

Article

Mathematical Description of the Groundwater Flow and that of the Impurity Spread, which Use Temporal Caputo or Riemann–Liouville Fractional Partial Derivatives, Is Non-Objective

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Abstract: In this paper, it is shown that the mathematical description of the bulk fluid flow and that of content impurity spread, which uses temporal Caputo or temporal Riemann–Liouville fractional order partial derivatives, having integral representation on a finite interval, in the case of a horizontal unconfined aquifer is non-objective. The basic idea is that different observers using this type of description obtain different results which cannot be reconciled, in other words, transformed into each other using only formulas that link the numbers representing a moment in time for two different choices from the origin of time measurement. This is not an academic curiosity; it is rather a problem to find which one of the obtained results is correct.

Keywords: mathematical description; groundwater flow; impurity spread; fractional order derivative

1. Introduction

The mathematical description of a real-world phenomenon is objective if it is independent of the observer. That is, it is possible to reconcile the observation of a phenomenon into a single coherent description of it. This requirement was pointed out by Galileo Galilee (1564–1642), Isaac Newton (1643–1727), and Albert Einstein (1879–1955) in the context of mathematical description of a mechanical movement: “The mechanical event is independent of the observer”. A possible and elementary understanding of the independence of the mechanical event from the observer is the independence of the event from the choice of the frame of reference and from the choice of the moment considered the origin for time measurement, made by an observer. What this means precisely in this paper is presented in the following. To describe mathematically the evolution of a mechanical event, an observer chooses a fixed orthogonal reference frame in the affine Euclidean space, a fixed moment in time (called origin for time measurement), and a unit for time measuring (second). For different observers this choice can be different. In this paper the “objectivity of a mathematical description” means that the description is independent of the choice of the fixed orthogonal reference frame and of the choice of origin for time measuring. This means that the results obtained by two different observers can be reconciled, in other words, transformed into each other, using only formulas that link the coordinates of a point in two fixed orthogonal reference frames and formulas that link the numbers representing a moment in time for two different choices of the origin of time measurement. This concept of “objectivity of a mathematical description” is different from the concept of “objectivity in physics” presented in [1]. The advantage of our kind of understanding of the “objectivity of a mathematical description”

used in this paper, is that it is less restrictive than Galilean invariance, Lorentz invariance or Einstein covariance and General covariance, it can be easily applied in a specific case and the reader does not need prior knowledge of either Galilean or Lorentz invariance, or Einstein or General covariance. Mathematical descriptions which depend on the choice of the fixed orthogonal reference frame or on the choice of the origin of time measurement are non-objective in the sense of this paper. In the case of descriptions which are non-objective, two observers who describe the same mechanical event obtain two different results that cannot be reconciled, in other words, cannot be transformed into each other, using only formulas that link the coordinates of a point in two fixed orthogonal reference frames and formulas that link the numbers representing the same moment in time for two different choices of the origin of time measurement. This concept of non-objective description can be easily applied in a specific case and the reader does not need prior knowledge of either Galilean or Lorentz invariance, or Einstein or General covariance. The majority of mathematical descriptions formulated in terms of integer-order derivatives or integer-order partial derivatives, reported in the literature (books of differential equations of mathematical physics), are objective in the sense of this manuscript.

In the following, the objectivity of the description formulated in terms of integer-order derivatives of some phenomena appearing in hydrology is illustrated. The details related to the verification of objectivity in the case of the mathematical description of the groundwater flow to the well and the spreading of impurities were introduced in the manuscript for several reasons:

In the literature, accessible to us for free, we have not found the justification of the objectivity of the mathematical description that uses integer-order derivatives.

We wanted to show how the objectivity of a mathematical description in terms of integer-order derivatives can be verified directly by a simple case occurring in nature.

In these details there are formulas, which represent partial results, that are also used in Sections 2–5 in which the non-objectivity is discussed in the case of the use of temporal Caputo or Riemann–Liouville fractional derivatives having integral representation on a finite interval.

The reasoning that is done to prove the objectivity of the mathematical description in the case of integer-order derivatives, helps in understanding the reasoning that is done to demonstrate the non-objectivity of the description in the case of using temporal Caputo or Riemann–Liouville fractional derivatives, having integral representation on a finite interval. In the case of using fractional derivatives, the reasoning is by "reduction ad absurdum". At the beginning it is assumed that the description is objective. After this, the steps are followed when demonstrating objectivity in the case of using integer-order derivatives. So an equality is obtained, which follows from the assumption of the objectivity. In general, the obtained equality is not true. Hence the conclusion that the hypothesis of the objectivity of the description (in this case of the use of fractional derivatives, having integral representation on a finite interval), is false.

In hydrology [2] and [3] the horizontal unconfined aquifer around the well is represented as a subset Ω of points of the affine Euclidean space E_3 . A vertical section of Ω is shown in Figure 1.

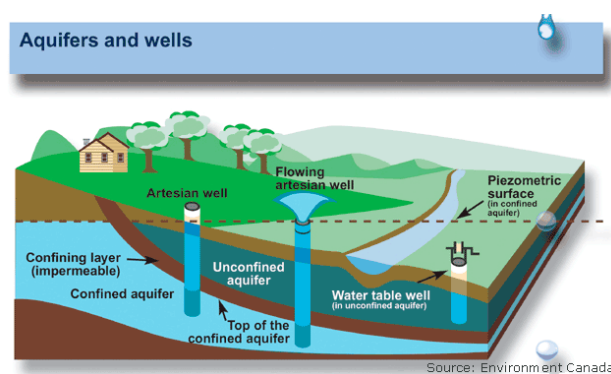


Figure 1. Aquifers and wells.

A particle Q of the bulk fluid at a moment in time M is represented by a point P of Ω . P is the place occupied by the fluid particle Q at the moment in time M . To describe the position of P , observer O chooses a fixed orthogonal reference frame $R_O = (O; \vec{e}_1, \vec{e}_2, \vec{e}_3)$ in E_3 and describes the position of P with the coordinates (x_1, x_2, x_3) of P respecting the reference frame R_O . The unit vector \vec{e}_3 is usually oriented vertically upward and the plane determined by the vectors \vec{e}_1, \vec{e}_2 (the datum plane) is under the bottom of the aquifer. To describe the time evolution, observer O chooses a moment in time M_O for fixing the origin for time measurement (the moment, when his stopwatch for time measurement, starts) and a unit for time measuring (second). A moment in time M that is earlier than M_O is represented by a negative real number $t_M < 0$ (representing the units of time between the moment M and the moment M_O), a moment in time M which is later than M_O is represented by a positive real number $t_M > 0$ (representing the units of time between the moment M and the moment M_O) and the moment in time M_O is represented by the real number $t_{M_O} = 0$.

Observer O describes the flow to the well in the horizontal unconfined aquifer with a real valued function $h_O = h_O(t_M, x_1, x_2, x_3)$ and a vector valued function $\vec{U}_O = \vec{U}_O(t_M, x_1, x_2, x_3)$. The function h_O is the piezometric head and the function \vec{U}_O is the associated flow velocity. The number $h_O(t_M, x_1, x_2, x_3)$ represents the piezometric head at the moment in time t_M in the point P of coordinates (x_1, x_2, x_3) . The vector $\vec{U}_O(t_M, x_1, x_2, x_3)$ represents the associated velocity of the fluid particle Q , that at the moment in time t_M is in the point P of coordinates (x_1, x_2, x_3) . To describe the position of P , observer O^* chooses a fixed orthogonal reference frame $R_{O^*} = (O^*; \vec{e}^*_1, \vec{e}^*_2, \vec{e}^*_3)$ in E_3 and describes the position of P with the coordinates (x^*_1, x^*_2, x^*_3) of P respecting the reference frame $R_{O^*} = (O^*; \vec{e}^*_1, \vec{e}^*_2, \vec{e}^*_3)$. The unit vector \vec{e}^*_3 is usually oriented vertically upward and the plane determined by the vectors \vec{e}^*_1, \vec{e}^*_2 (the datum plane) is under the bottom of the aquifer. To describe the time evolution, observer O^* chooses a moment in time M_{O^*} for fixing the origin for time measurement (the moment, when its stopwatch for time measurement starts) and a unit for time measuring (second). A moment in time M that is earlier than M_{O^*} is represented by a negative real number $t^*_M < 0$ (representing the units of time between the moment M and the moment M_{O^*}), a moment in time M which is later than M_{O^*} is represented by a positive real number $t^*_M > 0$ (representing the units of time between the moment M and the moment M_{O^*}) and the moment in time M_{O^*} is represented by the real number $t^*_{M_{O^*}} = 0$.

Observer O^* describes the flow to the well in the same horizontal unconfined aquifer with a real valued function $h_{O^*} = h_{O^*}(t^*_M, x^*_1, x^*_2, x^*_3)$ and a vector valued function $\vec{U}_{O^*} = \vec{U}_{O^*}(t^*_M, x^*_1, x^*_2, x^*_3)$. The function h_{O^*} is the piezometric head and the function \vec{U}_{O^*} is the associated flow velocity. The number $h_{O^*}(t^*_M, x^*_1, x^*_2, x^*_3)$ represents the piezometric head at the moment in time t^*_M in the point P of coordinates (x^*_1, x^*_2, x^*_3) . The vector $\vec{U}_{O^*}(t^*_M, x^*_1, x^*_2, x^*_3)$ represents the associated velocity of the fluid particle Q , that at the moment in time t^*_M is in the point P of coordinates (x^*_1, x^*_2, x^*_3) .

Note that a moment in time M in case of the observer O is described by the real number t_M and in case of the observer O^* by the real number t^*_M . For the numbers t_M and t^*_M the following relations hold:

$$t_M = t^*_M + t_{M_{O^*}} \quad (1)$$

$$t^*_M = t_M + t^*_{M_O} \quad (2)$$

In the above mentioned relations $t_{M_{O^*}}$ is the real number, that represents the moment M_{O^*} in the system of time measurement of the observer O and $t^*_{M_O}$ is the real number, that represents the moment M_O in the system of time measurement of the observer O^* .

At any moment in time M , the coordinates (x_1, x_2, x_3) with respect to R_O and (x^*_1, x^*_2, x^*_3) with respect to R_{O^*} represent the same point P in the three dimensional affine Euclidean space E_3 . Therefore, for the coordinates the following relations hold:

$$x_k = x_{kO^*} + \sum_{i=1}^{i=3} a_{ik} \cdot x^*_i \quad k = 1, 2, 3 \quad (3)$$

or equivalently

$$x^*_k = x^*_{kO} + \sum_{i=1}^{i=3} a_{ki} x_i \quad k = 1, 2, 3 \quad (4)$$

The significance of the quantities appearing in the above mentioned relations are:

$a_{ij} = \langle \vec{e}^*_i, \vec{e}_j \rangle = \text{constant} = \text{scalar product of the unit vectors } \vec{e}^*_i \text{ and } \vec{e}_j \text{ in } E_3$, in other words,

$$\vec{e}^*_i = \sum_{k=1}^{k=3} a_{ik} \vec{e}_k \quad \vec{e}_i = \sum_{k=1}^{k=3} a_{ki} \vec{e}^*_k \quad (5)$$

$(x_{1O^*}, x_{2O^*}, x_{3O^*})$ are the coordinates of the point O^* with respect to the reference frame R_O , $(x^*_{1O}, x^*_{2O}, x^*_{3O})$ are the coordinates of the point O with respect to the reference frame R_{O^*} . At any moment in time M , and at any point P the vectors $\vec{U}_O(t_M, x_1, x_2, x_3)$ and $\vec{U}_{O^*}(t^*_M, x^*_1, x^*_2, x^*_3)$ represent the velocity of the same flow and the scalars $h_O(t_M, x_1, x_2, x_3)$ with $h_{O^*}(t^*_M, x^*_1, x^*_2, x^*_3)$ represent the same piezometric head.

Therefore, the following relations hold:

$$U_{Ok}(t_M, x_1, x_2, x_3) = \sum_{i=1}^{i=3} a_{ik} U_{O^*i}(t^*_M, x^*_1, x^*_2, x^*_3) \quad k = 1, 2, 3 \quad (6)$$

$$U_{O^*k}(t^*_M, x^*_1, x^*_2, x^*_3) = \sum_{i=1}^{i=3} a_{ki} U_{Oi}(t_M, x_1, x_2, x_3) \quad k = 1, 2, 3 \quad (7)$$

$$h_O(t_M, x_1, x_2, x_3) = h_{O^*}(t_M + t^*_{M_{O^*}}, x^*_{1O} + \sum_{i=1}^{i=3} a_{1i} x_i, x^*_{2O} + \sum_{i=1}^{i=3} a_{2i} x_i, x^*_{3O} + \sum_{i=1}^{i=3} a_{3i} x_i) \quad (8)$$

$$h_{O^*}(t^*_M, x^*_1, x^*_2, x^*_3) = h_O(t^*_M + t_{M_{O^*}}, x_{1O^*} + \sum_{i=1}^{i=3} a_{i1} \cdot x^*_i, x_{2O^*} + \sum_{i=1}^{i=3} a_{i2} \cdot x^*_i, x_{3O^*} + \sum_{i=1}^{i=3} a_{i3} \cdot x^*_i) \quad (9)$$

Equations (1)–(4) and (6)–(9) reconcile the mathematical description of the bulk groundwater flow made by the two observers, and enable the flow description by the piezometric head h_O and the associated velocity \vec{U}_O , or by the piezometric head h_{O^*} and the associated velocity \vec{U}_{O^*} . This means that the above presented mathematical description of the bulk groundwater flow to the well, in a horizontal unconfined aquifer, is objective.

In classical theory of the 2D flow to the well, in a horizontal unconfined aquifer, [2] and [3], the real valued function $h_O = h_O(t_M, x_1, x_2)$ and the vector valued function $\vec{U}_O = \vec{U}_O(t_M, x_1, x_2)$ that describe the 2D flow in terms of the observer O , verify the equations:

$$S \cdot \frac{\partial h_O}{\partial t_M} + T \cdot \left(\frac{\partial^2 h_O}{\partial x_1^2} + \frac{\partial^2 h_O}{\partial x_2^2} \right) = Q_s \quad (10)$$

$$\vec{U}_O(t_M, x_1, x_2) = -\frac{K}{\varphi} \cdot \left(\frac{\partial h_O}{\partial x_1} \cdot \vec{e}_1 + \frac{\partial h_O}{\partial x_2} \cdot \vec{e}_2 \right) \quad (11)$$

where S is the storage coefficient; T is the transmissivity of the aquifer; Q_S is the leakage rate; K is the hydraulic conductivity and φ is the porosity.

The real valued function $h_{O^*} = h_{O^*}(t^*_{*M}, x^*_{*1}, x^*_{*2})$ and the vector valued function $\vec{U}_{O^*} = \vec{U}_{O^*}(t^*_{*M}, x^*_{*1}, x^*_{*2})$, that in terms of the observer O^* , describe the bulk fluid 2D flow to the well, verify the equations:

$$S \cdot \frac{\partial h_{O^*}}{\partial t^*_{*M}} + T \cdot \left(\frac{\partial^2 h_{O^*}}{\partial x^*_{*1}} + \frac{\partial^2 h_{O^*}}{\partial x^*_{*2}} \right) = Q_S \tag{12}$$

$$\vec{U}_{O^*}(t^*_{*M}, x^*_{*1}, x^*_{*2}) = -\frac{K}{\varphi} \cdot \left(\frac{\partial h_{O^*}}{\partial x^*_{*1}} \cdot \vec{e}_{*1} + \frac{\partial h_{O^*}}{\partial x^*_{*2}} \cdot \vec{e}_{*2} \right) \tag{13}$$

where S is the storage coefficient; T is the transmissivity of the aquifer; Q_S is the leakage rate; K is the hydraulic conductivity and φ is the porosity.

Description with Equations (10) and (11) is objective if and only if it describes the same flow as (12), (13).

The objectivity of the description can be proven showing that:

If $h_O(t_M, x_1, x_2)$ and $\vec{U}_O(t_M, x_1, x_2)$ verify Equations (10) and (11), then the functions $h_{O^*}(t^*_{*M}, x^*_{*1}, x^*_{*2})$ and $\vec{U}_{O^*}(t^*_{*M}, x^*_{*1}, x^*_{*2})$, defined by:

$$h_{O^*}(t^*_{*M}, x^*_{*1}, x^*_{*2}) = h_O(t^*_{*M} + t_{M_{O^*}}, x_{1_{O^*}} + \sum_{i=1}^{i=2} a_{i1} \cdot x^*_{*i}, x_{2_{O^*}} + \sum_{i=1}^{i=2} a_{i2} \cdot x^*_{*i}) \tag{14}$$

$$\vec{U}_{O^*}(t^*_{*M}, x^*_{*1}, x^*_{*2}) = \vec{U}_O(t^*_{*M} + t_{M_{O^*}}, x_{1_{O^*}} + \sum_{j=1}^{j=2} a_{j1} \cdot x^*_{*j}, x_{2_{O^*}} + \sum_{j=1}^{j=2} a_{j2} \cdot x^*_{*j})$$

verify Equations (12) and (13) and if $h_{O^*}(t^*_{*M}, x^*_{*1}, x^*_{*2})$ and $\vec{U}_{O^*}(t^*_{*M}, x^*_{*1}, x^*_{*2})$ verify Equations (12) and (13), then the functions $h_O(t_M, x_1, x_2)$ and $\vec{U}_O(t_M, x_1, x_2)$, defined by:

$$h_O(t_M, x_1, x_2) = h_{O^*}(t_M + t^*_{*M_{O^*}}, x_{*1_{O^*}} + \sum_{i=1}^{i=2} a_{i1} \cdot x_i, x_{*2_{O^*}} + \sum_{i=1}^{i=2} a_{i2} \cdot x_i) \tag{15}$$

$$\vec{U}_O(t_M, x_1, x_2) = \vec{U}_{O^*}(t_M + t^*_{*M_{O^*}}, x_{*1_{O^*}} + \sum_{j=1}^{j=2} a_{j1} \cdot x_j, x_{*2_{O^*}} + \sum_{j=1}^{j=2} a_{j2} \cdot x_j)$$

verify Equations (10) and (11).

A short proof of the objectivity of this description is the following: assume that the functions $h_O(t_M, x_1, x_2)$ and $\vec{U}_O(t_M, x_1, x_2)$ verify Equations (10) and (11) and consider the functions $h_{O^*}(t^*_{*M}, x^*_{*1}, x^*_{*2})$, $\vec{U}_{O^*}(t^*_{*M}, x^*_{*1}, x^*_{*2})$, defined by (14). Note that the following equalities hold:

$$\frac{\partial h_{O^*}}{\partial t^*_{*M}} = \frac{\partial h_O}{\partial t_M}; \frac{\partial h_{O^*}}{\partial x^*_{*i}} = \frac{\partial h_O}{\partial x_i} \cdot a_{i1} + \frac{\partial h_O}{\partial x_i} \cdot a_{i2}; \tag{16}$$

$$\frac{\partial^2 h_{O^*}}{\partial x^*_{*i}{}^2} = a_{i1} \cdot \left(\frac{\partial^2 h_O}{\partial x_1^2} \cdot a_{i1} + \frac{\partial^2 h_O}{\partial x_1 \partial x_2} \cdot a_{i2} \right) + a_{i2} \cdot \left(\frac{\partial^2 h_O}{\partial x_2 \partial x_1} \cdot a_{i1} + \frac{\partial^2 h_O}{\partial x_2^2} \cdot a_{i2} \right)$$

$$\frac{\partial^2 h_{O^*}}{\partial x^*_{*1}{}^2} + \frac{\partial^2 h_{O^*}}{\partial x^*_{*2}{}^2} = a_{11} \cdot \left(\frac{\partial^2 h_O}{\partial x_1^2} \cdot a_{11} + \frac{\partial^2 h_O}{\partial x_1 \partial x_2} \cdot a_{12} \right) + a_{12} \cdot \left(\frac{\partial^2 h_O}{\partial x_2 \partial x_1} \cdot a_{11} + \frac{\partial^2 h_O}{\partial x_2^2} \cdot a_{12} \right) +$$

$$a_{21} \cdot \left(\frac{\partial^2 h_O}{\partial x_1^2} \cdot a_{21} + \frac{\partial^2 h_O}{\partial x_1 \partial x_2} \cdot a_{22} \right) + a_{22} \cdot \left(\frac{\partial^2 h_O}{\partial x_2 \partial x_1} \cdot a_{21} + \frac{\partial^2 h_O}{\partial x_2^2} \cdot a_{22} \right) = \frac{\partial^2 h_O}{\partial x_1^2} + \frac{\partial^2 h_O}{\partial x_2^2}$$

$$-\frac{K}{\varphi} \cdot \left(\frac{\partial h_{O^*}}{\partial x^*_{*1}} \cdot \vec{e}_{*1} + \frac{\partial h_{O^*}}{\partial x^*_{*2}} \cdot \vec{e}_{*2} \right) = -\frac{K}{\varphi} \cdot \left(\left(\frac{\partial h_O}{\partial x_1} \cdot a_{11} + \frac{\partial h_O}{\partial x_2} \cdot a_{12} \right) \cdot \vec{e}_{*1} + \left(\frac{\partial h_O}{\partial x_1} \cdot a_{21} + \frac{\partial h_O}{\partial x_2} \cdot a_{22} \right) \cdot \vec{e}_{*2} \right) =$$

$$-\frac{K}{\varphi} \cdot \left(\frac{\partial h_O}{\partial x_1} \cdot (a_{11} \cdot \vec{e}_{*1} + a_{21} \cdot \vec{e}_{*2}) + \frac{\partial h_O}{\partial x_2} \cdot (a_{12} \cdot \vec{e}_{*1} + a_{22} \cdot \vec{e}_{*2}) \right) = -\frac{K}{\varphi} \cdot \left(\frac{\partial h_O}{\partial x_1} \cdot \vec{e}_1 + \frac{\partial h_O}{\partial x_2} \cdot \vec{e}_2 \right)$$

Using Equations (16) and replacing the terms in (12) and (13), follows that the functions $h_{O^*}(t^*_M, x^*_1, x^*_2)$, $\bar{U}_{O^*}(t^*_M, x^*_1, x^*_2)$, defined by (14), verify Equations (12) and (13). The second part of the proof is similar.

Therefore, the description with Equations (10) and (11) is objective. That is, different observers, describing the groundwater flows with these tools, obtain results that can be reconciled, in other words, transformed into each other, using Equations (1)–(4) that link the coordinates of a point in two fixed orthogonal reference frames and formulas that link the numbers representing a moment in time for two different choices of the origin of time measurement.

The spread of an impurity, contained in the bulk fluid flowing in a porous media, is described by the concentration of that impurity. Observer O describes the concentration with the real valued function $C_O = C_O(t_M, x_1, x_2, x_3)$, that verifies the partial differential equation (PDE):

$$\frac{\partial C_O}{\partial t_M} = \sum_{i=1}^{i=3} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{j=3} D^{ij}_O \cdot \frac{\partial C_O}{\partial x_j} \right) - \sum_{i=1}^{i=3} \frac{\partial}{\partial x_i} (U_{O_i}(t_M, x_1, x_2, x_3) \cdot C_O) + S_O \tag{17}$$

where $D_O = D_O(x_1, x_2, x_3)$ is the diffusion tensor, $U_{O_i}(t_M, x_1, x_2, x_3)$ are the components of the bulk fluid flow velocity, given by Equation (11), the term $\sum_{i=1}^{i=3} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{j=3} D^{ij}_O \cdot \frac{\partial C_O}{\partial x_j} \right)$ describes the impurity spread by diffusion, the term $-\sum_{i=1}^{i=3} \frac{\partial}{\partial x_i} (U_{O_i}(t_M, x_1, x_2, x_3) \cdot C_O)$ describes the impurity spread by convection, $S_O = S_O(t_M, x_1, x_2, x_3)$ describes the source or the sinks of the impurity. See [3–6].

In PDE (17) $D^{ij}_O = D^{ij}_O(x_1, x_2, x_3)$, $S_O = S_O(t_M, x_1, x_2, x_3)$, and $U_{O_i}(t_M, x_1, x_2, x_3)$ are assumed to be known and $C_O = C_O(t_M, x_1, x_2, x_3)$ is unknown.

Observer O^* describes the spread of the impurity by the real valued function $C_{O^*} = C_{O^*}(t^*_M, x^*_1, x^*_2, x^*_3)$, that verifies the following PDE:

$$\frac{\partial C_{O^*}}{\partial t^*_M} = \sum_{i=1}^{i=3} \frac{\partial}{\partial x^*_i} \left(\sum_{j=1}^{j=3} D^{ij}_{O^*} \cdot \frac{\partial C_{O^*}}{\partial x^*_j} \right) - \sum_{i=1}^{i=3} \frac{\partial}{\partial x^*_i} (U_{O^*_i}(t^*_M, x^*_1, x^*_2, x^*_3) \cdot C_{O^*}) + R_{O^*} \tag{18}$$

where $D^{ij}_{O^*} = D^{ij}_{O^*}(x^*_1, x^*_2, x^*_3) = D^{ij}_O(x_1, x_2, x_3)$ and $S_{O^*} = S_{O^*}(t^*_M, x^*_1, x^*_2, x^*_3) = S_O(t_M, x_1, x_2, x_3)$

In case of the 2D flow to a well, in a horizontal unconfined aquifer, for observer O , $C_O = C_O(t_M, x_1, x_2)$, $D^{ij}_O = D^{ij}_O(x_1, x_2)$, $S_O = S_O(t_M, x_1, x_2)$, and PDE (17) becomes:

$$\frac{\partial C_O}{\partial t_M} = \sum_{i=1}^{i=2} \frac{\partial}{\partial x_i} \left(\sum_{j=1}^{j=2} D^{ij}_O \cdot \frac{\partial C_O}{\partial x_j} \right) - \sum_{i=1}^{i=2} \frac{\partial}{\partial x_i} (U_{O_i}(t_M, x_1, x_2) \cdot C_O) + S_O \tag{19}$$

Under the same hypothesis, for observer O^* we have $C_{O^*} = C_{O^*}(t^*_M, x^*_1, x^*_2)$, $D^{ij}_{O^*} = D^{ij}_{O^*}(x^*_1, x^*_2)$, $S_{O^*} = S_{O^*}(t^*_M, x^*_1, x^*_2)$ and PDE (18) becomes:

$$\frac{\partial C_{O^*}}{\partial t^*_M} = \sum_{i=1}^{i=2} \frac{\partial}{\partial x^*_i} \left(\sum_{j=1}^{j=2} D^{ij}_{O^*} \cdot \frac{\partial C_{O^*}}{\partial x^*_j} \right) - \sum_{i=1}^{i=2} \frac{\partial}{\partial x^*_i} (U_{O^*_i}(t^*_M, x^*_1, x^*_2) \cdot C_{O^*}) + S_{O^*} \tag{20}$$

where $t^*_M = t_M + t^*_{M_0}$; $x^*_k = x^*_{kO} + \sum_{i=1}^{i=2} a_{ki}x_i$ $k = 1, 2$, $D_{O^*} = D_{O^*}(x^*_1, x^*_2) = D_O(x_1, x_2)$ and $S_{O^*} = S_{O^*}(t^*_M, x^*_1, x^*_2) = S_O(t_M, x_1, x_2)$.

Objectivity of the impurity spread description means that the solutions of PDEs (19) and (20) describe the same spread. Objectivity can be proven showing that if $C_O = C_O(t_M, x_1, x_2)$ is a solution of Equation (19), then the function $C_{O^*}(t^*_M, x^*_1, x^*_2)$, defined by

$$C_{O^*}(t^*_M, x^*_1, x^*_2) = C_O(t^*_M + t_{M_{O^*}}, x_{1_{O^*}} + \sum_{i=1}^{i=2} a_{i1}x_{i^*}, x_{2_{O^*}} + \sum_{i=1}^{i=2} a_{i2}x_{i^*}) \tag{21}$$

verifies Equation (20) and if $C_{O^*} = C_{O^*}(t^*_M, x^*_1, x^*_2)$ is a solution of Equation (20), then the function $C_O(t_M, x_1, x_2)$, defined by

$$C_O(t_M, x_1, x_2) = C_{O^*}(t_M + t^*_{M_{O^*}}, x^*_{1_{O^*}} + \sum_{i=1}^{i=2} a_{1i}x_i, x^*_{2_{O^*}} + \sum_{i=1}^{i=2} a_{2i}x_i) \tag{22}$$

verifying Equation (19).

In the following we provide a short proof of the objectivity of this description in the case, when the aquifer is homogeneous, isotropic, and D and S are constants.

In this case Equations (19) and (20) become:

$$\frac{\partial C_O}{\partial t_M} = D \cdot \sum_{i=1}^{i=2} \frac{\partial^2 C_O}{\partial x_i^2} - \sum_{i=1}^{i=2} U_{O_i}(t_M, x_1, x_2) \cdot \frac{\partial C_O}{\partial x_i} + S \tag{23}$$

$$\frac{\partial C_{O^*}}{\partial t^*_M} = D \cdot \sum_{i=1}^{i=2} \frac{\partial^2 C_{O^*}}{\partial x^*_i{}^2} - \sum_{i=1}^{i=2} U_{O^*_i}(t^*_M, x^*_1, x^*_2) \cdot \frac{\partial C_{O^*}}{\partial x^*_i} + S \tag{24}$$

We start with $C_O = C_O(t_M, x_1, x_2)$, the solution of Equation (23), and the function $C_{O^*}(t^*_M, x^*_1, x^*_2)$, defined by (21). For this function the following equalities hold:

$$\frac{\partial C_{O^*}}{\partial t^*_M} = \frac{\partial C_O}{\partial t_M}; \quad \frac{\partial C_{O^*}}{\partial x^*_i} = \sum_{k=1}^{k=2} a_{ik} \cdot \frac{\partial C_O}{\partial x_k} \tag{25}$$

$$\begin{aligned} \sum_{i=1}^{i=2} U_{O^*_i}(t^*_M, x^*_1, x^*_2) \cdot \frac{\partial C_{O^*}}{\partial x^*_i} &= \sum_{i=1}^{i=2} \left(\sum_{j=1}^{j=2} a_{ij} U_{O_j}(t_M, x_1, x_2) \right) \cdot \left(\sum_{k=1}^{k=2} a_{ik} \cdot \frac{\partial C_O}{\partial x_k} \right) = \\ \sum_{j=1}^{j=2} \sum_{k=1}^{k=2} U_{O_j}(t_M, x_1, x_2) \cdot \frac{\partial C_O}{\partial x_k} \cdot \left(\sum_{i=1}^{i=2} a_{ij} \cdot a_{ik} \right) &= \sum_{j=1}^{j=2} \sum_{k=1}^{k=2} U_{O_j}(t_M, x_1, x_2) \cdot \frac{\partial C_O}{\partial x_k} \cdot \delta_{jk} = \\ \sum_{k=1}^{k=2} U_{O_k}(t_M, x_1, x_2) \cdot \frac{\partial C_O}{\partial x_k} \end{aligned} \tag{26}$$

$$\begin{aligned} \sum_{i=1}^{i=2} \frac{\partial^2 C_{O^*}}{\partial x^*_i{}^2} &= \sum_{i=1}^{i=2} \frac{\partial}{\partial x^*_i} \left(\frac{\partial C_{O^*}}{\partial x^*_i} \right) = \sum_{i=1}^{i=2} \frac{\partial}{\partial x^*_i} \left(\sum_{k=1}^{k=2} a_{ik} \cdot \frac{\partial C_O}{\partial x_k} \right) = \sum_{i=1}^{i=2} \sum_{k=1}^{k=2} \sum_{l=1}^{l=2} a_{ik} \cdot a_{il} \cdot \frac{\partial^2 C_O}{\partial x_k \partial x_l} = \\ \sum_{k=1}^{k=2} \frac{\partial^2 C_O}{\partial x_k^2} \end{aligned} \tag{27}$$

Replacing in (20) the terms $\frac{\partial C_{O^*}}{\partial t^*_M}$; $\sum_{i=1}^{i=2} \frac{\partial^2 C_{O^*}}{\partial x^*_i{}^2}$ and $\sum_{i=1}^{i=2} U_{O^*_i}(t^*_M, x^*_1, x^*_2) \cdot \frac{\partial C_{O^*}}{\partial x^*_i}$ with those obtained in Equations (25)–(27), equality (20) is obtained. Therefore, the function $C_{O^*}(t^*_M, x^*_1, x^*_2)$, defined by (21), is a solution of Equation (20).

If we start with a solution $C_{O^*} = C_{O^*}(t^*_M, x^*_1, x^*_2)$ of Equation (20) and we consider the function $C_O(t_M, x_1, x_2)$ given by (22), then, in a similar way, we obtain that this function is a solution of Equation (19).

So, the description of the impurity spread with (23) is objective. That is, different observers, describing the impurity spread with these tools, obtain results that can be reconciled, in other words, transformed into each other, using Equations (1)–(4).

The objectivity of the above presented descriptions implies that different observers describing the same phenomenon, using integer-order partial derivatives, obtain results that can be reconciled, in other words, transformed into each other, using Equations (1)–(4).

Beside the objective mathematical descriptions of the bulk fluid flow and impurity spread in porous media (see references [1–7]) formulated in terms of integer-order partial derivatives, there are mathematical descriptions of the bulk fluid flow and impurity spread in porous media which use fractional order temporal or spatial partial derivatives. See, for instance, references [8–21]. In these works, the analysis of the objectivity is missing. At first, we thought that in the case of the description with fractional derivatives, the objectivity is fulfilled and therefore it is ignored. However, curiosity has pushed us to see how the fulfillment of the objectivity condition (in sense of our manuscript) can be proven mathematically. For this special issue, we have chosen fractional behavior in nature for the very simple case of groundwater flow to well and the spread of impurities. Thus were “born” Sections 2 and 3 of the manuscript in which we analyzed the objectivity of the description of the groundwater flow and Sections 4 and 5, in which we analyzed the of objectivity of the description of the spread of impurities, instead of using the integer-order derivatives temporal Caputo or Riemann–Liouville fractional partial derivatives, having integral representation on finite interval. The purpose of the present paper is to show that, in case of an unconfined horizontal aquifer, the mathematical descriptions, that use Caputo or Riemann–Liouville fractional order temporal partial derivatives, having integral representation on a finite interval, generally are non-objective. Two observers, who use fractional derivatives, obtain different results that cannot be reconciled, in other words, transformed into each other, using Equations (1)–(4).

Remember that for a continuously differentiable function $f : [0, \infty) \times [0, \infty) \rightarrow R$ the Caputo temporal fractional partial derivative of order α , $0 < \alpha$, is defined with the following integral representation on a finite interval (see [22]):

$${}^C_0D_t^\alpha f(t, x) = \frac{1}{\Gamma(n - \alpha)} \cdot \int_0^t \frac{\frac{\partial^n f}{\partial \tau^n}(\tau, x)}{(t - \tau)^{\alpha+1-n}} d\tau \quad (28)$$

Note that the derivative defined with (28) was considered by other people before Caputo, like Gherasimov (see [22]). So, the name of Caputo used in this paper may not be appropriate.

For a continuously differentiable function $f : [0, \infty) \times [0, \infty) \rightarrow R$ the Riemann–Liouville temporal fractional partial derivative of order α , $0 < \alpha$, is defined with the following integral representation on a finite interval (see [22]):

$${}^{R-L}_0D_t^\alpha f(t, x) = \frac{1}{\Gamma(n - \alpha)} \cdot \frac{\partial^n}{\partial t^n} \int_0^t \frac{f(\tau, x)}{(t - \tau)^{\alpha+1-n}} d\xi \quad (29)$$

In Equations (28) and (29), Γ is the Euler gamma function and $n = [\alpha] + 1$, $[\alpha]$ is the integer part of α .

2. In Case of an Unconfined Horizontal Aquifer the Piezometric Head Dynamics Description, Using Temporal Caputo Fractional Order Partial Derivatives, with Integral Representation on a Finite Interval, Is Non-Objective

Assume that in the piezometric head dynamics description that the temporal Caputo fractional partial derivative of order α , $0 < \alpha < 1$, with integral representation on a finite interval, is used.

In case of the 2D flow to a well, in a horizontal unconfined aquifer, Equation (10) for observer O and Equation (12) for observer O^* become:

$$S \cdot {}^C_0D_{t_M}^\alpha h_O + T \cdot \left(\frac{\partial^2 h_O}{\partial^2 x_1} + \frac{\partial^2 h_O}{\partial^2 x_2} \right) = Q_S \quad (30)$$

$$S \cdot {}^C_0D_{t^*_M}^\alpha h_{O^*} + T \cdot \left(\frac{\partial^2 h_{O^*}}{\partial^2 x^*_1} + \frac{\partial^2 h_{O^*}}{\partial^2 x^*_2} \right) = Q_S \quad (31)$$

Objectivity of the piezometric head dynamics description means that the solutions of the fractional partial differential Equations (30) and (31) describe the same dynamics.

That is:

if $h_O(t_M, x_1, x_2)$ verifies Equation (30), then the function $h_{O^*}(t^*_M, x^*_1, x^*_2)$, defined by (14), verifies Equation (31) and if $h_{O^*}(t^*_M, x^*_1, x^*_2)$ verifies Equation (31), then the function $h_O(t_M, x_1, x_2)$, defined by (15), verifies Equation (30).

Assume that the reference frames R_O and R_{O^*} of the observers O and O^* coincide, in other words, $x^*_1 = x_1$ and $x^*_2 = x_2$. Assume also that the piezometric head dynamics described by Equation (30) is objective. Start with a solution $h_O(t_M, x_1, x_2)$ of Equation (30) and consider for this particular situation the function $h_{O^*}(t^*_M, x_1, x_2)$, defined by (14). Note that, in this particular situation, the following equalities hold:

$${}^C_0D_{t^*_M}^\alpha h_{O^*}(t^*_M, x_1, x_2) = {}^C_0D_{t_M}^\alpha h_O(t_M, x_1, x_2) + \frac{1}{\Gamma(1-\alpha)} \cdot \int_0^{t^*_M} \frac{\frac{\partial h_{O^*}}{\partial \tau^*}(\tau^*, x_1, x_2)}{(t^*_M - \tau^*)^\alpha} d\tau^* \quad (32)$$

$$T \cdot \left(\frac{\partial^2 h_O}{\partial^2 x_1} + \frac{\partial^2 h_O}{\partial^2 x_2} \right) = T \cdot \left(\frac{\partial^2 h_{O^*}}{\partial^2 x^*_1} + \frac{\partial^2 h_{O^*}}{\partial^2 x^*_2} \right)$$

Using Equation (32) and replacing the terms in (31), it follows that: if the function $h_{O^*}(t^*_M, x_1, x_2)$ (defined by (14)) verifies Equation (31), then the following equality holds:

$$\frac{1}{\Gamma(1-\alpha)} \cdot \int_0^{t^*_M} \frac{\frac{\partial h_{O^*}}{\partial \tau^*}(\tau^*, x_1, x_2)}{(t^*_M - \tau^*)^\alpha} d\tau^* \quad (33)$$

Equation (33) is a consequence of the assumption that the mathematical description (30) is objective. However, generally (33) it is not verified. This means that the assumption that the piezometric head dynamics description with Equation (30) is objective, is false. It follows that the mathematical description with Equation (30) is non-objective. That is, observers O and O^* , describing the same piezometric head dynamics, with (30) and (31) respectively, obtain different results which cannot be reconciled, in other words, transformed into each other, using Equations (1) and (2), that link the numbers representing the same moment in time for two different choices of the origin of time measurement. The problem is to find which one of the results is correct.

3. In Case of an Unconfined Horizontal Aquifer the Piezometric Head Dynamics Description which Uses Temporal Riemann–Liouville Fractional Order Partial Derivatives, with Integral Representation on a Finite Interval, Is Non-Objective

Assume that in the piezometric head dynamics description that the temporal Riemann–Liouville fractional partial derivative of order α , $0 < \alpha < 1$, with integral representation on a finite interval,

is used. In case of the 2D flow to a well, in a horizontal unconfined aquifer, Equation (10) for observer O and Equation (12) for observer O^* become:

$$S \cdot {}^{R-L}D_{t_M}^\alpha h_O + T \cdot \left(\frac{\partial^2 h_O}{\partial^2 x_1} + \frac{\partial^2 h_O}{\partial^2 x_2} \right) = Q_S \quad (34)$$

$$S \cdot {}^{R-L}D_{t^*_M}^\alpha h_{O^*} + T \cdot \left(\frac{\partial^2 h_{O^*}}{\partial^2 x^*_1} + \frac{\partial^2 h_{O^*}}{\partial^2 x^*_2} \right) = Q_S \quad (35)$$

Objectivity of the piezometric head dynamics description means that the solutions of the fractional partial differential Equations (34) and (35) describe the same dynamics.

That is:

if $h_O(t_M, x_1, x_2)$ verifies Equation (34), then the function $h_{O^*}(t^*_M, x^*_1, x^*_2)$, defined by (14), verifies Equation (35) and if $h_{O^*}(t^*_M, x^*_1, x^*_2)$ verifies Equation (35), then the function $h_O(t_M, x_1, x_2)$, defined by (15), verifies Equation (34).

Assume that the reference frames R_O and R_{O^*} of the observers O and O^* coincide, in other words, $x^*_1 = x_1$ and $x^*_2 = x_2$. Assume also that the piezometric head dynamics described by Equation (34) is objective. Start with a solution $h_O(t_M, x_1, x_2)$ of Equation (34) and consider for this particular situation the function $h_{O^*}(t^*_M, x_1, x_2)$, defined by (14). Note that, in this particular situation, the following equalities hold:

$${}^{R-L}D_{t^*_M}^\alpha h_{O^*}(t^*_M, x_1, x_2) = {}^{R-L}D_{t_M}^\alpha h_O(t_M, x_1, x_2) + \frac{1}{\Gamma(1-\alpha)} \cdot \frac{\partial}{\partial t^*_M} \int_0^{t^*_M} \frac{h_{O^*}(\tau^*, x_1, x_2)}{(t^*_M - \tau^*)^\alpha} d\tau^* \quad (36)$$

$$T \cdot \left(\frac{\partial^2 h_O}{\partial^2 x_1} + \frac{\partial^2 h_O}{\partial^2 x_2} \right) = T \cdot \left(\frac{\partial^2 h_{O^*}}{\partial^2 x^*_1} + \frac{\partial^2 h_{O^*}}{\partial^2 x^*_2} \right)$$

Using (36) and replacing the terms in (35) it follows that:

if the function $h_{O^*}(t^*_M, x_1, x_2)$ (defined by (14)) verifies Equation (35), then the following equality holds:

$$\frac{1}{\Gamma(1-\alpha)} \cdot \frac{\partial}{\partial t^*_M} \int_0^{t^*_M} \frac{h_{O^*}(\tau^*, x_1, x_2)}{(t^*_M - \tau^*)^\alpha} d\tau^* = 0 \quad (37)$$

Equation (37) is a consequence of the assumption that the mathematical description (34) is objective. However, generally, (37) is not verified. This means that the assumption that the piezometric head dynamics described by Equation (34) is objective, is false. It follows that the mathematical description with Equation (34) is non-objective. That is, observers O and O^* , describing the same piezometric head dynamics, with (34) and (35), respectively, obtain different results, that cannot be reconciled, in other words, transformed into each other, using Equations (1) and (2): that link the numbers representing a moment in time for two different choices of the origin of time measurement. The problem is to find which one of the results is correct.

4. In Case of an Unconfined Horizontal Aquifer the Impurity Spread Description which Uses Temporal Caputo fractional Order Partial Derivatives, with Integral Representation on a Finite Interval, Is Non-Objective

Assume that in the impurity spread dynamics description the temporal Caputo fractional partial derivative of order α , $0 < \alpha < 1$, with integral representation on a finite interval, is used. In case of the 2D flow to a well, in a horizontal unconfined isotropic homogeneous aquifer, Equation (23) for observer O and Equation (24) for observer O^* becomes:

$$C_0 D_{t_M}^\alpha C_O = D \cdot \sum_{i=1}^{i=2} \frac{\partial^2 C_O}{\partial x_i^2} - \sum_{i=1}^{i=2} U_{O_i}(t_M, x_1, x_2) \cdot \frac{\partial C_O}{\partial x_i} + S \quad (38)$$

$$\frac{\partial C_{O^*}}{\partial t^{*M}} = D \cdot \sum_{i=1}^{i=2} \frac{\partial^2 C_{O^*}}{\partial x^{*i}{}^2} - \sum_{i=1}^{i=\#} U_{O^*i}(t^{*M}, x^{*1}, x^{*2}) \cdot \frac{\partial C_{O^*}}{\partial x^{*i}} + S \tag{39}$$

Objectivity of the impurity spread description means that the solutions of the fractional partial differential Equations (38) and (39) describe the same dynamics.

That is:

if $C_O(t_M, x_1, x_2)$ verifies Equation (38), then the function $C_{O^*}(t^{*M}, x^{*1}, x^{*2})$, defined by (21), verifies Equation (39) and if $C_{O^*}(t^{*M}, x^{*1}, x^{*2})$; verifies Equation (39), then the function, $C_O(t_M, x_1, x_2)$ defined by (22), verifies Equation (38).

Assume that the reference frames R_O and R_{O^*} of the observers O and O^* coincide, in other words, $x^{*1} = x_1$ and $x^{*2} = x_2$. Also assume that the impurity spread dynamics described with Equation (38) is objective. Start with a solution $C_O(t_M, x_1, x_2)$ of Equation (38) and consider for this particular situation the function $C_{O^*}(t^{*M}, x_1, x_2)$, defined by (21). Note that, in this particular situation, the following equalities hold:

$${}^C_0D_{t^*M}^\alpha C_{O^*}(t^{*M}, x_1, x_2) = {}^C_0D_{t_M}^\alpha C_O + \frac{1}{\Gamma(1-\alpha)} \cdot \int_0^{t^*M_O} \frac{\frac{\partial C_{O^*}}{\partial \tau^*}(\tau^*, x_1, x_2)}{(t^*M - \tau^*)^\alpha} d\tau^* \tag{40}$$

$$D \cdot \sum_{i=1}^{i=2} \frac{\partial^2 C_O}{\partial x_i^2} - \sum_{i=1}^{i=2} U_{O_i}(t_M, x_1, x_2) \cdot \frac{\partial C_O}{\partial x_i} + S = D \cdot \sum_{i=1}^{i=2} \frac{\partial^2 C_{O^*}}{\partial x^{*i}{}^2} - \sum_{i=1}^{i=2} U_{O^*i}(t^{*M}, x^{*1}, x^{*2}) \cdot \frac{\partial C_{O^*}}{\partial x^{*i}} + S$$

Using Equation (40) and replacing the terms in (39) it follows that: if the function $C_{O^*}(t^{*M}, x_1, x_2)$ defined by (21), verifies Equation (39), then the following equality holds:

$$\frac{1}{\Gamma(1-\alpha)} \cdot \int_0^{t^*M_O} \frac{\frac{\partial C_{O^*}}{\partial \tau^*}(\tau^*, x_1, x_2)}{(t^*M - \tau^*)^\alpha} d\tau^* = 0 \tag{41}$$

Equation (41) is a consequence of the assumption that the mathematical description with Equation (38) is objective. However, generally (41) is not verified. This means that the assumption that, the impurity spread dynamics description with Equation (38) is objective, is false. It follows that the mathematical description with Equation (38) is non-objective. That is, observers O and O^* , describing the dynamics of the impurity spread, with (38) and (39), respectively, obtain different results which cannot be reconciled, in other words, transformed into each other, using Equations (1) and (2): that link the numbers representing a moment in time for two different choices of the origin of time measurement. The problem is to find which one of the results is correct.

5. In Case of an Unconfined Horizontal Aquifer, the Impurity Spread Description which Uses Temporal Riemann–Liouville Fractional Order Partial Derivatives, with Integral Representation on a Finite Interval, Is Non-Objective

Assume that, in the impurity spread dynamics description the temporal Riemann–Liouville fractional partial derivative of order α , $0 < \alpha < 1$, with integral representation on finite interval, is used. In case of the 2D flow to a well, in a horizontal unconfined isotropic homogeneous aquifer, Equation (23) for observer O and Equation (24) for observer O^* become:

$${}^{R-L}_0D_{t_M}^\alpha C_O = D \cdot \sum_{i=1}^{i=2} \frac{\partial^2 C_O}{\partial x_i^2} - \sum_{i=1}^{i=2} U_{O_i}(t_M, x_1, x_2) \cdot \frac{\partial C_O}{\partial x_i} + S \tag{42}$$

$${}^{R-L}D_{t^*M} = D \cdot \sum_{i=1}^{i=2} \frac{\partial^2 C_{O^*}}{\partial x^{*i}{}^2} - \sum_{i=1}^{i=\#} U_{O^*i}(t^{*M}, x^{*1}, x^{*2}) \cdot \frac{\partial C_{O^*}}{\partial x^{*i}} + S \tag{43}$$

Objectivity of the impurity spread description means that the solutions of the fractional partial differential Equations (42) and (43) describe the same dynamics.

That is:

if $C_O(t_M, x_1, x_2)$ verifies Equation (42), then the function $C_{O^*}(t^*_M, x^*_1, x^*_2)$, defined by (21), verifies Equation (43) and if $C_{O^*}(t^*_M, x^*_1, x^*_2)$ verifies Equation (43), then the function $C_O(t_M, x_1, x_2)$, defined by (22), verifies Equation (42).

Assume that the reference frames R_O and R_{O^*} of the observers O and O^* coincide, in other words, $x^*_1 = x_1$ and $x^*_2 = x_2$. Also assume, that the impurity spread dynamics described by Equation (42) is objective. Start with a solution $C_O(t_M, x_1, x_2)$ of Equation (42) and consider for this particular situation the function $C_{O^*}(t^*_M, x_1, x_2)$, defined by (21). Note that, in this particular situation, the following equalities hold:

$$\begin{aligned} {}^{R-L}{}_0D_{t^*_M}{}^\alpha C_{O^*}(t^*_M, x_1, x_2) &= {}^{R-L}{}_0D_{t_M}{}^\alpha C_O(t_M, x_1, x_2) + \\ &\frac{1}{\Gamma(1-\alpha)} \cdot \frac{\partial}{\partial t^*_M} \int_0^{t^*_M} \frac{C_{O^*}(\tau^*, x_1, x_2)}{(t^*_M - \tau^*)^\alpha} d\tau^* \quad (44) \\ D \cdot \sum_{i=1}^{i=2} \frac{\partial^2 C_O}{\partial x_i^2} - \sum_{i=1}^{i=2} U_{O_i}(t_M, x_1, x_2) \cdot \frac{\partial C_O}{\partial x_i} + S &= D \cdot \sum_{i=1}^{i=2} \frac{\partial^2 C_{O^*}}{\partial x_i^2} - \sum_{i=1}^{i=2} U_{O^*_i}(t^*_M, x^*_1, x^*_2) \cdot \frac{\partial C_{O^*}}{\partial x_i} + S \end{aligned}$$

Using Equation (44) and replacing the terms in (43), it follows that: if the functions $C_{O^*}(t^*_M, x_1, x_2)$, defined by (21), verify Equation (43), then the following equality holds:

$$\frac{1}{\Gamma(1-\alpha)} \cdot \frac{\partial}{\partial t^*_M} \int_0^{t^*_M} \frac{C_{O^*}(\tau^*, x_1, x_2)}{(t^*_M - \tau^*)^\alpha} d\tau^* = 0 \quad (45)$$

Equation (45) is a consequence of the assumption that the mathematical description with Equation (42) is objective. However, generally (45) is not verified. This means, the assumption that the impurity spread dynamics described by Equation (42) is objective, is false. It follows that the mathematical description with Equation (42) is non-objective. That is, observers O and O^* , describing the dynamics of the impurity spread, with (42) and (43), respectively, obtain different results which cannot be reconciled, in other words, transformed into each other, using Equations (1) and (2): that link the numbers that represent a moment in time for two different choices of the origin of time measurement. The problem is to find which one of the results is correct.

6. Conclusions and Comments

1. Mathematical descriptions of the bulk groundwater flow to well in a horizontal unconfined aquifer and that of the spread of the contained impurity, which use integer-order partial derivatives are objective. This means, that the results obtained by different observers can be reconciled, in other words, transformed into each other, using Equations (1)–(4) that link the coordinates of a point in two fixed orthogonal reference frames and formulas that link the numbers representing a moment in time for two different choices of the origin of time measurement.
2. Mathematical descriptions of the bulk groundwater flow to a well in a horizontal unconfined aquifer and that of the spread of the contained impurity which use temporal Caputo or Riemann–Liouville fractional order partial derivatives, having integral representation on a finite interval, are non-objective, in other words, they depend on the choice of the origin of time measurement. Due to that, two observers describing the groundwater flow and spread of impurity with these tools, generally obtain different results that cannot be reconciled, in other words, transformed into each other using Equations (1) and (2) that link the numbers representing a moment in time for two different choices of the origin of time measurement. This is not an academic curiosity, it is rather a challenge to find which one of the reported results is correct.

3. The results obtained by us in Sections 2 and 3 can be instructive for the authors of some of the papers [8–15], because they show that the use of temporal Caputo and Riemann–Liouville fractional partial derivatives affect the objectivity of the description of the flow in porous media. It is an argument for why the analysis of the objectivity of the mathematical description of the flow in porous media proposed in the papers that use temporal Caputo and Riemann–Liouville fractional order partial derivatives, having integral representation on a finite interval, is necessary. The results obtained by us in Sections 4 and 5 can be instructive for the authors of some of the papers [16–21], because they show that the use of temporal Caputo and Riemann–Liouville fractional partial derivatives affect the objectivity of the description of the spread of impurities in porous media. There is an argument for why the analysis of the objectivity of the mathematical description of the spread of impurities in porous media proposed in the papers that use temporal Caputo and Riemann–Liouville fractional order partial derivatives, having integral representation on a finite interval, is necessary.
4. In the early 2000s a discussion started about the initialization problems in [23–28]. Some published results in [26] and [27] concluded in the inconsistency of Caputo and Riemann–Liouville’s definition to take into account initial conditions if these definitions are used in fractional partial differential equations or in ordinary differential equations. In [23], [26], and [27] a time shift was used to highlight the above mentioned problem. Our approach to the question: why can integer-order derivatives not simply be replaced by fractional-order derivatives to develop the fractional-order theories? is different. What we know from the scientific literature is that the assertion “integer-order derivatives cannot be simply replaced by fractional-order derivatives to develop the fractional-order theories” has not been proven so far. In fact, the general assertion, as formulated, refers to all the equations of mathematical physics and we do not think it will be proven soon. However, what we think is fact is that this statement can be demonstrated in some proper cases. In this paper we actually demonstrate this statement in the case of describing fluid flow in porous media and impurity spread also showing the cause, in other words, that the objectivity of the description is lost.
5. A given mathematical tool is not necessarily appropriate for the mathematical description of a certain real word phenomenon.

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