fractal and fractional

Article

Fractional Derivatives and Dynamical Systems in Material Instability

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Abstract: Loss of stability is studied extensively in nonlinear investigations, and classified as generic bifurcations. It requires regularity, being connected with non-locality. Such behavior comes from gradient terms in constitutive equations. Most fractional derivatives are non-local, thus by using them in defining strain, a non-local strain appears. In such a way, various versions of non-localities are obtained by using various types of fractional derivatives. The study aims for constitutive modeling via instability phenomena, that is, by observing the way of loss of stability of material, we can be informed about the proper form of its mathematical model.

Keywords: fractional derivatives; dynamical system; bifurcation

1. Introduction

There are materials where tests justify models with fractional order derivatives. Moreover, all fractional derivatives are non-local. Thus, by using strain calculated by fractional derivation of the displacement field, a non-local quantity appears instead of the conventional (local) strain. Fractional calculus appears to be a powerful tool to deal with forms of non-localities. When the term non-locality is used in its original meaning, the value of some quantity in an internal point of the body is determined by the values of other quantities in a whole region around that location.

In material instability problems, like shear band or neck formation [1], the post-critical behavior plays an essential role. Especially for numerical analysis [2], the form and thickness of instability zones should be mesh-independent for a proper material model. Such property implies a generic method of the loss of stability in the sense of dynamical systems theory [3,4]. Thus, the complete mathematical description of the solid continuum (a set of differential equations) should undergo a generic bifurcation. While that model is more or less fixed, the only element of the set of basic equations to be varied is the constitutive equation. Consequently, one should use a constitutive equation, which, by adding the equations of motion and the kinematic equation, forms a system, having generic bifurcation at loss of stability. Conventionally, such constitutive equations are formed by gradient regularization [5], which can also be interpreted as addition of non-locality.

Non-locality is an essential part of continuum physics from the early period of rational mechanics. Two forms are distinguished, the weak and the strong non-localities [6]. The earliest distinction was introduced by Maugin in 1977 at a conference in Warsaw [7]. Weak non-locality concepts that use gradient models could be better referred to as gradient theories of the n-th order [6]. At strong non-locality, constitutive equations become integral expressions over space, perhaps with a more or less rapid attenuation with the distance of the spatial kernel [6]. In a continuum physicist's point of view, the use of fractional derivatives fits well into the generalization of strong non-locality theory [8].

Weak non-localities can be introduced like Aifantis [5], by adding gradient term(s) to constitutive equations

$$\sigma = C\varepsilon - l^2\nabla^2\varepsilon. \tag{1}$$

Term $\nabla^2\varepsilon$ is used in Equation (1) as in [5], to express second "gradient dependence" of material [8]. (Obviously, such an operator is the Laplace operator, $\nabla^2 \equiv \Delta$). Gradient effects have several physical interpretation including internal lengths or micro-structural effect theories. Early studies from the 1960s use the notion of polar bodies [9] and couple stress effect and then a gradient of deformation gradient appears, which can be interpreted as a constitutive equation containing a first order strain gradient.

In case of strong non-locality, local stress is determined by the strain in a neighborhood [10]. Similarly to [11], assume that such interval in a uniaxial case is $[a, b]$. Then, integration is used to encounter that with attenuation function $c(\xi)$, which reads

$$\sigma(x) = C\varepsilon(x) + \int_a^b c(x-\xi)\,\varepsilon(\xi)\,d\xi,$$

by transforming into the displacement field and assuming attenuation function as

$$c(\xi) = \frac{1}{2\Gamma(1-\alpha)\,|\xi|^{\alpha}}$$

with $0 < \alpha < 1$,

$$\sigma = C\frac{\partial u(x)}{\partial x} + \frac{1}{2\Gamma(1-\alpha)}\int_a^b \frac{1}{|x-\xi|^{\alpha}}\frac{\partial u(\xi)}{\partial \xi}d\xi. \tag{2}$$

In [11], one could see how Formula (2) can easily be generalized to Caputo's fractional derivative. It shows why non-locality is often modeled by using non-local derivatives [12] and then constitutive equations are in the forms of fractional differential equations [13]. This work deals with the consequences of fractional material models in material instability investigations.

The structure of this paper is as follows. Firstly, a short overview of basic fractional calculus is presented, including Euler's gamma function. A wide range of detailed monographs of that topic is available like [14] or [13] and the reference in them. Secondly, dynamical systems theory is introduced and applied to material instability problems to explain the importance of a property called generic in bifurcation analysis. The most important part is the following section, in which conditions of generic bifurcation behavior are studied for various types of constitutive equations using various fractional derivatives.

2. Elements of Fractional Calculus

2.1. Fractional Integral Operators and Fractional Derivatives

Fractional derivatives are constructed from a fractional order generalization of Cauchy's integra formula. In such a way, Riemann–Liouville's fractional integral operator

$${}_aI_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x f(\xi)\,(x-\xi)^{\alpha-1}\,d\xi \tag{3}$$

is defined, where $0 < \alpha < 1$ and Euler's gamma function $\Gamma(\alpha)$ is an integral of the two most important functions of analysis. These are power function and exponential functions and then

$$\Gamma(\alpha) = \int_0^{\infty} \xi^{\alpha-1}e^{-\xi}d\xi. \tag{4}$$

It can also be interpreted as a generalization of factorial to non-integers because, for integer $\alpha = n$, definition (4) results in being factorial,

$$\Gamma(n) = (n-1)!\,. \tag{5}$$

Integration in Formula (3) goes from left to right, $(a < x)$; for that reason, it is called the left integral operator. By taking a derivative of the left Riemann–Liouville integral operator, a fractional derivative

$$ {}_aD_x^\alpha f(x) := \frac{d}{dx}\,{}_aI_x^{1-\alpha} f(x) \tag{6}$$

can be defined. The usual form of the left Riemann–Liouville derivative is obtained from Formulas (3) and (6)

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x f(\xi)(x-\xi)^{-\alpha}\, d\xi, \tag{7}$$

which is a non-local fractional derivative for interval $[a, x]$.

By varying the interval of integral operator in Formula (7) to $[x, b]$, the so-called right Riemann–Liouville derivative can be defined as

$$ {}_xD_b^\alpha f(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b f(\xi)(\xi-x)^{-\alpha}\, d\xi. $$

By changing operators of derivation and integration in Formula (6), left Caputo's derivative is defined

$$ {}_a^CD_x^\alpha f(x) := {}_aI_x^{1-\alpha} \left(\frac{d}{d\xi} f\right)(x) $$

and then its integro-differential operator formula reads

$$ {}_a^CD_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{df(\xi)}{d\xi}(x-\xi)^{-\alpha}\, d\xi \tag{8}$$

being the left (non-local) Caputo derivative for interval $[a, x]$. Hence, the right Caputo derivative for interval $[x, b]$ reads

$$ {}_x^CD_b^\alpha f(x) = -\frac{1}{\Gamma(1-\alpha)} \int_x^b \frac{df(\xi)}{d\xi}(\xi-x)^{-\alpha}\, d\xi. \tag{9}$$

2.2. Symmetric Fractional Derivatives

For most applications in non-local mechanics, fractional derivative

$$ \frac{\partial^\alpha}{\partial x^\alpha} $$

should be a symmetric derivative. For example, such derivative can be constructed from left and right Riemann–Liouville

$$ \frac{\partial^\alpha f(x)}{\partial x^\alpha} = \frac{1}{2}\left({}_aD_x^\alpha f(x) - {}_xD_b^\alpha f(x)\right) \tag{10}$$

or Caputo derivatives

$$ \frac{\partial^\alpha f(x)}{\partial x^\alpha} = \frac{1}{2}\left({}_a^CD_x^\alpha f(x) - {}_x^CD_b^\alpha f(x)\right) \tag{11}$$

or similarly from other types of left and right fractional derivatives. As a famous example in fractional mechanics, a symmetric derivative [15] is presented. Let us start from equivalent forms of the left and right Caputo derivatives like [16]. From (8),

$$ {}_a^CD_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_a^x (f(\xi) - f(a))(x-\xi)^{-\alpha}\, d\xi $$

is obtained and (9) is equivalent to

$$ {}_x^CD_b^\alpha f(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^b (f(\xi) - f(b))(\xi-x)^{-\alpha}\, d\xi. $$

Then, take "asymptotic cases" ($a \to \infty$, $b \to \infty$) and combine them into a symmetric form of Liouville–Caputo derivatives, like

$$^{C}D^{\alpha}_{+}f\left(x\right) - {}^{C}D^{\alpha}_{-}f\left(x\right),$$

which is closely related to the Riesz potential [15].

2.3. On a Two-Sided Fractional Derivative as Infinite Series

There is a more efficient way to construct symmetric fractional derivatives. Its origin is at Grünwald–Letnikov formulation. For repeated n^{th} backward differentiation [17]

$$D^{n}_{x}f(x) = \lim_{h\to 0}\frac{\nabla^{n}_{h}f(x)}{h^{n}},$$

where

$$\nabla^{n}_{h}f(x) = \sum_{k=0}^{n}(-1)^{k}\begin{pmatrix} n \\ k \end{pmatrix} f\left(x-kh\right).$$

Then, instead of integer (n), take fractional order (α) difference as an infinite series

$$\nabla^{\alpha}_{h}f(x) = \sum_{k=0}^{\infty}(-1)^{k}\begin{pmatrix} \alpha \\ k \end{pmatrix} f\left(x-kh\right),$$

where

$$\begin{pmatrix} \alpha \\ k \end{pmatrix} = \frac{(-1)^{k+1}\,\Gamma\left(k-\alpha\right)}{\Gamma\left(1-\alpha\right)\Gamma\left(k+1\right)}.$$

Now, for $h > 0$, the left- and right-sided Grünwald–Letnikov derivatives are

$$^{GL}D^{\alpha}_{x+} = \lim_{h\to 0}\frac{\nabla^{\alpha}_{h}f(x)}{h^{\alpha}}, \quad {}^{GL}D^{\alpha}_{x-} = \lim_{h\to 0}\frac{\nabla^{\alpha}_{-h}f(x)}{h^{\alpha}},$$

and a symmetric derivative can be constructed by adding them together. In a similar way, so-called centered derivatives could be defined and even integral forms are available [18]. As a generalized version, in this study, the two-sided fractional derivatives (TSFD) are used as it was introduced by Ortigueira [19]

$$D^{\beta}_{\theta}f\left(x\right) := \lim_{h\to 0^{+}} h^{-\beta}\sum_{n=-\infty}^{+\infty}\frac{(-1)^{n}\,\Gamma\left(\beta+1\right)}{\Gamma\left(\frac{\beta+\theta}{2}-n+1\right)\Gamma\left(\frac{\beta-\theta}{2}+n+1\right)}f\left(x-nh\right), \tag{12}$$

where $\beta > -1$. The solution of the eigenvalue – eigenvector problem for differential operator (12) is

$$D^{\beta}_{\theta}e^{i\kappa x} = \left|\kappa\right|^{\beta}e^{i\frac{\pi}{2}\theta sgn(\kappa)e^{i\kappa x}} \tag{13}$$

and this fact will be used extensively in our stability investigation.

3. A Dynamical Systems Approach of Material Instability

The classical description of continuum mechanics consists of the set of basic equations of continua. In this section, a dynamical system is constructed from these equations and material instability of a state of material is studied as an instability of a solution of that dynamical system.

In the simplest possible case, a uniaxial problem is studied with small deformations. Then, basic equations are

- Cauchy's equations of motion
$$\rho \dot{v} = \sigma \nabla, \tag{14}$$
- the kinematic equation in rate form
$$\dot{\varepsilon} = \frac{1}{2} \left(v\nabla + \nabla v \right), \tag{15}$$
- and the constitutive equation
$$F \left(\varepsilon, \sigma, \dot{\varepsilon}, \dot{\sigma}, \ldots \right) = 0. \tag{16}$$

Note that constitutive Equation (16) is used here in its general form, a function of strain field ε, stress field σ, and various derivatives of them. Moreover, in addition to the variables shown in Equation (16), further dependence may and will appear later in this paper. Strain gradient and fractional strain gradient may also be included into the constitutive equation.

From the basic Equations (14)–(16), a dynamical system can be defined
$$\frac{d}{dt} \begin{bmatrix} v \\ \varepsilon \\ \sigma \end{bmatrix} = \mathbf{F} \left(v, \varepsilon, \sigma \right), \tag{17}$$
where
$$\mathbf{F} \left(v, \varepsilon, \sigma \right) = \begin{bmatrix} \frac{1}{\rho} \sigma \nabla \\ \frac{1}{2} \left(v\nabla + \nabla v \right) \\ -\left(\frac{\partial F}{\partial \dot{\sigma}} \right)^{-1} \left(\left(\frac{\partial F}{\partial \varepsilon} \right) \varepsilon + \left(\frac{\partial F}{\partial \sigma} \right) \sigma + \left(\frac{\partial F}{\partial \dot{\varepsilon}} \right) \frac{1}{2} \left(v\nabla + \nabla v \right) \right) + \ldots \end{bmatrix}. \tag{18}$$

By studying stability and bifurcation, let us assume that
$$\mathbf{y_0} = \left(v_0, \varepsilon_0, \sigma_0 \right)$$
is an equilibrium of Equation (17),
$$\mathbf{F} \left(v_0, \varepsilon_0, \sigma_0 \right) = 0.$$

Then, by introducing homogeneous perturbations,
$$\begin{aligned} v &= v_0 + \widetilde{v}, \\ \varepsilon &= \varepsilon_0 + \widetilde{\varepsilon}, \\ \sigma &= \sigma_0 + \widetilde{\sigma} \end{aligned}$$
and
$$\mathbf{y} = \mathbf{y_0} + \widetilde{\mathbf{y}},$$
the linearization of Equations (17) and (18) is satisfied for the perturbations. In that sense, homogeneous perturbation for a uniaxial continuum of length L means that
$$\widetilde{v}(-\frac{L}{2}) = 0, \quad \widetilde{v}(\frac{L}{2}) = 0, \tag{19}$$
$$\widetilde{\varepsilon}(-\frac{L}{2}) = 0, \quad \widetilde{\varepsilon}(\frac{L}{2}) = 0, \tag{20}$$
$$\widetilde{\sigma}(-\frac{L}{2}) = 0, \quad \widetilde{\sigma}(\frac{L}{2}) = 0. \tag{21}$$

In mechanics, the case of adiabatic localization [20] may justify the case of infinite length [15], ($L \to \infty$), especially for material instability problems. Such case assumes that the effect of the boundaries can be eliminated sufficiently far from both ends of a rod, which correlates the experimental observations. Shear bands or necking regions appear generally "in the middle" of a specimen, sufficiently far from both ends.

As it is presented in [21], the eigenvalues and eigenvectors of linearized operator

$$\mathbf{LF} := \left(\frac{d}{d\mathbf{y}} \mathbf{F} \right) \Big|_{\mathbf{y}=\mathbf{y_0}} \tag{22}$$

play the key role in stability and bifurcation analysis. For this reason, the characteristic equation of Formula (22)

$$(\mathbf{LF} - \lambda \mathbf{I}) \, \widetilde{y} = 0 \tag{23}$$

should be solved and stability conditions are formed for its solutions λ_i in a usual way:

- state $\mathbf{y_0}$ of the material is stable, if
$$Re\,(\lambda_i) < 0$$
for all solutions of Equation (23),
- loss-of-stability happens, when at least for one i,
$$Re\,(\lambda_i) = 0.$$

Two ways of instabilities can be distinguished,

- the static bifurcation $(\lambda = 0)$, that is,
$$(\mathbf{LF}) \, \widetilde{y}_c = 0, \tag{24}$$
and
- the dynamic bifurcation $(Re\,(\lambda) = 0, Im\,(\lambda) \neq 0)$,
$$(\mathbf{LF}) \, \widetilde{y}_c = i\omega \widetilde{y}_c, \tag{25}$$

for non-trivial $\widetilde{y}_c$.

Nonlinear analysis can be performed, if operator defined by Formula (22) has non-trivial critical eigenspace, thus Equation (23) should have a non-trivial solution $\widetilde{y}_c$ at $Re\,(\lambda) = 0$. Such case is referred to as generic bifurcation and nonlinearity should be projected to this eigenspace resulting in bifurcation equations [21]. For this reason, the existence and exact determination of $\widetilde{y}_c$ are of critical importance. The following section deals with that problem in the case of various constitutive equations.

Characteristic Equation (23) is a partial differential equation, while Formula (22) is a differential operator. Its solution, or its solvability is far from obvious even in the case of integer order derivatives. A kind of simplifying approximation is to restrict the problem to some proper functions. Such functions are harmonics $e^{i\kappa x}$ in the so-called periodic perturbation technique [21]. When homogeneous boundary conditions given by Formulas (19)–(21) are taken into consideration, the perturbation frequencies are

$$\kappa = \frac{n\pi}{L}, \quad n = 1, 2, \ldots \, .$$

4. Bifurcations for Constitutive Equations with Fractional Derivatives

4.1. Constitutive Equation with Non-Local Strain

Firstly, strong non-locality is assumed. In such case, local stress σ is determined by the strain ε in a neighborhood [10]. A generalization of such approach is the use of a non-local strain, where, instead of local strain

$$\varepsilon = \frac{1}{2} \left(u \nabla + \nabla u \right),$$

a non-local one is used, defined by a symmetric fractional derivative

$$\varepsilon_\alpha = \frac{\partial^\alpha u}{\partial |x|^\alpha} \tag{26}$$

(see Formulas (10)–(12)). In Formula (26), absolute value points out the symmetric nature of derivative. Then, a static bifurcation is studied at non-local strain as in [15].

Let the constitutive equation be the so-called Malvern–Cristescu equation

$$\sigma + D\dot{\sigma} = E\varepsilon_\alpha + H\dot{\varepsilon}_\alpha \tag{27}$$

and let us extend the scope of investigation to dynamic bifurcation too. Now, characteristic Equation (23) has the form

$$-D\rho\lambda^3 y_1(x) - \rho\lambda^2 y_1(x) + H\frac{\partial}{\partial x}\frac{\partial^\alpha}{\partial |x|^\alpha}\lambda y_1(x) + E\frac{\partial}{\partial x}\frac{\partial^\alpha}{\partial |x|^\alpha} y_1(x) = 0. \tag{28}$$

Static bifurcation condition in Formula (24) to Equation (28) result in

$$E\frac{\partial}{\partial x}\frac{\partial^\alpha}{\partial |x|^\alpha} y_1(x) = 0. \tag{29}$$

4.1.1. Malvern–Cristescu Constitutive Equation at TSFD

By using TSFD, static bifurcation condition reads Formula (29)

$$E\frac{\partial}{\partial x}\, D_\theta^\beta\,(x) = 0$$

and for all periodic perturbations $e^{i\kappa x}$ the critical tangent stiffness is $E = 0$. Here, a kind of lack of generic behavior is that all perturbations are critical. This means that no finite dimensional critical eigenspace can be determined.

For the dynamic bifurcation (condition in Formula (25)), by substituting $\lambda = i\omega$ into Equation (28),

$$\mathrm{i}\, H\,\omega\,\frac{\partial}{\partial y}\, D_\theta^\beta\,(y) + \mathrm{E}\,\frac{\partial}{\partial y}\, D_\theta^\beta\,(y) + \mathrm{i}\,\mathrm{D}\,\rho\, y\,\omega^3 + \rho\, y\,\omega^2 = 0 \tag{30}$$

is obtained. Assume that $\kappa > 0$; then, by substituting Formula (13) into Equation (30), the bifurcation condition is

$$\omega^2\,\rho + \mathrm{D}\,\omega^3\,\rho\,\mathrm{i} + \mathrm{E}\,\kappa\,\kappa^\beta\,\mathrm{e}^{\frac{\pi\,\theta\,\mathrm{i}}{2}}\,\mathrm{i} - H\,\kappa\,\kappa^\beta\,\omega\,\mathrm{e}^{\frac{\pi\,\theta\,\mathrm{i}}{2}} = 0. \tag{31}$$

The next step is to get real and imaginary partition of Equation (31). The real part leads to

$$\omega^2\,\rho - \mathrm{E}\,\kappa\,\kappa^\beta \sin\left(\frac{\pi\,\theta}{2}\right) - H\,\kappa\,\kappa^\beta\,\omega\cos\left(\frac{\pi\,\theta}{2}\right) = 0, \tag{32}$$

while the imaginary part implies

$$\mathrm{D}\,\omega^3\,\rho + \mathrm{E}\,\kappa\,\kappa^\beta \cos\left(\frac{\pi\,\theta}{2}\right) - H\,\kappa\,\kappa^\beta\,\omega\sin\left(\frac{\pi\,\theta}{2}\right) = 0. \tag{33}$$

The solution of the "real part" for ω reads

$$\omega_{1,2} = \frac{H\,\kappa\,\kappa^\beta\cos\left(\frac{\pi\,\theta}{2}\right) \pm \sqrt{\kappa\,\kappa^\beta\left(4\,\mathrm{E}\,\rho\sin\left(\frac{\pi\,\theta}{2}\right) + H^2\,\kappa\,\kappa^\beta\cos\left(\frac{\pi\,\theta}{2}\right)^2\right)}}{2\,\rho}$$

One might introduce

$$\bar{H} = \frac{H\,\kappa\,\kappa^{\beta}}{\rho} \equiv \frac{H}{\rho}\kappa^{1+\beta}, \quad \bar{E} = \frac{E\,\kappa\,\kappa^{\beta}}{\rho} \equiv \frac{E}{\rho}\kappa^{1+\beta}$$

Then, the imaginary part results

$$\mathrm{D}\,\omega^3 - \omega\,\bar{H}\sin\left(\frac{\pi\,\theta}{2}\right) + \bar{E}\cos\left(\frac{\pi\,\theta}{2}\right) = 0, \tag{34}$$

and the solution for the "real part" is

$$\omega_{1,2} = \frac{\bar{H}\cos\left(\frac{\pi\theta}{2}\right) \pm \sqrt{4\,\bar{E}\sin\left(\frac{\pi\theta}{2}\right) + \bar{H}^2\cos\left(\frac{\pi\theta}{2}\right)^2}}{2}. \tag{35}$$

Now, ω should be substituted from Formula (35) into Equation (34) and conditions should be formed for constitutive parameters E, D, H for various values of perturbing frequency κ. The next subsection, for the sake of simplicity, deals with the case of Riesz derivative as a special case of TSFD.

4.1.2. Malvern–Cristescu at Riesz Derivative

To study the dynamic bifurcation of Malvern-Cristescu equation, one should return to Equations (32) and (33). In such case, these are

$$\omega^2\,\rho - H\,\kappa\,\kappa^{\beta}\,\omega = 0, \tag{36}$$

and

$$\mathrm{D}\,\omega^3\,\rho + \mathrm{E}\,\kappa\,\kappa^{\beta} = 0. \tag{37}$$

While $\omega \neq 0$, from Equation (36)

$$\omega = \frac{H}{\rho}\,\kappa^{\beta+1}, \tag{38}$$

the dynamic bifurcation condition then reads

$$\mathrm{D}\,\frac{H^2}{\rho^2}\,\kappa^{2(\beta+1)} + \frac{E}{\rho} = 0. \tag{39}$$

Remark that either $D = 0$ or $H = 0$ leads to the "classical" coexistent dynamic and static bifurcation case at $E = 0$.

In Formula (39), either D or E should be negative, but the sign of H has no effect on dynamic bifurcation. By expressing E from Formula (39), its value is plotted in Figure 1, with data $D = -1$, $H = 1, \rho = 0.01, \beta_1 = -0.2, \beta_2 = -0.55, \beta_3 = -0.8$. Now, for a critical perturbation frequency κ_{crit}, the critical tangent stiffness can be determined and vice versa.

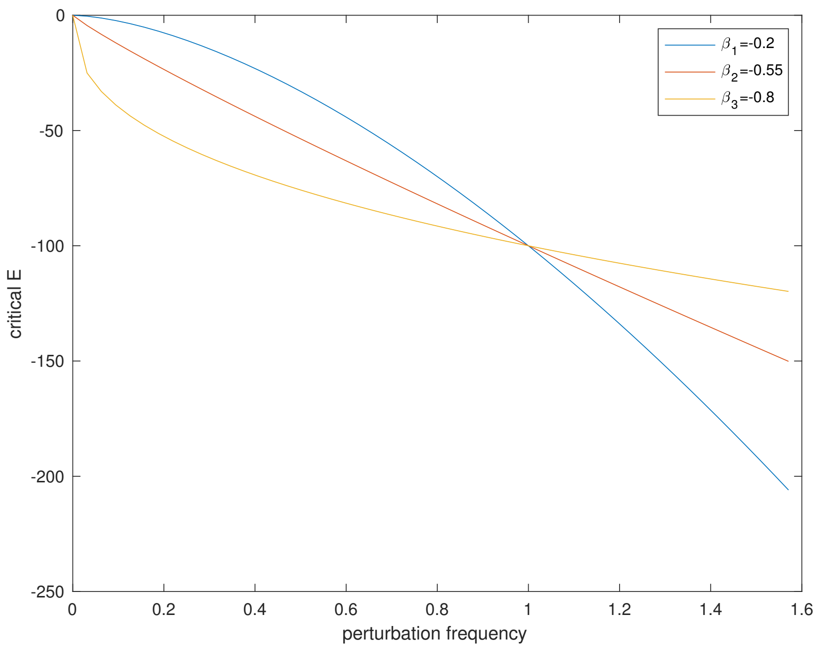

Figure 1. Dynamic bifurcation of Malvern–Cristescu material: tangent stiffness as function of κ at various orders of derivative β.

4.1.3. Classical Visco-Elasto-Plastic Case at Riesz Derivative

For a classical visco-elasto-plastic material, $D = 0$. While $\theta = 0, \beta > -1$ represents the case of symmetric Riesz derivative [19], Equation (34) implies

$$\bar{E} = 0 \tag{40}$$

and Equation (32) implies

$$\omega \rho - \bar{H} = 0. \tag{41}$$

Unfortunately, Formula (40) is the same as the static bifurcation condition, thus such constitutive equation results in non-generic coexistent static and dynamic bifurcations, consequently unavailable for a dynamic stability analysis. Additionally, non-generic behavior appears also in the fact that all periodic perturbations are critical (all values of κ can be selected as a critical perturbing frequency).

4.2. Fractional Gradient Material

As an example of strong non-locality, second gradient materials are studied. As it is used by Aifantis [5], a second gradient term should be included into constitutive equations and Formula (1) is obtained. In the Aifantis–Tarasov material model [22], the idea is to use fractional derivative in Formula (1). In uniaxial form, such equation reads

$$\sigma = B\varepsilon + C\frac{\partial^\alpha}{\partial x^\alpha}\varepsilon. \tag{42}$$

However, non-locality is already involved in Equation (42) and the use of local "fractional" derivatives might even be possible in the sense of non-local mechanics ([23,24]). However, such derivatives are widely criticized [25]; for this reason, TSFD will be used again:

$$\sigma = B\varepsilon + CD_\theta^\beta\left(\varepsilon\right). \tag{43}$$

In this case, characteristic Equation (23) reads

$$\lambda^2 y_1 - \left(B + CD_\theta^\beta\right) \frac{\partial^2}{\partial x^2} y_1 = 0. \tag{44}$$

Then, for periodic perturbations $e^{\kappa\, x\, \mathrm{i}}$ (selected to be the eigenfunctions of D_θ^β),

$$\lambda^2\, \mathrm{e}^{\kappa\, x\, \mathrm{i}} + B\, \kappa^2\, \mathrm{e}^{\kappa\, x\, \mathrm{i}} + C\, \kappa^\beta\, \kappa^2\, \mathrm{e}^{\frac{\pi\,\theta\,\mathrm{i}}{2}}\, \mathrm{e}^{\kappa\, x\, \mathrm{i}} = 0 \tag{45}$$

is obtained. Equation (45) can be solved to λ

$$\lambda_{1,2} = \pm\kappa\sqrt{-B - C\,\kappa^\beta\, \mathrm{e}^{\frac{\pi\,\theta\,\mathrm{i}}{2}}}. \tag{46}$$

From Formula (46), static bifurcation condition is

$$B = -C\,\kappa^\beta\, \mathrm{e}^{\frac{\pi\,\theta\,\mathrm{i}}{2}},$$

in case of symmetric Riesz derivative $(\theta = 0,\ \beta > -1)$

$$B = -C\,\kappa^\beta. \tag{47}$$

By expressing B from Equation (47) as a function of perturbation frequency κ, its graph is plotted in Figure 2 at $(C = -1)$.

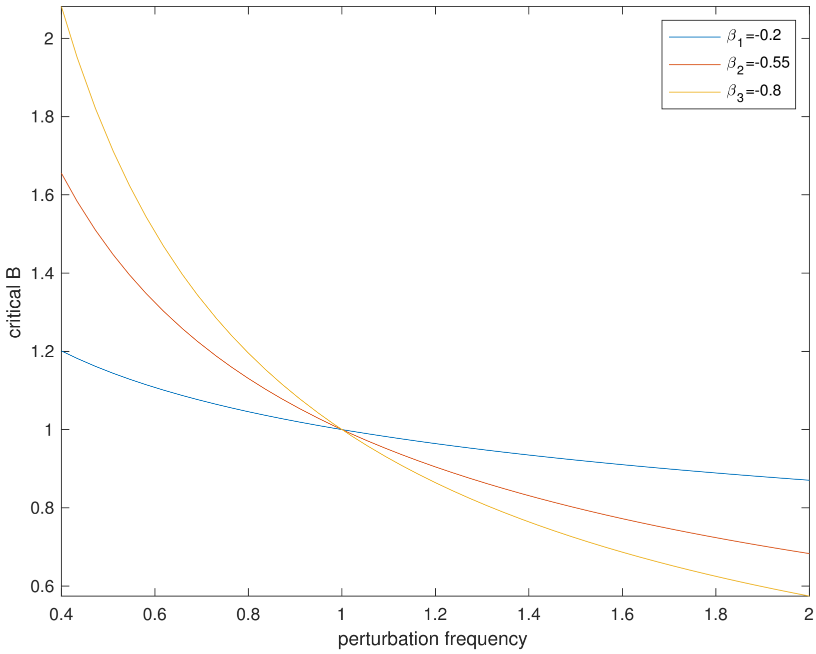

Figure 2. Static bifurcation of Aifantis–Tarasov material: critical tangent stiffness as function of κ at various orders of derivative β.

Dynamic bifurcation study requires adding rate dependence to the constitutive Equation (43)

$$\sigma = B\,\varepsilon + D\,\dot{\varepsilon} + C\,D_\theta^\beta\,\varepsilon \tag{48}$$

Then, the characteristic equation is

$$\rho\,\lambda^2 y_1 - \left(B\ \ + \mathrm{D}\,\lambda\ \ + C\,D_\theta^\beta\ \right) \frac{\partial^2}{\partial x^2} y_1 = 0 \tag{49}$$

for eigenfunctions $e^{\kappa\, x\, \mathrm{i}}$

$$\lambda^2\, \rho + \kappa^2\, (B + \mathrm{D}\, \lambda) + C\, \kappa^\beta\, \kappa^2\, \mathrm{e}^{\frac{\pi\, \theta\, \mathrm{i}}{2}} = 0. \tag{50}$$

Equation (50) can be solved for λ

$$\lambda_{1,2} = -\frac{\mathrm{D}\, \kappa^2 \pm \kappa\, \sqrt{\mathrm{D}^2\, \kappa^2 - 4\, B\, \rho - 4\, C\, \kappa^\beta\, \rho}}{2\, \rho}. \tag{51}$$

A necessary condition for dynamic bifurcation is

$$D = 0, \quad \text{while} \quad 4\, B\, \rho + 4\, C\, \kappa^\beta\, \rho\, \mathrm{e}^{\frac{\pi\, \theta\, \mathrm{i}}{2}} > 0. \tag{52}$$

Notice that the inequality in Formula (52) prescribes that parameters should be below the curves in Figure 2. For example, the stability chart for $\beta = -0.8$ is presented in Figure 3. When $D > 0$, and the parameters are in the shaded region, the state of the material is stable.

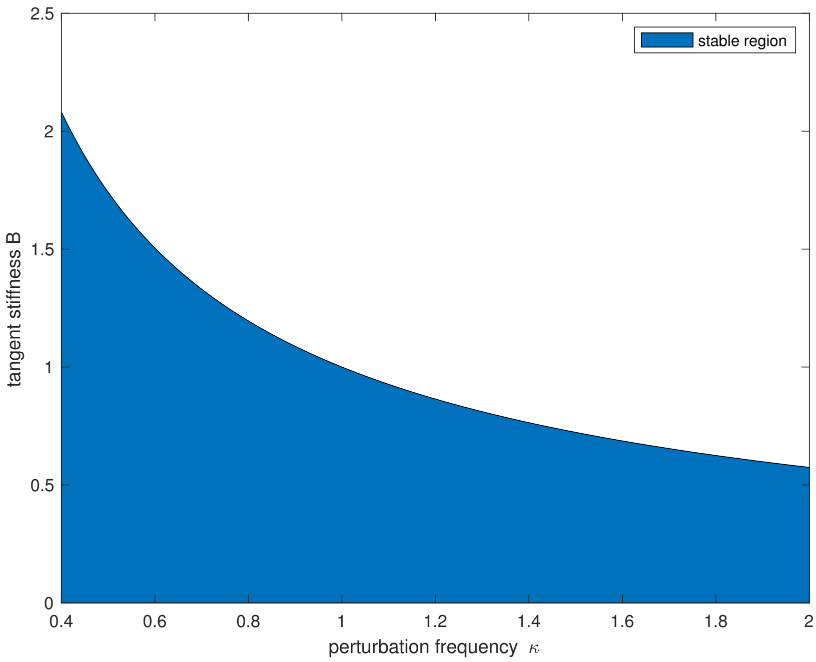

Figure 3. Stability chart for $\beta = -0.8$.

4.3. Non-Local Strain Gradient Material

In case of a generalization of the classical second gradient dependent material [5], the constitutive equation reads

$$\sigma = B\varepsilon_\alpha + C\frac{\partial^2}{\partial x^2}\varepsilon_\alpha. \tag{53}$$

In material model of Formula (53), both weak (second gradient term) and strong non-localities (fractional strain) are present.

Having been derived characteristic Equation (23) for Formula (53),

$$\rho\lambda^2 y_1(x) = \left(B\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial}{\partial x} + C\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial^3}{\partial x^3} \right) y_1(x)$$

is obtained.

In this case, a critical perturbation can be identified. Then, static bifurcation condition in Formula (24) for periodic perturbations

$$y_1(x) = \exp(i\omega x)$$

results in being

$$\frac{\partial^\alpha}{\partial x^\alpha}\frac{\partial}{\partial x}\left(\left(B-\omega^2 C\right)y_1(x)\right)=0. \tag{54}$$

Equation (54) implies that

$$B_{crit} = \omega^2 C. \tag{55}$$

Then, critical perturbation frequency is

$$\omega = \sqrt{\frac{B_{crit}}{C}} \tag{56}$$

and the non-trivial eigenfunction is

$$y_1(x) = \exp\left(i\sqrt{\frac{B_{crit}}{C}}x\right). \tag{57}$$

From Formulas (55)–(57), we find no dependence on the type of fractional derivative, and the result is the same as the integer order gradient problem [21].

5. Conclusions

Stability analysis was performed for non-local materials. Non-locality was modeled by using fractional derivatives. The main aim was to guarantee non-trivial critical eigenfunctions at the loss of stability. Such eigenfunctions play an important role in nonlinear studies because nonlinearities are projected to them to form bifurcation equations. This case is referred to as generic bifurcation. The existence of non-trivial critical eigenfunctions was studied for various non-local constitutive equations.

Two main groups distinguished the strong and the weak types of non-localities. In the case of strongly non-local materials, the introduction of fractional strain via fractional derivative seems to be an obvious generalization of the classical theories. Fractional derivatives here should be non-local and symmetric. The so-called two-sided fractional derivatives by [19] satisfy all these requirements and its special version, the Riesz derivative, was used in all bifurcation investigations.

Constitutive relations were classified as equations with non-local strain and fractional gradient dependent ones. The first type was represented by the Malvern–Cristescu equation and a simplified version of it, called the classical visco-elasto-plastic material. The second group was represented by the fractional gradient material (Aifantis–Tarasov model), its rate dependent version. At the end, the non-local strain gradient material was studied.

The main results are as follows:

- For the Malvern–Cristescu equation at static bifurcation in non-generic because no finite dimensional critical eigenspace exists. By studying dynamic bifurcation, generic behavior was detected as it can be observed in (39) and in Figure 1.
- In case of classical visco-elasto-plastic material, no generic nature can be assumed. Both infinite dimensional critical eigenspace and coexistent static and dynamic bifurcations are found. Thus, such equation cannot be used for any nonlinear stability analysis.
- For fractional gradient materials, a generic static bifurcation exists with condition (47). In Figure 2, the curve of critical tangent stiffness and critical eigenfunction was presented. Unfortunately, dynamic bifurcation is non-generic, but a stability chart can be drawn, Figure 3.

- In the case of non-local strain gradient material, both strong and weak non-localities are present in the constitutive equation. The results are the same as the simple second gradient material [21].

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