions. In precise terms, we investigate the following boundary value problem:

$$\begin{cases} ({}^C\mathcal{D}^{\alpha} + \lambda {}^C\mathcal{D}^{\alpha-1})u(t) = f(t, u(t), v(t), {}^C\mathcal{D}^{\xi}v(t)),\ 1 < \alpha \leq 2,\ 0 < \xi < 1,\ \lambda > 0, \\ ({}^C\mathcal{D}^{\beta} + \lambda {}^C\mathcal{D}^{\beta-1})v(t) = g(t, u(t), {}^C\mathcal{D}^{\bar{\xi}}u(t), v(t)), 1 < \beta \leq 2,\ 0 < \bar{\xi} < 1, \\ u(1) = 0,\ a_1\mathcal{I}^{\gamma_1}v(\eta_1) + b_1u(T) = K_1,\ \gamma_1 > 0,\ 1 < \eta_1 < T, \\ v(1) = 0,\ a_2\mathcal{I}^{\gamma_2}u(\eta_2) + b_2v(T) = K_2,\ \gamma_2 > 0,\ 1 < \eta_2 < T, \end{cases} \tag{1}$$

where ${}^C\mathcal{D}^{(\cdot)}$ and $\mathcal{I}^{(\cdot)}$ respectively denote the Caputo-Hadamard fractional derivative and Hadamard fractional integral (to be defined later), $f, g : [1, T] \times \mathbb{R}^3 \to \mathbb{R}$ are given appropriate functions and $a_i, b_i, K_i, (i = 1, 2)$ are real constants.

The rest of the paper is organized as follows. In Section 2, we recall the background material related to the topic under investigation and prove an auxiliary lemma which plays a key role in deriving the desired results. Section 3 contains the main results.

2. Preliminaries

In this section, we recall some preliminary concepts of Hadamard and Caputo-Hadamard fractional calculus related to our work. We also prove an auxiliary lemma, which plays a key role in converting the given problem into a fixed point problem.

Definition 1 ([6,7]). *The Hadamard fractional integral of order $q \in \mathbb{C}$, $\mathcal{R}(q) > 0$, for a function $g \in L^p[a, b]$, $0 \leq a \leq t \leq b \leq \infty$, is defined as*

$$\begin{aligned} I^q_{a+}g(t) &= \frac{1}{\Gamma(q)} \int_a^t \left(\log \frac{t}{s}\right)^{q-1} \frac{g(s)}{s} ds, \\ I^q_{b-}g(t) &= \frac{1}{\Gamma(q)} \int_t^b \left(\log \frac{s}{t}\right)^{q-1} \frac{g(s)}{s} ds. \end{aligned}$$

Definition 2 ([6,7]). *Let $[a, b] \subset \mathbb{R}$, $\delta = t\frac{d}{dt}$ and $AC^n_\delta[a, b] = \{g : [a, b] \to \mathbb{R} : \delta^{n-1}(g(t)) \in AC[a, b]\}$. The Hadamard derivative of fractional order q for a function $g \in AC^n_\delta[a, b]$ is defined as*

$$\begin{aligned} D^q_{a+}g(t) &= \delta^n(I^{n-q}_{a+})(t) = \frac{1}{\Gamma(n-q)} \left(t\frac{d}{dt}\right)^n \int_a^t \left(\log \frac{t}{s}\right)^{n-q-1} \frac{g(s)}{s} ds, \\ D^q_{b-}g(t) &= (-\delta)^n(I^{n-q}_{b-})(t) = \frac{1}{\Gamma(n-q)} \left(-t\frac{d}{dt}\right)^n \int_t^b \left(\log \frac{s}{t}\right)^{n-q-1} \frac{g(s)}{s} ds, \end{aligned}$$

where $n - 1 < q < n$, $n = [q] + 1$ and $[q]$ denotes the integer part of the real number q and $\log(\cdot) = \log_e(\cdot)$.

Definition 3 ([21]). *For $\mathcal{R}(q) > 0$, $n = [\mathcal{R}(q)] + 1$, and $g \in AC^n_\delta[a, b]$ $0 \leq a \leq t \leq b \leq \infty$, the Caputo-type modification of the Hadamard fractional derivative is defined by*

$${}^C\mathcal{D}^q_{a+}g(t) = D^q_{a+}\left[g(s) - \sum_{k=0}^{n-1} \frac{\delta^k g(a)}{k!}\left(\log \frac{s}{a}\right)^k\right](t),$$

$${}^C\mathcal{D}^q_{b-}g(t) = D^q_{b-}\left[g(s) - \sum_{k=0}^{n-1} \frac{(-1)^k\delta^k g(b)}{k!}\left(\log \frac{b}{s}\right)^k\right](t).$$

Theorem 1 ([21]). *Let $\mathcal{R}(q) \geq 0, n = [\mathcal{R}(q)] + 1$ and $g \in AC^n_\delta[a,b]$, $0 \leq a \leq t \leq b \leq \infty$. Then ${}^C\mathcal{D}^q_{a+}g(t)$ and ${}^C\mathcal{D}^q_{b-}g(t)$ exist everywhere on $[a,b]$ and*

(a) *if $q \notin \mathbb{N}_0$,*

$$ {}^C\mathcal{D}^q_{a+}g(t) = \frac{1}{\Gamma(n-q)} \int_a^t \left(\log\frac{t}{s}\right)^{n-q-1} \delta^n g(s)\frac{ds}{s} = (I^{n-q}_{a+})\delta^n g(t), $$

$$ {}^C\mathcal{D}^q_{b-}g(t) = \frac{(-1)^n}{\Gamma(n-q)} \int_t^b \left(\log\frac{s}{t}\right)^{n-q-1} \delta^n g(s)\frac{ds}{s} = (-1)^n(I^{n-q}_{b-})\delta^n g(t); $$

(b) *if $q = n \in \mathbb{N}_0$,*

$$ {}^C\mathcal{D}^q_{a+}g(t) = \delta^n g(t), \qquad {}^C\mathcal{D}^q_{b-}g(t) = (-1)^n\delta^n g(t). $$

In particular,

$$ {}^C\mathcal{D}^0_{a+}g(t) = {}^C\mathcal{D}^0_{b-}g(t) = g(t). $$

Remark 1 ([29]). *For $q \in \mathbb{C}$ such that $0 < q < 1$, the Caputo-Hadamard fractional derivative is defined as*

$$ {}^C\mathcal{D}^q_{a+}g(t) = \frac{1}{\Gamma(1-q)} \int_a^t \left(\log\frac{t}{s}\right)^{-q} g'(s)ds, $$

$$ {}^C\mathcal{D}^q_{b-}g(t) = \frac{-1}{\Gamma(1-q)} \int_t^b \left(\log\frac{s}{t}\right)^{-q} g'(s)ds. $$

Lemma 1 ([21]). *Let $\mathcal{R}(q) \geq 0, n = [\mathcal{R}(q)] + 1$ and $g \in C[a,b]$. If $\mathcal{R}(q) \neq 0$ or $q \in \mathbb{N}$, then*

$$ {}^C\mathcal{D}^q_{a^+}(I^q_{a+}g)(t) = g(t), \quad {}^C\mathcal{D}^q_{b^-}(I^q_{b-}g)(t) = g(t). $$

Lemma 2 ([21]). *Let $g \in AC^n_\delta[a,b]$ or $C^n_\delta[a,b]$ and $q \in \mathbb{C}$, then*

$$ I^q_{a^+}({}^C\mathcal{D}^q_{a+}g)(t) = g(t) - \sum_{k=0}^{n-1} \frac{\delta^k g(a)}{k!}\left(\log\frac{t}{a}\right)^k, $$

$$ I^q_{b^-}({}^C\mathcal{D}^q_{b-}g)(t) = g(t) - \sum_{k=0}^{n-1} \frac{\delta^k g(b)}{k!}\left(\log\frac{b}{t}\right)^k. $$

Now we present an auxiliary lemma dealing with the linear variant of the problem (1).

Lemma 3. *Let $h_1, h_2 \in AC^n_\delta[1,T]$. Then the solution of the linear system of fractional differential equations:*

$$ \begin{aligned} ({}^C\mathcal{D}^\alpha + \lambda {}^C\mathcal{D}^{\alpha-1})u(t) &= h_1(t), \\ ({}^C\mathcal{D}^\beta + \lambda {}^C\mathcal{D}^{\beta-1})v(t) &= h_2(t), \end{aligned} \tag{2} $$

supplemented with the boundary conditions:

$$ \begin{aligned} u(1) = 0 \quad &, \quad a_1\mathcal{I}^{\gamma_1}v(\eta_1) + b_1u(T) = K_1,\ \gamma_1 > 0,\ 1 < \eta_1 < T, \\ v(1) = 0 \quad &, \quad a_2\mathcal{I}^{\gamma_2}u(\eta_2) + b_2v(T) = K_2,\ \gamma_2 > 0,\ 1 < \eta_2 < T, \end{aligned} \tag{3} $$

is given by

$$ \begin{aligned} u(t) &= \frac{(1-t^{-\lambda})}{\lambda\Delta}\Big\{(K_2A_2 - K_1B_2) + T^{-\lambda}\Big[b_1B_2\int_1^T s^{\lambda-1}\mathcal{I}^{\alpha-1}h_1(s)ds - b_2A_2\int_1^T s^{\lambda-1}\mathcal{I}^{\beta-1}h_2(s)ds\Big] \\ &+ \frac{a_1B_2}{\Gamma(\gamma_1)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{s}\right)^{\gamma_1-1} s^{-(\lambda+1)}\Big(\int_1^s m^{\lambda-1}\mathcal{I}^{\beta-1}h_2(m)dm\Big)ds \end{aligned} $$

$$
\begin{aligned}
&- \frac{a_2 A_2}{\Gamma(\gamma_2)} \int_1^{\eta_2} \left(\log\frac{\eta_2}{s}\right)^{\gamma_2-1} s^{-(\lambda+1)} \left(\int_1^s m^{\lambda-1}\mathcal{I}^{\alpha-1} h_1(m)dm\right)ds \Bigg\} \\
&+ t^{-\lambda}\int_1^t s^{\lambda-1}\mathcal{I}^{\alpha-1}h_1(s)ds, \qquad (4)
\end{aligned}
$$

and

$$
\begin{aligned}
v(t) &= \frac{(1-t^{-\lambda})}{\lambda\Delta}\Bigg\{(K_1B_1 - K_2A_1) + T^{-\lambda}\Big[b_2A_1\int_1^T s^{\lambda-1}\mathcal{I}^{\beta-1}h_2(s)ds - b_1B_1\int_1^T s^{\lambda-1}\mathcal{I}^{\alpha-1}h_1(s)ds\Big] \\
&+ \frac{a_2 A_1}{\Gamma(\gamma_2)} \int_1^{\eta_2} \left(\log\frac{\eta_2}{s}\right)^{\gamma_2-1} s^{-(\lambda+1)} \left(\int_1^s m^{\lambda-1}\mathcal{I}^{\alpha-1} h_1(m)dm\right)ds \\
&- \frac{a_1 B_1}{\Gamma(\gamma_1)} \int_1^{\eta_1} \left(\log\frac{\eta_1}{s}\right)^{\gamma_1-1} s^{-(\lambda+1)} \left(\int_1^s m^{\lambda-1}\mathcal{I}^{\beta-1} h_2(m)dm\right)ds \Bigg\} \\
&+ t^{-\lambda}\int_1^t s^{\lambda-1}\mathcal{I}^{\beta-1}h_2(s)ds. \qquad (5)
\end{aligned}
$$

where

$$\Delta = B_1A_2 - A_1B_2 \neq 0, \qquad (6)$$

$$A_1 = \frac{b_1}{\lambda}(1-T^{-\lambda}), \quad A_2 = \frac{a_1}{\Gamma(\gamma_1+1)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{s}\right)^{\gamma_1} s^{-(\lambda+1)}ds, \qquad (7)$$

$$B_1 = \frac{a_2}{\Gamma(\gamma_2+1)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{s}\right)^{\gamma_2} s^{-(\lambda+1)}ds, \quad B_2 = \frac{b_2}{\lambda}(1-T^{-\lambda}). \qquad (8)$$

Proof. In view of Theorem 1 and lemma 2, the general solution of the system (2) can be written as

$$u(t) = c_0 t^{-\lambda} + \frac{c_1}{\lambda}(1-t^{-\lambda}) + t^{-\lambda}\int_1^t s^{\lambda-1}\mathcal{I}^{\alpha-1}h_1(s)ds, \qquad (9)$$

$$v(t) = d_0 t^{-\lambda} + \frac{d_1}{\lambda}(1-t^{-\lambda}) + t^{-\lambda}\int_1^t s^{\lambda-1}\mathcal{I}^{\beta-1}h_2(s)ds, \qquad (10)$$

where $c_i,\ d_i (i = 0,1)$ are unknown arbitrary constants. Using the data $u(1) = 0,\ v(1) = 0$ given by (3) in (9) and (10), we find that $c_0 = 0$ and $d_0 = 0$. Thus (9) and (10) take the form:

$$
\begin{aligned}
u(t) &= \frac{c_1}{\lambda}(1-t^{-\lambda}) + t^{-\lambda}\int_1^t s^{\lambda-1}\mathcal{I}^{\alpha-1}h_1(s)ds, \qquad (11) \\
v(t) &= \frac{d_1}{\lambda}(1-t^{-\lambda}) + t^{-\lambda}\int_1^t s^{\lambda-1}\mathcal{I}^{\beta-1}h_2(s)ds. \qquad (12)
\end{aligned}
$$

Using the nonlocal integral boundary conditions: $a_1\mathcal{I}^{\gamma_1}v(\eta_1) + b_1u(T) = K_1$ and $a_2\mathcal{I}^{\gamma_2}u(\eta_2) + b_2v(T) = K_2$ in (11) and (12), we obtain

$$A_1c_1 + A_2d_1 = \mathcal{J}_1,\ B_1c_1 + B_2d_1 = \mathcal{J}_2, \qquad (13)$$

where A_i and B_i $(i = 1,2)$ are respectively given by (7) and (8), and

$$
\begin{aligned}
\mathcal{J}_1 &= K_1 - \frac{a_1}{\Gamma(\gamma_1)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{s}\right)^{\gamma_1-1} s^{-(\lambda+1)}\left(\int_1^s m^{\lambda-1}\mathcal{I}^{\beta-1}h_2(m)dm\right)ds \\
&\quad -b_1T^{-\lambda}\int_1^T s^{\lambda-1}\mathcal{I}^{\alpha-1}h_1(s)ds, \qquad (14) \\
\mathcal{J}_2 &= K_2 - \frac{a_2}{\Gamma(\gamma_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{s}\right)^{\gamma_2-1} s^{-(\lambda+1)}\left(\int_1^s m^{\lambda-1}\mathcal{I}^{\alpha-1}h_1(m)dm\right)ds \\
&\quad -b_2T^{-\lambda}\int_1^T s^{\lambda-1}\mathcal{I}^{\beta-1}h_2(s)ds. \qquad (15)
\end{aligned}
$$

Solving the system (13) for c_1 and d_1, we find that

$$\begin{aligned} c_1 &= \frac{(K_2A_2 - K_1B_2)}{\Delta} + \frac{T^{-\lambda}}{\Delta}\Big[b_1B_2\int_1^T s^{\lambda-1}\mathcal{I}^{\alpha-1}h_1(s)ds - b_2A_2\int_1^T s^{\lambda-1}\mathcal{I}^{\beta-1}h_2(s)ds\Big] \\ &+ \frac{a_1B_2}{\Delta\Gamma(\gamma_1)}\int_1^{\eta_1}\Big(\log\frac{\eta_1}{s}\Big)^{\gamma_1-1}s^{-(\lambda+1)}\Big(\int_1^s m^{\lambda-1}\mathcal{I}^{\beta-1}h_2(m)dm\Big)ds \\ &- \frac{a_2A_2}{\Delta\Gamma(\gamma_2)}\int_1^{\eta_2}\Big(\log\frac{\eta_2}{s}\Big)^{\gamma_2-1}s^{-(\lambda+1)}\Big(\int_1^s m^{\lambda-1}\mathcal{I}^{\alpha-1}h_1(m)dm\Big)ds, \end{aligned} \tag{16}$$

$$\begin{aligned} d_1 &= \frac{(K_1B_1 - K_2A_1)}{\Delta} + \frac{T^{-\lambda}}{\Delta}\Big[b_2A_1\int_1^T s^{\lambda-1}\mathcal{I}^{\beta-1}h_2(s)ds - b_1B_1\int_1^T s^{\lambda-1}\mathcal{I}^{\alpha-1}h_1(s)ds\Big] \\ &+ \frac{a_2A_1}{\Delta\Gamma(\gamma_2)}\int_1^{\eta_2}\Big(\log\frac{\eta_2}{s}\Big)^{\gamma_2-1}s^{-(\lambda+1)}\Big(\int_1^s m^{\lambda-1}\mathcal{I}^{\alpha-1}h_1(m)dm\Big)ds \\ &- \frac{a_1B_1}{\Delta\Gamma(\gamma_1)}\int_1^{\eta_1}\Big(\log\frac{\eta_1}{s}\Big)^{\gamma_1-1}s^{-(\lambda+1)}\Big(\int_1^s m^{\lambda-1}\mathcal{I}^{\beta-1}h_2(m)dm\Big)ds, \end{aligned} \tag{17}$$

where Δ is given by (6). Substituting the values of c_1 and d_1 in (11) and (12), we obtain the solution (4) and (5). This completes the proof. □

3. Existence and Uniqueness Results

This section is concerned with the main results of the paper. First of all, we fix our terminology. Let $X = \{x : x \in C([1,T],\mathbb{R})$ and ${}^C\mathcal{D}^{\bar{\xi}}x \in C([1,T],\mathbb{R})\}$ and $Y = \{y : y \in C([1,T],\mathbb{R})$ and ${}^C\mathcal{D}^{\xi}y \in C([1,T],\mathbb{R})\}$ be the spaces respectively equipped with the norms $\|x\|_X = \|x\| + \|{}^C\mathcal{D}^{\bar{\xi}}x\| = \sup_{t\in[1,T]}|x(t)| + \sup_{t\in[1,T]}|{}^C\mathcal{D}^{\bar{\xi}}x(t)|$ and $\|y\|_Y = \|y\| + \|{}^C\mathcal{D}^{\xi}y\| = \sup_{t\in[1,T]}|y(t)| + \sup_{t\in[1,T]}|{}^C\mathcal{D}^{\xi}y(t)|$. Observe that $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ are Banach spaces. In consequence, the product space $(X\times Y, \|.\|_{X\times Y})$ is a Banach space endowed with the norm $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$ for $(x,y) \in X\times Y$.

Using Lemma 3, we introduce an operator $T : X\times Y \to X\times Y$ as follows:

$$T(u,v)(t) := (T_1(u,v)(t), T_2(u,v)(t)), \tag{18}$$

where

$$\begin{aligned} T_1(u,v)(t) &= \frac{(1-t^{-\lambda})}{\lambda\Delta}\Big\{(K_2A_2 - K_1B_2) + T^{-\lambda}\Big[b_1B_2\int_1^T s^{\lambda-1}\mathcal{I}^{\alpha-1}f(s,u(s),v(s),{}^C\mathcal{D}^{\xi}v(s))ds \\ &- b_2A_2\int_1^T s^{\lambda-1}\mathcal{I}^{\beta-1}g(s,u(s),{}^C\mathcal{D}^{\bar{\xi}}u(s),v(s))ds\Big] \\ &+ \frac{a_1B_2}{\Gamma(\gamma_1)}\int_1^{\eta_1}\Big(\log\frac{\eta_1}{s}\Big)^{\gamma_1-1}s^{-(\lambda+1)}\Big(\int_1^s m^{\lambda-1}\mathcal{I}^{\beta-1}g(m,u(m),{}^C\mathcal{D}^{\bar{\xi}}u(m),v(m))dm\Big)ds \\ &- \frac{a_2A_2}{\Gamma(\gamma_2)}\int_1^{\eta_2}\Big(\log\frac{\eta_2}{s}\Big)^{\gamma_2-1}s^{-(\lambda+1)}\Big(\int_1^s m^{\lambda-1}\mathcal{I}^{\alpha-1}f(m,u(m),v(m),{}^C\mathcal{D}^{\xi}v(m))dm\Big)ds\Big\} \\ &+ t^{-\lambda}\int_1^t s^{\lambda-1}\mathcal{I}^{\alpha-1}f(s,u(s),v(s),{}^C\mathcal{D}^{\xi}v(s))ds, \end{aligned} \tag{19}$$

$$\begin{aligned} T_2(u,v)(t) &= \frac{(1-t^{-\lambda})}{\lambda\Delta}\Big\{(K_1B_1 - K_2A_1) + T^{-\lambda}\Big[b_2A_1\int_1^T s^{\lambda-1}\mathcal{I}^{\beta-1}g(s,u(s),{}^C\mathcal{D}^{\bar{\xi}}u(s),v(s))ds \\ &- b_1B_1\int_1^T s^{\lambda-1}\mathcal{I}^{\alpha-1}f(s,u(s),v(s),{}^C\mathcal{D}^{\xi}v(s))ds\Big] \\ &+ \frac{a_2A_1}{\Gamma(\gamma_2)}\int_1^{\eta_2}\Big(\log\frac{\eta_2}{s}\Big)^{\gamma_2-1}s^{-(\lambda+1)}\Big(\int_1^s m^{\lambda-1}\mathcal{I}^{\alpha-1}f(m,u(m),v(m),{}^C\mathcal{D}^{\xi}v(m))dm\Big)ds \\ &- \frac{a_1B_1}{\Gamma(\gamma_1)}\int_1^{\eta_1}\Big(\log\frac{\eta_1}{s}\Big)^{\gamma_1-1}s^{-(\lambda+1)}\Big(\int_1^s m^{\lambda-1}\mathcal{I}^{\beta-1}g(m,u(m),{}^C\mathcal{D}^{\bar{\xi}}u(m),v(m))dm\Big)ds\Big\} \end{aligned} \tag{20}$$

$$+ \quad t^{-\lambda}\int_1^t s^{\lambda-1}\mathcal{I}^{\beta-1}g(s,u(s),{}^C\mathcal{D}^{\bar{\xi}}u(s),v(s))ds.$$

Next we enlist the assumptions that we need in the sequel.

(H_1) Let $f,g:[1,T]\times\mathbb{R}^3\to\mathbb{R}$ be continuous functions and there exist real constants $\mu_j,\tau_j\geq 0$ $(j=1,2,3)$ and $\mu_0>0,\ \tau_0>0$ such that

$$\begin{aligned} |f(t,x_1,x_2,x_3)| &\leq \mu_0+\mu_1|x_1|+\mu_2|x_2|+\mu_3|x_3|,\\ |g(t,x_1,x_2,x_3)| &\leq \tau_0+\tau_1|x_1|+\tau_2|x_2|+\tau_3|x_3|, \forall x_j\in\mathbb{R},\ j=1,2,3. \end{aligned}$$

(H_2) There exist positive constants $l,\ l_1$ such that

$$\begin{aligned} |f(t,x_1,x_2,x_3)-f(t,y_1,y_2,y_3)| &\leq l\left(|x_1-y_1|+|x_2-y_2|+|x_3-y_3|\right),\\ |g(t,x_1,x_2,x_3)-g(t,y_1,y_2,y_3)| &\leq l_1\left(|x_1-y_1|+|x_2-y_2|+|x_3-y_3|\right), \forall t\in[1,T],\ x_j,\ y_j\in\mathbb{R}. \end{aligned}$$

For computational convenience, we set

$$\rho = \sup_{t\in[1,T]}\left|1-t^{-\lambda}\right| = \left|1-T^{-\lambda}\right|, \tag{21}$$

$$\Theta_1 = \frac{\rho|K_2A_2-K_1B_2|}{\lambda|\Delta|},\quad \overline{\Theta}_1=\frac{|K_2A_2-K_1B_2|}{|\Delta|}\Big(\log T\Big)^{1-\bar{\xi}}, \tag{22}$$

$$\Theta_2 = \frac{\rho|K_1B_1-K_2A_1|}{\lambda|\Delta|},\quad \overline{\Theta}_2=\frac{|K_1B_1-K_2A_1|}{|\Delta|}\Big(\log T\Big)^{1-\xi}, \tag{23}$$

$$M_1 = \frac{\rho}{\lambda|\Delta|\Gamma(\alpha+1)}\Big[|b_1||B_2|(\log T)^\alpha+\frac{|a_2||A_2|}{\Gamma(\gamma_2+1)}\Big(\log\eta_2\Big)^{\alpha+\gamma_2}\Big]+\frac{\Big(\log T\Big)^\alpha}{\Gamma(\alpha+1)}, \tag{24}$$

$$\overline{M}_1 = \frac{\Big(\log T\Big)^{1-\bar{\xi}}}{|\Delta|\Gamma(\alpha+1)}\Big[|b_1||B_2|(\log T)^\alpha+\frac{|a_2||A_2|}{\Gamma(\gamma_2+1)}\Big(\log\eta_2\Big)^{\alpha+\gamma_2}+\lambda|\Delta|\Big(\log T\Big)^\alpha+\alpha|\Delta|\Big(\log T\Big)^{\alpha-1}\Big], \tag{25}$$

$$M_2 = \frac{\rho}{\lambda|\Delta|\Gamma(\beta+1)}\Big[\frac{|a_1||B_2|}{\Gamma(\gamma_1+1)}\Big(\log\eta_1\Big)^{\beta+\gamma_1}+|b_2||A_2|\Big(\log T\Big)^\beta\Big], \tag{26}$$

$$\overline{M}_2 = \frac{\Big(\log T\Big)^{1-\bar{\xi}}}{|\Delta|\Gamma(\beta+1)}\Big[|b_2||A_2|\Big(\log T\Big)^\beta+\frac{|a_1||B_2|}{\Gamma(\gamma_1+1)}\Big(\log\eta_1\Big)^{\beta+\gamma_1}\Big], \tag{27}$$

$$N_1 = \frac{\rho}{\lambda|\Delta|\Gamma(\alpha+1)}\Big[|b_1||B_1|(\log T)^\alpha+\frac{|a_2||A_1|}{\Gamma(\gamma_2+1)}\Big(\log\eta_2\Big)^{\alpha+\gamma_2}\Big], \tag{28}$$

$$\overline{N}_1 = \frac{\Big(\log T\Big)^{1-\xi}}{|\Delta|\Gamma(\alpha+1)}\Big[|b_1||B_1|(\log T)^\alpha+\frac{|a_2||A_1|}{\Gamma(\gamma_2+1)}\Big(\log\eta_2\Big)^{\alpha+\gamma_2}\Big], \tag{29}$$

$$N_2 = \frac{\rho}{\lambda|\Delta|\Gamma(\beta+1)}\Big[\frac{|a_1||B_1|}{\Gamma(\gamma_1+1)}\Big(\log\eta_1\Big)^{\beta+\gamma_1}+|b_2||A_1|\Big(\log T\Big)^\beta\Big]+\frac{\Big(\log T\Big)^\beta}{\Gamma(\beta+1)}, \tag{30}$$

$$\overline{N}_2 = \frac{\Big(\log T\Big)^{1-\xi}}{|\Delta|\Gamma(\beta+1)}\Big[|b_2||A_1|\Big(\log T\Big)^\beta+\frac{|a_1||B_1|}{\Gamma(\gamma_1+1)}\Big(\log\eta_1\Big)^{\beta+\gamma_1}+\lambda|\Delta|\Big(\log T\Big)^\beta+\beta|\Delta|\Big(\log T\Big)^{\beta-1}\Big], \tag{31}$$

$$\begin{aligned} \varpi_1 &= \Theta_1+\Theta_2+\frac{\overline{\Theta}_1}{\Gamma(2-\bar{\xi})}+\frac{\overline{\Theta}_2}{\Gamma(2-\xi)}+\mu_0\Big(M_1+N_1+\frac{\overline{M}_1}{\Gamma(2-\bar{\xi})}+\frac{\overline{N}_1}{\Gamma(2-\xi)}\Big)\\ &+ \tau_0\Big(M_2+N_2+\frac{\overline{M}_2}{\Gamma(2-\bar{\xi})}+\frac{\overline{N}_2}{\Gamma(2-\xi)}\Big), \end{aligned} \tag{32}$$

$$\varpi_2 = \mu_1\Big(M_1+N_1+\frac{\overline{M}_1}{\Gamma(2-\bar{\xi})}+\frac{\overline{N}_1}{\Gamma(2-\xi)}\Big)+\max\{\tau_1,\tau_2\}\Big(M_2+N_2+\frac{\overline{M}_2}{\Gamma(2-\bar{\xi})}+\frac{\overline{N}_2}{\Gamma(2-\xi)}\Big), \tag{33}$$

$$\varpi_3 = \max\{\mu_2,\mu_3\}\Big(M_1+N_1+\frac{\overline{M}_1}{\Gamma(2-\bar{\xi})}+\frac{\overline{N}_1}{\Gamma(2-\xi)}\Big)+\tau_3\Big(M_2+N_2+\frac{\overline{M}_2}{\Gamma(2-\bar{\xi})}+\frac{\overline{N}_2}{\Gamma(2-\xi)}\Big). \tag{34}$$

Now, we are in a position to present our first existence result for the boundary value problem (1), which is based on Leray-Schauder alternative.

Lemma 4 (Leray-Schauder alternative [30]). *Let $F : E \to E$ be a completely continuous operator. Let $\varepsilon(F) = \{x \in E : x = \kappa F(x)$ for some $0 < \kappa < 1\}$. Then either the set $\varepsilon(F)$ is unbounded or F has at least one fixed point.*

Theorem 2. *Assume that (H_1) holds and that $\max\{\varpi_2, \varpi_3\} < 1$, where ϖ_2 and ϖ_3 are given by (33) and (34) respectively. Then the boundary value problem (1) has at least one solution on $[1, T]$.*

Proof. In the first step, we establish that the operator $T : X \times Y \to X \times Y$ is completely continuous. By continuity of the functions f and g, it follows that the operators T_1 and T_2 are continuous. In consequence, the operator T is continuous. In order to show that the operator T is uniformly bounded, let $\Omega \subset X \times Y$ be a bounded set. Then there exist positive constants L_1 and L_2 such that $|f(t, u(t), v(t), {}^C\mathcal{D}^{\xi}v(t))| \le L_1$, $|g(t, u(t), {}^C\mathcal{D}^{\bar{\xi}}u(t), v(t))| \le L_2$, $\forall (u, v) \in \Omega$. Then, for any $(u, v) \in \Omega$, we have

$$\begin{aligned}
|T_1(u,v)(t)| \le\; & \frac{|K_2A_2 - K_1B_2|\rho}{\lambda|\Delta|} + \frac{\rho L_1}{\lambda|\Delta|}\Bigg\{ \frac{|b_1||B_2|T^{-\lambda}}{\Gamma(\alpha-1)} \int_1^T s^{\lambda-1}\Big(\int_1^s \Big(\log\frac{s}{m}\Big)^{\alpha-2}\frac{dm}{m}\Big)ds \\
&+ \frac{|a_2||A_2|}{\Gamma(\gamma_2)\Gamma(\alpha-1)} \int_1^{\eta_2} \Big(\log\frac{\eta_2}{s}\Big)^{\gamma_2-1} s^{-(\lambda+1)}\Big(\int_1^s m^{\lambda-1}\Big(\log\frac{m}{r}\Big)^{\alpha-2}\frac{dr}{r}\Big)dm\Big)ds \Bigg\} \\
&+ \frac{L_1|t^{-\lambda}|}{\Gamma(\alpha-1)} \int_1^t s^{\lambda-1}\Big(\int_1^s \Big(\log\frac{s}{m}\Big)^{\alpha-2}\frac{dm}{m}\Big)ds \\
&+ \frac{\rho L_2}{\lambda|\Delta|}\Bigg\{ \frac{|b_2||A_2|T^{-\lambda}}{\Gamma(\beta-1)} \int_1^T s^{\lambda-1}\Big(\int_1^s \Big(\log\frac{s}{m}\Big)^{\beta-2}\frac{dm}{m}\Big)ds \\
&+ \frac{|a_1||B_2|}{\Gamma(\gamma_1)\Gamma(\beta-1)} \int_1^{\eta_1} \Big(\log\frac{\eta_1}{s}\Big)^{\gamma_1-1} s^{-(\lambda+1)}\Big(\int_1^s m^{\lambda-1}\Big(\log\frac{m}{r}\Big)^{\beta-2}\frac{dr}{r}\Big)dm\Big)ds \Bigg\}, \\
\le\; & \frac{|K_2A_2 - K_1B_2|\rho}{\lambda|\Delta|} + \frac{\rho L_1}{\lambda|\Delta|\Gamma(\alpha+1)}\Big[|b_1||B_2|(\log T)^{\alpha} + \frac{|a_2||A_2|}{\Gamma(\gamma_2+1)}\Big(\log\eta_2\Big)^{\alpha+\gamma_2}\Big] \\
&+ \frac{L_1}{\Gamma(\alpha+1)}\Big(\log T\Big)^{\alpha} + \frac{\rho L_2}{\lambda|\Delta|\Gamma(\beta+1)}\Big[\frac{|a_1||B_2|}{\Gamma(\gamma_1+1)}\Big(\log\eta_1\Big)^{\beta+\gamma_1} + |b_2A_2|\Big(\log T\Big)^{\beta}\Big],
\end{aligned}$$

which, on taking the norm for $t \in [1, T]$ and using (22), (24) and (26) yields

$$\|T_1(u,v)\| \le \Theta_1 + L_1M_1 + L_2M_2.$$

Since $0 < \bar{\xi} < 1$, we use Remark 1 to get

$$|{}^C\mathcal{D}^{\bar{\xi}}T_1(u,v)(t)| \le \frac{1}{\Gamma(1-\bar{\xi})}\int_1^t \Big(\log\frac{t}{s}\Big)^{-\bar{\xi}}\Big|T_1'(u,v)(s)\Big|\frac{ds}{s} \le \frac{1}{\Gamma(2-\bar{\xi})}\left(\overline{\Theta}_1 + L_1\overline{M}_1 + L_2\overline{M}_2\right),$$

where $\overline{\Theta}_1$, $\overline{M}_1$ and $\overline{M}_2$ are respectively given by (22), (25) and (27). Hence

$$\|T_1(u,v)\|_X = \|T_1(u,v)\| + \|{}^C\mathcal{D}^{\bar{\xi}}T_1(u,v)\| \le \Theta_1 + L_1M_1 + L_2M_2 + \frac{1}{\Gamma(2-\bar{\xi})}\left(\overline{\Theta}_1 + L_1\overline{M}_1 + L_2\overline{M}_2\right). \quad (35)$$

Similarly, using (23), (28) and (30), we obtain

$$\begin{aligned}
|T_2(u,v)(t)| \le\; & \frac{\rho|K_1B_1 - K_2A_1|}{\lambda|\Delta|} + \frac{\rho L_1}{\lambda|\Delta|\Gamma(\alpha+1)}\Big[|b_1||B_1|(\log T)^{\alpha} + \frac{|a_2||A_1|}{\Gamma(\gamma_2+1)}\Big(\log\eta_2\Big)^{\alpha+\gamma_2}\Big] \\
&+ \frac{\rho L_2}{\lambda|\Delta|\Gamma(\beta+1)}\Big[\frac{|b_2||A_1|}{\Gamma(\gamma_1+1)}\Big(\log\eta_1\Big)^{\beta+\gamma_1} + |b_2||A_1|\Big(\log T\Big)^{\beta}\Big] + \frac{L_2}{\Gamma(\beta+1)}\Big(\log T\Big)^{\beta}
\end{aligned}$$

$$\leq \quad \Theta_2 + L_1 N_1 + L_2 N_2.$$

As before, one can find that

$$|^C\mathcal{D}^{\xi} T_2(u,v)(t)| \quad \leq \quad \frac{1}{\Gamma(2-\xi)}\left(\overline{\Theta}_2 + L_1\overline{N}_1 + L_2\overline{N}_2\right),$$

where $\overline{\Theta}_2$, $\overline{N}_1$ and $\overline{N}_2$ are respectively given by (23), (29) and (31).

In consequence, we get

$$\|T_2(u,v)\|_Y = \|T_2(u,v)\| + \|^C\mathcal{D}^{\xi} T_2(u,v)\| \quad \leq \quad \Theta_2 + L_1 N_1 + L_2 N_2 + \frac{1}{\Gamma(2-\xi)}\left(\overline{\Theta}_2 + L_1\overline{N}_1 + L_2\overline{N}_2\right). \quad (36)$$

From the inequalities (35) and (36), we deduce that T_1 and T_2 are uniformly bounded, which implies that the operator T is uniformly bounded.

Next, we show that T is equicontinuous. Let $t_1,\ t_2 \in [1,T]$ with $t_1 < t_2$. Then we have

$$\begin{aligned}
& |T_1(u,v)(t_2) - T_1(u,v)(t_1)| \\
\leq\ & \frac{\left|t_1^{-\lambda} - t_2^{-\lambda}\right|}{\lambda|\Delta|}\Bigg\{|K_2A_2 - K_1B_2| + T^{-\lambda}\Big[|b_1||B_2|\int_1^T s^{\lambda-1}\mathcal{I}^{\alpha-1}|f(s,u(s),v(s),{}^C\mathcal{D}^{\xi}v(s))|ds \\
+\ & |b_2||A_2|\int_1^T s^{\lambda-1}\mathcal{I}^{\beta-1}|g(s,u(s),{}^C\mathcal{D}^{\bar{\xi}}u(s),v(s))|ds\Big] \\
+\ & \frac{|a_1||B_2|}{\Gamma(\gamma_1)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{s}\right)^{\gamma_1-1}s^{-(\lambda+1)}\Big(\int_1^s m^{\lambda-1}\mathcal{I}^{\beta-1}|g(m,u(m),{}^C\mathcal{D}^{\bar{\xi}}u(m),v(m))|dm\Big)ds \\
+\ & \frac{|a_2||A_2|}{\Gamma(\gamma_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{s}\right)^{\gamma_2-1}s^{-(\lambda+1)}\Big(\int_1^s m^{\lambda-1}\mathcal{I}^{\alpha-1}|f(m,u(m),v(m),{}^C\mathcal{D}^{\xi}v(m))|dm\Big)ds\Bigg\} \\
+\ & \left|t_2^{-\lambda} - t_1^{-\lambda}\right|\int_1^{t_1} s^{\lambda-1}\mathcal{I}^{\alpha-1}|f(s,u(s),v(s),{}^C\mathcal{D}^{\xi}v(s))|ds \\
+\ & t_2^{-\lambda}\int_{t_1}^{t_2} s^{\lambda-1}\mathcal{I}^{\alpha-1}|f(s,u(s),v(s),{}^C\mathcal{D}^{\xi}v(s))|ds \\
\to\ & 0 \text{ as } t_2 \to t_1,
\end{aligned}$$

independent of (u,v) on account of $|f(t,u(t),v(t),{}^C\mathcal{D}^{\xi}v(t))| \leq L_1$ and $|g(t,u(t),{}^C\mathcal{D}^{\bar{\xi}}u(t),v(t))| \leq L_2$. Also we have

$$\begin{aligned}
& |^C\mathcal{D}^{\bar{\xi}}T_1(u,v)(t_2) - {}^C\mathcal{D}^{\bar{\xi}}T_1(u,v)(t_1)| \\
\leq\ & \frac{1}{\Gamma(2-\bar{\xi})}\left|\int_1^{t_2}\left(\log\frac{t_2}{s}\right)^{-\bar{\xi}}T_1'(u,v)(s)ds - \int_1^{t_1}\left(\log\frac{t_1}{s}\right)^{-\bar{\xi}}T_1'(u,v)(s)ds\right| \\
\leq\ & \frac{1}{\Gamma(1-\bar{\xi})}\left\{\int_1^{t_1}\left|\left(\log\frac{t_2}{s}\right)^{-\bar{\xi}} - \left(\log\frac{t_1}{s}\right)^{-\bar{\xi}}\right|s^{-\lambda-1}ds + \int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{-\bar{\xi}}s^{-(\lambda+1)}ds\right\}\times \\
\times\ & \Bigg\{|K_2A_2 - K_1B_2| + T^{-\lambda}\Big[|b_1||B_2|\int_1^T s^{\lambda-1}\mathcal{I}^{\alpha-1}|f(s,u(s),v(s),{}^C\mathcal{D}^{\xi}v(s))|ds \\
+\ & |b_2||A_2|\int_1^T s^{\lambda-1}\mathcal{I}^{\beta-1}|g(s,u(s),{}^C\mathcal{D}^{\bar{\xi}}u(s),v(s))|ds\Big] \\
+\ & \frac{|a_1||B_2|}{\Gamma(\gamma_1)}\int_1^{\eta_1}\left(\log\frac{\eta_1}{s}\right)^{\gamma_1-1}s^{-(\lambda+1)}\Big(\int_1^s m^{\lambda-1}\mathcal{I}^{\beta-1}|g(m,u(m),{}^C\mathcal{D}^{\bar{\xi}}u(m),v(m))|dm\Big)ds \\
+\ & \frac{|a_2||A_2|}{\Gamma(\gamma_2)}\int_1^{\eta_2}\left(\log\frac{\eta_2}{s}\right)^{\gamma_2-1}s^{-(\lambda+1)}\Big(\int_1^s m^{\lambda-1}\mathcal{I}^{\alpha-1}|f(m,u(m),v(m),{}^C\mathcal{D}^{\xi}v(m))|dm\Big)ds\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{\lambda}{\Gamma(1-\bar{\xi})}\int_{1}^{t_1}\Big|\Big(\log\frac{t_2}{s}\Big)^{-\bar{\xi}}\\
&- \Big(\log\frac{t_1}{s}\Big)^{-\bar{\xi}}\Big|s^{-(\lambda+1)}\Big(\int_1^s m^{\lambda-1}\mathcal{I}^{\alpha-1}|f(m,u(m),v(m),{}^C\mathcal{D}^{\xi}v(m))|dm\Big)ds\\
&+ \frac{\lambda}{\Gamma(1-\bar{\xi})}\int_{t_1}^{t_2}\Big(\log\frac{t_2}{s}\Big)^{-\bar{\xi}}s^{-(\lambda+1)}\Big(\int_1^s m^{\lambda-1}\mathcal{I}^{\alpha-1}|f(m,u(m),v(m),{}^C\mathcal{D}^{\xi}v(m))|dm\Big)ds\\
&+ \frac{1}{\Gamma(1-\bar{\xi})}\int_{1}^{t_1}\Big|\Big(\log\frac{t_2}{s}\Big)^{-\bar{\xi}}-\Big(\log\frac{t_1}{s}\Big)^{-\bar{\xi}}\Big|s^{-1}\mathcal{I}^{\alpha-1}|f(s,u(s),v(s),{}^C\mathcal{D}^{\xi}v(s)|ds\\
&+ \frac{1}{\Gamma(1-\bar{\xi})}\int_{t_1}^{t_2}\Big(\log\frac{t_2}{s}\Big)^{-\bar{\xi}}s^{-1}\mathcal{I}^{\alpha-1}|f(s,u(s),v(s),{}^C\mathcal{D}^{\xi}v(s))|ds\to 0 \text{ as } t_2\to t_1,
\end{aligned}
$$

independent of (u,v). In a similar manner, one can obtain that

$$|T_2(u,v)(t_2)-T_2(u,v)(t_1)|\to 0 \text{ and } |{}^C\mathcal{D}^{\xi}T_2(u,v)(t_2)-{}^C\mathcal{D}^{\xi}T_2(u,v)(t_1)|\to 0$$

as $t_2\to t_1$ independent of (u,v) on account of the boundedness of f and g. Thus the operator T is equicontinuous in view of equicontinuity of T_1 and T_2. Therefore, by Arzela-Ascoli's theorem, it follows that the operator T is compact (completely continuous).

Finally, it will be shown that the set $\varepsilon(T)=\{(u,v)\in X\times Y:(u,v)=\kappa T(u,v)\,;\,0\le\kappa\le 1\}$ is bounded. Let $(u,v)\in\varepsilon(T)$. Then $(u,v)=\kappa T(u,v)$. For any $t\in[1,T]$, we have $u(t)=\kappa T_1(u,v)(t)$, $v(t)=\kappa T_2(u,v)(t)$. Using (H_1) in (19), we get

$$
\begin{aligned}
&|u(t)|\\
\le\;& \frac{\rho}{\lambda|\Delta|}\Bigg\{|K_2A_2-K_1B_2|+T^{-\lambda}\Big[\frac{|b_1||B_2|}{\Gamma(\alpha-1)}\int_1^T s^{\lambda-1}\Big(\int_1^s\Big(\log\frac{s}{m}\Big)^{\alpha-2}\times\\
\times\;& \Big(\mu_0+\mu_1|u(m)|+\mu_2|v(m)|+\mu_3|{}^C\mathcal{D}^{\xi}v(m)|\Big)\frac{dm}{m}\Big)ds\\
+\;& \frac{|b_2||A_2|}{\Gamma(\beta-1)}\int_1^T s^{\lambda-1}\Big(\int_1^s\Big(\log\frac{s}{m}\Big)^{\beta-2}\Big(\tau_0+\tau_1|u(m)|+\tau_2|{}^C\mathcal{D}^{\bar{\xi}}u(m)|+\tau_3|v(m)|\Big)\frac{dm}{m}\Big)ds\Big]\\
+\;& \frac{|a_1||B_2|}{\Gamma(\gamma_1)\Gamma(\beta-1)}\int_1^{\eta_1}\Big(\log\frac{\eta_1}{s}\Big)^{\gamma_1-1}s^{-(\lambda+1)}\times\\
\times\;& \Big(\int_1^s m^{\lambda-1}\Big(\int_1^m\Big(\log\frac{m}{r}\Big)^{\beta-2}\Big[\tau_0+\tau_1|u(r)|+\tau_2|{}^C\mathcal{D}^{\bar{\xi}}u(r)|+\tau_3|v(r)|\Big]\frac{dr}{r}\Big)dm\Big)ds\\
+\;& \frac{|a_2||A_2|}{\Gamma(\gamma_2)\Gamma(\alpha-1)}\int_1^{\eta_2}\Big(\log\frac{\eta_2}{s}\Big)^{\gamma_2-1}s^{-(\lambda+1)}\Big(\int_1^s m^{\lambda-1}\Big(\int_1^m\Big(\log\frac{m}{r}\Big)^{\alpha-2}\times\\
\times\;& \Big(\mu_0+\mu_1|u(r)|+\mu_2|v(r)|+\mu_3|{}^C\mathcal{D}^{\xi}v(r)|\Big)\frac{dr}{r}\Big)dm\Big)ds\Bigg\}\\
+\;& \frac{|t^{-\lambda}|}{\Gamma(\alpha-1)}\int_1^t s^{\lambda-1}\Big(\int_1^s\Big(\log\frac{s}{m}\Big)^{\alpha-2}\Big[\mu_0+\mu_1|u(m)|+\mu_2|v(m)|+\mu_3|{}^C\mathcal{D}^{\xi}v(m)|\Big]\frac{dm}{m}\Big)ds,
\end{aligned}
$$

which, on taking the norm for $t\in[1,T]$, yields

$$
\begin{aligned}
\|u\| \le\;& \Theta_1+\Big(\mu_0+\mu_1\|u\|_X+\max\{\mu_2,\mu_3\}\|v\|_Y\Big)M_1\\
+\;& \Big(\tau_0+\max\{\tau_1,\tau_2\}\|u\|_X+\tau_3\|v\|_Y\Big)M_2.
\end{aligned}
$$

Similarly one can find that

$$\|{}^C\mathcal{D}^{\bar{\xi}}u\| \le \frac{1}{\Gamma(2-\bar{\xi})}\Big\{\overline{\Theta}_1+\Big(\mu_0+\mu_1\|u\|_X+\max\{\mu_2,\mu_3\}\|v\|_Y\Big)\overline{M}_1$$

$$+ \quad \Big(\tau_0 + \max\{\tau_1, \tau_2\}\|u\|_X + \tau_3\|v\|_Y\Big)\overline{M}_2\Big\}.$$

Consequently, we have

$$\begin{aligned} \|u\|_X &= \|u\| + \|{}^C\mathcal{D}^{\bar{\xi}}u\| \\ &\leq \Theta_1 + \frac{\overline{\Theta}_1}{\Gamma(2-\bar{\xi})} + \Big(M_1 + \frac{\overline{M}_1}{\Gamma(2-\bar{\xi})}\Big)\Big(\mu_0 + \mu_1\|u\|_X + \max\{\mu_2, \mu_3\}\|v\|_Y\Big) \\ &+ \Big(M_2 + \frac{\overline{M}_2}{\Gamma(2-\bar{\xi})}\Big)\Big(\tau_0 + \max\{\tau_1, \tau_2\}\|u\|_X + \tau_3\|v\|_Y\Big). \end{aligned} \tag{37}$$

Likewise, we can derive that

$$\begin{aligned} \|v\|_Y &\leq \Theta_2 + \frac{\overline{\Theta}_2}{\Gamma(2-\xi)} + \Big(N_1 + \frac{\overline{N}_1}{\Gamma(1-\xi)}\Big)\Big(\mu_0 + \mu_1\|u\|_X + \max\{\mu_2, \mu_3\}\|v\|_Y\Big) \\ &+ \Big(N_2 + \frac{\overline{N}_2}{\Gamma(2-\xi)}\Big)\Big(\tau_0 + \max\{\tau_1, \tau_2\}\|u\|_X + \tau_3\|v\|_Y\Big). \end{aligned} \tag{38}$$

From (37) and (38), we get

$$\begin{aligned} & \|u\|_X + \|v\|_Y = \Theta_1 + \Theta_2 + \frac{\overline{\Theta}_1}{\Gamma(2-\bar{\xi})} + \frac{\overline{\Theta}_2}{\Gamma(2-\xi)} \\ + \;& \mu_0\Big(M_1 + N_1 + \frac{\overline{M}_1}{\Gamma(2-\bar{\xi})} + \frac{\overline{N}_1}{\Gamma(2-\xi)}\Big) + \tau_0\Big(M_2 + N_2 + \frac{\overline{M}_2}{\Gamma(2-\bar{\xi})} + \frac{\overline{N}_2}{\Gamma(1-\xi)}\Big) \\ + \;& \|u\|_X\Big[\mu_1\Big(M_1 + N_1 + \frac{\overline{M}_1}{\Gamma(2-\bar{\xi})} + \frac{\overline{N}_1}{\Gamma(2-\xi)}\Big) + \max\{\tau_1, \tau_2\}\Big(M_2 + N_2 + \frac{\overline{M}_2}{\Gamma(2-\bar{\xi})} + \frac{\overline{N}_2}{\Gamma(2-\xi)}\Big)\Big] \\ + \;& \|v\|_X\Big[\max\{\mu_2, \mu_3\}\Big(M_1 + N_1 + \frac{\overline{M}_1}{\Gamma(2-\bar{\xi})} + \frac{\overline{N}_1}{\Gamma(2-\xi)}\Big) + \tau_3\Big(M_2 + N_2 + \frac{\overline{M}_2}{\Gamma(2-\bar{\xi})} + \frac{\overline{N}_2}{\Gamma(2-\xi)}\Big)\Big] \\ \leq \;& \varpi_1 + \max\{\varpi_2, \varpi_3\}\|(u, v)\|_{X\times Y}, \end{aligned} \tag{39}$$

which, together with $\|(u, v)\|_{X\times Y} = \|u\|_X + \|v\|_Y$, yields

$$\|(u, v)\|_{X\times Y} \leq \frac{\varpi_1}{1 - \max\{\varpi_2, \varpi_3\}}.$$

This shows that $\varepsilon(T)$ is bounded. Thus, Lemma 4 applies and that T has at least one fixed point. This implies that the boundary value problem (1) has at least one solution on $[1, T]$. The proof is completed. □

Example 1. *Consider the following coupled system of Caputo-Hadamard type sequential fractional differential equations*

$$\begin{aligned} ({}^C\mathcal{D}^{\frac{3}{2}} + \frac{1}{2}\,{}^C\mathcal{D}^{\frac{1}{2}})x(t) &= f(t, x(t), y(t), {}^C\mathcal{D}^{\frac{1}{3}}y(t)), \quad t \in [1, 10], \\ ({}^C\mathcal{D}^{\frac{5}{4}} + \frac{1}{2}\,{}^C\mathcal{D}^{\frac{1}{4}})y(t) &= g(t, x(t), {}^C\mathcal{D}^{\frac{1}{4}}x(t), y(t)), \quad t \in [1, 10], \end{aligned} \tag{40}$$

equipped with nonlocal coupled non-conserved boundary conditions:

$$\begin{aligned} u(1) &= 0, \; -2\mathcal{I}^{\frac{3}{2}}v(2) + u(10) = 3, \\ v(1) &= 0, \; -\mathcal{I}^{\frac{1}{4}}u(3) + 2v(10) = 7. \end{aligned} \tag{41}$$

Here, $\lambda = 1/2, \alpha = 3/2, \beta = 5/4, T = 10, a_1 = -2, a_2 = -1, b_1 = 1, b_2 = 2, K_1 = 3, K_2 = 7, \eta_1 = 2, \eta_2 = 3, \gamma_1 = 3/2, \gamma_2 = 1/4, \xi = 1/3, \bar{\xi} = 1/4,$

$$f(t, x(t), y(t), {}^C\mathcal{D}^{\frac{1}{3}}y(t)) = \frac{1}{2(24+t^2)}\left(3(t-1) + \frac{1}{2}\sin(x(t)) + |y(t)| + |{}^C\mathcal{D}^{\frac{1}{3}}y(t))|\right)$$

and

$$g(t, x(t), {}^C\mathcal{D}^{\frac{1}{4}}x(t), y(t)) = \frac{1}{49t}\left(\frac{1-t}{2} + |x(t)| + \frac{|{}^C\mathcal{D}^{\frac{1}{4}}x(t)|}{1+|{}^C\mathcal{D}^{\frac{1}{4}}x(t)|} + \sin(y(t))\right).$$

Clearly, the functions f *and* g *satisfy the condition* (H_1) *with* $\mu_0 = \frac{27}{50}, \mu_1 = \frac{1}{100}, \mu_2 = \mu_3 = \frac{1}{50}, \tau_0 = \frac{9}{98}, \tau_1 = \tau_2 = \tau_3 = \frac{1}{49}$. *Using the given data, we find that* $A_1 \approx 1.3675, |A_2| \approx 0.2186,$ $|B_1| \approx 0.7865,$ $B_2 \approx 2.7351,$ $|\Delta| \approx 3.5684,$ $\rho \approx 0.6838,$ $\Theta_1 \approx 2.5581,$ $\overline{\Theta}_1 \approx 3.49653,$ $\Theta_2 \approx 2.76436,$ $\overline{\Theta}_2 \approx 3.52477,$ $M_1 \approx 5.4654,$ $M_2 \approx 0.9275,$ $\overline{M}_1 \approx 9.5348,$ $\overline{M}_2 \approx 1.2677,$ $N_1 \approx 1.8178,$ $N_2 \approx 5.2756,$ $\overline{N}_1 \approx 1.6640,$ $\overline{N}_2 \approx 7.9915,$ $\varpi_1 \approx 25.0711,$ $\varpi_2 \approx 0.530375, \varpi_3 \approx 0.725385$. *With* $max\{\varpi_2, \varpi_3\} < 1$, *all the conditions of Theorem 2 are satisfied. Therefore, the problem (40) and (41) has a solution on on* $[1, 10]$.

The next result deals with the uniqueness of solutions for the problem (1) and relies on Banach contraction mapping principle. For computational convenience, we introduce the notations:

$$\begin{aligned} \Phi_1 &= \Theta_1 + r_1 M_1 + r_2 M_2,\ \Psi_1 = \ell M_1 + \ell_1 M_2,\ \Phi_2 = \Theta_2 + r_1 N_1 + r_2 N_2,\ \Psi_2 = \ell N_1 + \ell_1 N_2, \\ \overline{\Phi}_1 &= \overline{\Theta}_1 + r_1 \overline{M}_1 + r_2 \overline{M}_2,\ \overline{\Psi}_1 = \ell \overline{M}_1 + \ell_1 \overline{M}_2,\ \overline{\Phi}_2 = \overline{\Theta}_2 + r_1 \overline{N}_1 + r_2 \overline{N}_2,\ \overline{\Psi}_2 = \ell \overline{N}_1 + \ell_1 \overline{N}_2, \\ r_1 &= \sup_{t\in[1,T]} f(t,0,0,0) < \infty,\ r_2 = \sup_{t\in[1,T]} g(t,0,0,0) < \infty. \end{aligned} \tag{42}$$

Theorem 3. *Assume that* (H_2) *holds. Then the boundary value problem (1) has a unique solution on* $[1, T]$, *provided that*

$$\Psi_1 + \frac{\overline{\Psi}_1}{\Gamma(2-\bar{\xi})} < \frac{1}{2} \quad and \quad \Psi_2 + \frac{\overline{\Psi}_2}{\Gamma(2-\xi)} < \frac{1}{2}, \tag{43}$$

where Ψ_i *and* $\overline{\Psi}_i$ $(i = 1, 2)$ *are given by (42).*

Proof. Let us fix

$$r \geq \max\left\{\frac{\Phi_1 + \frac{\overline{\Phi}_1}{\Gamma(2-\bar{\xi})}}{\frac{1}{2} - (\Psi_1 + \frac{\overline{\Psi}_1}{\Gamma(2-\bar{\xi})})}, \frac{\Phi_2 + \frac{\overline{\Phi}_2}{\Gamma(2-\xi)}}{\frac{1}{2} - (\Psi_2 + \frac{\overline{\Psi}_2}{\Gamma(2-\xi)})}\right\},$$

where Φ_i, $\overline{\Phi}_i$, and Ψ_i, $\overline{\Psi}_i$ $(i = 1, 2)$ are given by (42). Then we show that $TB_r \subset B_r$, where

$$B_r = \{(u, v) \in X \times X : \|(u, v)\|_{X\times Y} \leq r\}.$$

For $(u, v) \in B_r$, we have

$$\begin{aligned} |f(t, u(t), v(t), {}^C\mathcal{D}^{\xi}v(t))| &\leq |f(t, u(t), v(t), {}^C\mathcal{D}^{\xi}v(t)) - f(t, 0, 0, 0)| + |f(t, 0, 0, 0)| \\ &\leq \ell[|u(t)| + |v(t)| + |{}^C\mathcal{D}^{\xi}v(t)|] + r_1 \\ &\leq \ell[\|u\|_X + \|v\|_Y] + r_1 \leq \ell\|(u, v)\|_{X\times Y} + r_1 \leq \ell r + r_1. \end{aligned}$$

Similarly, we can find that

$$|g(t, u(t), {}^C\mathcal{D}^{\bar{\xi}}u(t), v(t))| \leq \ell_1 r + r_2.$$

Then

$$|T_1(u, v)(t)| \leq \Theta_1 + r_1 M_1 + r_2 M_2 + (\ell M_1 + \ell_1 M_2)r \leq \Phi_1 + \Psi_1 r,$$

and

$$|^C\mathcal{D}^{\bar{\xi}}T_1(u,v)(t)| \leq \frac{1}{\Gamma(2-\bar{\xi})}\Big[\overline{\Theta}_1 + r_1\overline{M}_1 + r_2\overline{M}_2 + (\ell\overline{M}_1 + \ell_1\overline{M}_2)r\Big] \leq \frac{1}{\Gamma(2-\bar{\xi})}\Big[\overline{\Phi}_1 + \overline{\Psi}_1 r\Big].$$

Therefore,

$$\|T_1(u,v)\|_X = \|T_1(u,v)\| + \|^C\mathcal{D}^{\bar{\xi}}T_1(u,v)\| \leq \Phi_1 + \frac{\overline{\Phi}_1}{\Gamma(2-\bar{\xi})} + \Big[\Psi_1 + \frac{\overline{\Psi}_1}{\Gamma(2-\bar{\xi})}\Big]r \leq \frac{r}{2}. \tag{44}$$

In similar manner, we obtain

$$|T_2(u,v)(t)| \leq \Phi_2 + \Psi_2 r, \qquad |^C\mathcal{D}^{\xi}T_2(u,v)(t)| \leq \frac{1}{\Gamma(2-\xi)}\Big[\overline{\Phi}_2 + \overline{\Psi}_2 r\Big].$$

In consequence, we get

$$\|T_2(u,v)\|_Y = \|T_2(u,v)\| + \|^C\mathcal{D}^{\xi}T_2(u,v)\| \leq \Phi_2 + \frac{\overline{\Phi}_2}{\Gamma(2-\xi)} + \Big[\Psi_2 + \frac{\overline{\Psi}_2}{\Gamma(2-\xi)}\Big]r \leq \frac{r}{2}. \tag{45}$$

Thus, it follows from (44) and (45) that

$$\|T(u,v)\|_{X\times Y} = \|T_1(u,v)\|_X + \|T_2(u,v)\|_X \leq r,$$

which implies that $TB_r \subset B_r$.

Next we show that the operator T is a contraction. For that, let $u_i,\ v_i \in B_r$ $(i = 1,2)$. Then, for each $t \in [1,T]$, we have

$$\begin{aligned}
& |T_1(u_1,v_1)(t) - T_1(u_2,v_2)(t)| \\
\leq\ & \frac{|1-t^{-\lambda}|}{\lambda|\Delta|}\Bigg\{T^{-\lambda}\Big[\frac{|b_1|B_2|}{\Gamma(\alpha-1)}\int_1^T s^{\lambda-1}\Big(\int_1^s\Big(\log\frac{s}{m}\Big)^{\alpha-2} \times \\
\times\ & \Big|f(m,u_1(m),v_1(m),^C\mathcal{D}^{\xi}v_1(m)) - f(m,u_2(m),v_2(m),^C\mathcal{D}^{\xi}v_2(m))\Big|\frac{dm}{m}\Big)ds \\
+\ & \frac{|b_2A_2|}{\Gamma(\beta-1)}\int_1^T s^{\lambda-1}\Big(\int_1^s\Big(\log\frac{s}{m}\Big)^{\beta-2} \times \\
\times\ & |g(m,u_1(m),^C\mathcal{D}^{\bar{\xi}}u_1(m),v_1(m)) - g(m,u_2(m),^C\mathcal{D}^{\bar{\xi}}u_2(m),v_2(m))|\frac{dm}{m}\Big)ds\Big] \\
+\ & \frac{|a_1B_2|}{\Gamma(\gamma_1)\Gamma(\beta-1)}\int_1^{\eta_1}\Big(\log\frac{\eta_1}{s}\Big)^{\gamma_1-1}s^{-(\lambda+1)}\Big(\int_1^s m^{\lambda-1}\Big(\int_1^m\Big(\log\frac{m}{r}\Big)^{\beta-2} \times \\
\times\ & |g(r,u_1(r),^C\mathcal{D}^{\bar{\xi}}u_1(r),v_1(r)) - g(r,u_2(r),^C\mathcal{D}^{\bar{\xi}}u_2(r),v_2(r))|\frac{dr}{r}\Big)dm\Big)ds \\
+\ & \frac{|a_2A_2|}{\Gamma(\gamma_2)\Gamma(\alpha-1)}\int_1^{\eta_2}\Big(\log\frac{\eta_2}{s}\Big)^{\gamma_2-1}s^{-(\lambda+1)}\Big(\int_1^s m^{\lambda-1}\Big(\int_1^m\Big(\log\frac{m}{r}\Big)^{\alpha-2} \times \\
\times\ & \Big|f(r,u_1(r),v_1(r),^C\mathcal{D}^{\xi}v_1(r)) - f(r,u_2(r),v_2(r),^C\mathcal{D}^{\xi}v_2(r))\Big|\frac{dr}{r}\Big)dm\Big)ds\Bigg\} \\
+\ & \frac{t^{-\lambda}}{\Gamma(\alpha-1)}\int_1^t s^{\lambda-1}\Big(\int_1^s\Big(\log\frac{s}{m}\Big)^{\alpha-2} \times \\
\times\ & \Big|f(m,u_1(m),v_1(m),^C\mathcal{D}^{\xi}v_1(m)) - f(m,u_2(m),v_2(m),^C\mathcal{D}^{\xi}v_2(m))\Big|\frac{dm}{m}\Big)ds \\
\leq\ & M_1\ell\Big[\|u_1-u_2\| + \|v_1-v_2\| + \|^C\mathcal{D}^{\xi}v_1 - {}^C\mathcal{D}^{\xi}v_2\|\Big] \\
& +M_2\ell_1\Big[\|u_1-u_2\| + \|^C\mathcal{D}^{\bar{\xi}}u_1 - {}^C\mathcal{D}^{\bar{\xi}}u_2\| + \|v_1-v_2\|\Big]
\end{aligned}$$

$$\leq \quad \Psi_1\left[\|u_1-u_2\|_X+\|v_1-v_2\|_Y\right].$$

Also we have

$$\begin{aligned}|{}^C\mathcal{D}^{\bar{\xi}}T_1(u_1,v_1)(t)-{}^C\mathcal{D}^{\bar{\xi}}T_1(u_2,v_2)(t)| &\leq \frac{1}{\Gamma(1-\bar{\xi})}\int_1^t\left(\log\frac{t}{s}\right)^{-\bar{\xi}}\left|T_1'(u_1,v_1)(s)-T_1'(u_2,v_2)(s)\right|ds\\ &\leq \frac{\overline{\Psi}_1}{\Gamma(2-\bar{\xi})}[\|u_1-u_2\|_X+\|v_1-v_2\|_Y].\end{aligned}$$

From the foregoing inequalities, we get

$$\begin{aligned}\|T_1(u_1,v_1)-T_1(u_2,v_2)\|_X &= \|T_1(u_1,v_1)-T_1(u_2,v_2)\|+\|{}^C\mathcal{D}^{\bar{\xi}}T_1(u_1,v_1)-{}^C\mathcal{D}^{\bar{\xi}}T_1(u_2,v_2)\|\\ &\leq \left[\Psi_1+\frac{\overline{\Psi}_1}{\Gamma(1-\bar{\xi})}\right][\|u_1-u_2\|_X+\|v_1-v_2\|_Y]. \qquad (46)\end{aligned}$$

Similarly, we can find that

$$\|T_2(u_1,v_1)-T_2(u_2,v_2)\|_Y \leq \left[\Psi_2+\frac{\overline{\Psi}_2}{\Gamma(2-\xi)}\right][\|u_1-u_2\|_X+\|v_1-v_2\|_Y] \qquad (47)$$

Consequently, it follows from (46) and (47) that

$$\begin{aligned}\|T(u_1,v_1)-T(u_2,v_2)\|_{X\times Y} &= \|T_1(u_1,v_1)-T_1(u_2,v_2)\|_X+\|T_2(u_1,v_1)-T_2(u_2,v_2)\|_X\\ &\leq \left[\Psi_1+\Psi_2+\frac{\overline{\Psi}_1}{\Gamma(2-\bar{\xi})}+\frac{\overline{\Psi}_2}{\Gamma(2-\xi)}\right][\|u_1-u_2\|_X+\|v_1-v_2\|_Y].\end{aligned}$$

This shows that T is a contraction by (43). Hence, by Banach fixed point theorem, the operator T has a unique fixed point which corresponds to a unique solution of problem (1). This completes the proof. □

Example 2. *Consider the following coupled system of fractional differential equations*

$$\begin{aligned}({}^C\mathcal{D}^{\frac{3}{2}}+\frac{1}{2}\,{}^C\mathcal{D}^{\frac{1}{2}})x(t) &= \frac{1}{2(24+t^2)}\left(3+\sin(x(t))+|y(t)|+\tan^{-1}({}^C\mathcal{D}^{\frac{1}{3}}y(t))\right), \quad t\in[1,10]\\ ({}^C\mathcal{D}^{\frac{5}{4}}+\frac{1}{2}\,{}^C\mathcal{D}^{\frac{1}{4}})y(t) &= \frac{1}{49t}\left(\frac{t}{2}+|x(t)|+\frac{|{}^C\mathcal{D}^{\frac{1}{4}}x(t)|}{1+|{}^C\mathcal{D}^{\frac{1}{4}}x(t)|}+\sin(y(t))\right), \qquad (48)\end{aligned}$$

supplemented with nonlocal coupled non-conserved boundary conditions:

$$\begin{aligned}u(1) &= 0, \ -2\mathcal{I}^{\frac{3}{2}}v(2)+u(10)=3,\\ v(1) &= 0, \ -\mathcal{I}^{\frac{1}{4}}u(3)+2v(10)=7. \qquad (49)\end{aligned}$$

Here, $\lambda=1/2,\alpha=3/2,\beta=5/4,T=10,a_1=-2,a_2=-1,b_1=1,b_2=2,K_1=3,K_2=7,\eta_1=2,\eta_2=3,\gamma_1=3/2,\gamma_2=1/4,\xi=1/3,\bar{\xi}=1/4,$

$$f(t,x(t),y(t),{}^C\mathcal{D}^{\xi}y(t))=\frac{1}{2(24+t^2)}(3+\sin(x(t))+|y(t)|+\tan^{-1}({}^C\mathcal{D}^{\frac{1}{3}}y(t)))$$

and

$$g(t,x(t),{}^C\mathcal{D}^{\bar{\xi}}x(t),y(t))=\frac{1}{49t}\left(\frac{t}{2}+|x(t)|+\frac{|{}^C\mathcal{D}^{\frac{1}{4}}x(t)|}{1+|{}^C\mathcal{D}^{\frac{1}{4}}x(t)|}+\sin(y(t))\right).$$

From the inequalities:

$$|f(t,x_1(t),y_1(t),{}^C\mathcal{D}^{\frac{1}{3}}y_1(t))-f(t,x_2(t),y_2(t),{}^C\mathcal{D}^{\frac{1}{3}}y_2(t))|$$

$$\leq \frac{1}{50}\left(|x_1(t)-x_2(t)|+|y_1(t)-y_2(t)|+|{}^C\mathcal{D}^{\frac{1}{3}}y_1(t)-{}^C\mathcal{D}^{\frac{1}{3}}y_2(t)|\right),$$

$$|g(t,x_1(t),{}^C\mathcal{D}^{\frac{1}{4}}x_1(t),y_1(t))-g(t,x_2(t),{}^C\mathcal{D}^{\frac{1}{4}}x_2(t),y_2(t))$$
$$\leq \frac{1}{49}\left(|x_1(t)-x_2(t)|+|{}^C\mathcal{D}^{\frac{1}{4}}x_1(t)-{}^C\mathcal{D}^{\frac{1}{4}}x_2(t)|+|y_1(t)-y_2(t)|\right),$$

we have $l=\frac{1}{50}$ *and* $l_1=\frac{1}{49}$. *Using the given data, we find that* $A_1\approx 1.3675, |A_2|\approx 0.2186,\ |B_1|\approx 0.7865,\ B_2\approx 2.7351,\ |\Delta|\approx 3.5684,\ \rho\approx 0.6838,\ M_1\approx 5.4654,\ M_2\approx 0.9275,\ \Psi_1\approx 0.1282,\ \overline{M}_1\approx 9.5348,\ \overline{M}_2\approx 1.2677, \overline{\Psi}_1\approx 0.2166,\ N_1\approx 1.8178,\ N_2\approx 5.2756,\ \Psi_2\approx 0.1439,\ \overline{N}_1\approx 1.6640,\ \overline{N}_2\approx 7.9915, \overline{\Psi}_2\approx 0.1964$. *Further*

$$\Psi_1+\frac{\overline{\Psi}_1}{\Gamma(7/4)}\approx 0.3639<0.5,\ \Psi_2+\frac{\overline{\Psi}_2}{\Gamma(5/3)}\approx 0.3615<0.5.$$

Thus all the conditions of Theorem 3 are satisfied. In consequence, by Theorem 3, there exists a unique solution for the problem (48) and (49) on $[1,10]$.

4. Conclusions

We have developed the existence theory for a nonlocal integral boundary value problem of coupled sequential fractional differential equations involving Caputo-Hadamard fractional derivatives and Hadamard fractional integrals. Several results follow as special cases by fixing the values of the parameters involved in the problem. For example, by taking $a_i=-1, b_i=1, K_1=0=K_2$ and $T=e$, our results correspond to the ones associated with coupled strip boundary conditions of the form:

$$u(1)=0,\ \ u(T)=\mathcal{I}^{\gamma_1}v(\eta_1),\ \gamma_1>0,\ 1<\eta_1<e,$$
$$v(1)=0,\ \ v(T)=\mathcal{I}^{\gamma_2}u(\eta_2),\ \gamma_2>0,\ 1<\eta_2<e.$$

If we take $a_1=0=a_2$ in the results of this paper, we obtain the ones for a coupled system of Caputo-Hadamard fractional differential equations and uncoupled Dirichlet boundary conditions. We emphasize that the main results as well as the special cases presented in this paper are new and enrich the existing literature on the topic.

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