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Modified Mittag-Leffler Functions with Applications in Complex Formulae for Fractional Calculus

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Abstract: Mittag-Leffler functions and their variations are a popular topic of study at the present time, mostly due to their applications in fractional calculus and fractional differential equations. Here we propose a modification of the usual Mittag-Leffler functions of one, two, or three parameters, which is ideally suited for extending certain fractional-calculus operators into the complex plane. Complex analysis has been underused in combination with fractional calculus, especially with newly developed operators like those with Mittag-Leffler kernels. Here we show the natural analytic continuations of these operators using the modified Mittag-Leffler functions defined in this paper.

Keywords: Mittag-Leffler functions; Prabhakar fractional calculus; Atangana–Baleanu fractional calculus; complex integrals; analytic continuation

MSC: 33E12; 26A33; 30B40

1. Introduction

The study of special functions has been a significant subfield of mathematical analysis for decades, connecting with other areas such as differential equations, fractional calculus, and mathematical physics [1–3]. One important class of special functions consists of the so-called Mittag-Leffler function and its extensions. These have been intensively studied, with at least one whole textbook dedicated to them [4], along with many book chapters and important research papers [5–8]. They are particularly useful due to their connections with fractional calculus, having been called "fractional exponential functions" and arising naturally in solutions to various fractional differential equations [7,9,10], including some which are useful in applications such as viscoelasticity and evolution processes [11,12].

The original Mittag-Leffler function $E_\alpha(z)$ depends on one variable z and one parameter α, and it is defined by [13]

$$E_\alpha(z)=\sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+1)}, \tag{1}$$

where the series is locally uniformly convergent for any $z\in\mathbb{C}$ and any $\alpha\in\mathbb{C}$ with $\mathrm{Re}(\alpha)>0$.

This definition has been extended in various ways. The best-known extensions are the functions $E_{\alpha,\beta}(z)$ and $E^{\gamma}_{\alpha,\beta}(z)$, depending on one variable z and two or three parameters α, β, and γ. These are defined as follows [4,14]:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}, \tag{2}$$

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \binom{-\gamma}{n} \cdot \frac{(-z)^n}{\Gamma(n\alpha + \beta)} = \sum_{n=0}^{\infty} \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)\Gamma(n\alpha + \beta)} \cdot \frac{z^n}{n!}, \tag{3}$$

where again both series are locally uniformly convergent for any $z \in \mathbb{C}$ and any $\alpha, \beta, \gamma \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$. Other extensions involve even more than three parameters, or replacing the single variable z by multiple variables [10,15–17].

In fractional calculus—the study of the integral and derivative operators of calculus taken to non-integer orders [18–20]—most studies take place only in the real line. The standard Riemann–Liouville definition of a fractional integral to order α is

$${}^{RL}_{\ a}I^{\alpha}_{x} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - \xi)^{\alpha-1} f(\xi) \,\mathrm{d}\xi,$$

where $f(x)$ is a function defined on a real interval $x \in [a, b]$ but α is permitted to be complex (with positive real part). The fractional derivative is then defined as an extension of this, by means of the following formula for $\mathrm{Re}(\alpha) \geq 0$:

$${}^{RL}_{\ a}D^{\alpha}_{x} f(x) = \frac{\mathrm{d}^m}{\mathrm{d}x^m} \, {}^{RL}_{\ a}I^{m-\alpha}_{x} f(x), \qquad m := \lfloor \mathrm{Re}(\alpha) \rfloor + 1.$$

By treating the parameter α as an independent complex variable, it can be shown that ${}^{RL}_{\ a}D^{-\alpha}_{x} f(x) = {}^{RL}_{\ a}I^{\alpha}_{x} f(x)$ is an analytic extension of ${}^{RL}_{\ a}I^{\alpha}_{x} f(x)$ from the right half-plane to the left one.

For analytic complex-valued functions f, there is another formula equivalent to Riemann–Liouville which is more useful in the context of complex analysis [19,21]. Namely, the fractional differintegral (valid for all $\alpha \in \mathbb{C} \backslash \mathbb{Z}^-$) of $f(z)$ is

$${}^{\mathbb{C}}_{a}D^{\alpha}_{z} f(z) = \frac{\Gamma(\alpha + 1)}{2i\pi} \int_{H^z_a} (\zeta - z)^{-\alpha-1} f(\zeta) \,\mathrm{d}\zeta, \tag{4}$$

where the complex contour of integration H^z_a is the Hankel-type contour which starts above a on the branch cut from z, wraps around z in a counterclockwise sense, and returns to a.

There are many other ways to define fractional integrals and fractional derivatives, often inspired by or related to the Riemann–Liouville definition. Some of these are discussed in [22–24], with reference to some general classes into which such operators can be classified. In pure mathematics, ideally we consider the most general possible setting in which a particular result or behaviour can be proved. In applications, of course it is necessary to consider specific types of fractional calculus for the modelling of a given real-world problem.

We have already mentioned how Mittag-Leffler functions emerge naturally from the study of fractional calculus and fractional differential equations. They also appear frequently as the kernels of fractional integral and derivative operators. Many such operators are special cases of the Prabhakar fractional calculus [14,25], which is based on the 3-parameter Mittag-Leffler function (3), and which itself can be seen as a special case of some even more general operators [17,26].

Some of these special cases were defined without realising them as special cases, and hence they were given their own names independently. Among the most intensively used types of fractional calculus in the last few years are the so-called Atangana–Baleanu operators, defined in [27] using the 1-parameter Mittag-Leffler function (1). Although integral operators using 1-parameter Mittag-Leffler kernels were already considered years earlier [28–33], the so-called AB operators have become very popular with over 350 papers published on them between 2016 and April 2020 [34]. One mathematical development in this setting has been, in [35], the extension of complex contour integral formulae like (4) to other types of fractional calculus. Doing this for AB derivatives involved the introduction of a modified Mittag-Leffler function, related but different to the original function defined by (1).

In the current work, we seek to extend the notion of this modified 1-parameter Mittag-Leffler function to define similarly modified Mittag-Leffler functions with two and three parameters. We shall perform a rigorous analysis of these modified Mittag-Leffler functions, their domains and convergence properties, and use them to extend the Atangana–Baleanu and Prabhakar fractional-calculus operators into the setting of complex variables.

Specifically, the organisation of this paper is as follows. Section 2 introduces the modified Mittag-Leffler functions, firstly re-checking the 1-parameter function in Section 2.1 and then defining the 2-parameter and 3-parameter extensions in Section 2.2. Section 3 examines how they may be used in fractional calculus, firstly for the Prabhakar operators in Section 3.1 and then for the Atangana–Baleanu operators in Section 3.2, with some further related remarks about extensions of fractional-calculus operators in Section 3.3. Finally, Section 4 concludes the paper.

2. Modified Mittag-Leffler Functions

2.1. A Rigorous Recap of the 1-Parameter Case

In this section, we re-analyse the 1-parameter modified Mittag-Leffler function defined in [35]. It is necessary to do this because there were some omissions in the work of [35]: specifically, the problems arising from the $n = 0$ term. In fact, the function cannot actually be defined in exactly the way it was in [35], because $\Gamma(-n\alpha)$ is not defined at $n = 0$. Therefore, we consider here a slightly different version which starts from $n = 1$.

Definition 1 ([35])**.** *The modified Mittag-Leffler function $E^{\alpha}(z)$ is defined by the following series for all $z \in \mathbb{C}$ and $\alpha \in \mathbb{C}\backslash\mathbb{R}$ with $\mathrm{Re}(\alpha) > 0$:*

$$E^{\alpha}(z) = \sum_{n=1}^{\infty} \Gamma(-n\alpha) z^n, \tag{5}$$

and by analytic continuation for all $\alpha \in \mathbb{C}\backslash\mathbb{R}$.

The reason for defining and studying this function is only to demonstrate the convergence principles and methods which will then be used for the 2-parameter and 3-parameter modified Mittag-Leffler functions in Section 2.2 below. In itself, this function may not be important, because of the missing $n = 0$ term, but we can see it as a practice "toy" case for establishing the ideas to be used later.

Of course, changing the definition of the modified 1-parameter Mittag-Leffler function will affect the results of [35] on the Atangana–Baleanu fractional derivatives. We resolve this issue in Section 3 below by finding new complex contour formulae for the Atangana–Baleanu fractional derivatives.

The following result was already proved in [35]. We reproduce the proof here, with a little more detail, and also give an alternative method of proof which will be useful later in this paper.

Proposition 1 ([35])**.** *The infinite power series (5) is locally uniformly convergent for all $z \in \mathbb{C}$, for any fixed $\alpha \in \mathbb{C}\backslash\mathbb{R}$ with $\mathrm{Re}(\alpha) > 0$.*

Proof using reflection formula [35]. We rewrite the power series (5) using the reflection formula for the gamma function:

$$\begin{aligned} E^{\alpha}(z) &= \sum_{n=1}^{\infty} \frac{\pi}{\sin(-\pi n\alpha)} \cdot \frac{1}{\Gamma(n\alpha+1)} z^n \\ &= 2\pi i \sum_{n=1}^{\infty} \frac{1}{\exp(-i\pi n\alpha) - \exp(i\pi n\alpha)} \cdot \frac{z^n}{\Gamma(n\alpha+1)}. \end{aligned}$$

The latter series is identical to the original Mittag-Leffler function (1) except for the extra factor $\frac{1}{\exp(-i\pi n\alpha)-\exp(i\pi n\alpha)}$. We split into two cases to consider the behaviour of this factor:

- If $\mathrm{Im}(\alpha) > 0$, then $\exp(-i\pi n\alpha) - \exp(i\pi n\alpha) \sim \exp(-i\pi n\alpha)$ and so $\frac{1}{\exp(-i\pi n\alpha)-\exp(i\pi n\alpha)} \sim \exp(i\pi n\alpha)$ has exponential decay.
- If $\mathrm{Im}(\alpha) < 0$, then $\exp(-i\pi n\alpha) - \exp(i\pi n\alpha) \sim \exp(i\pi n\alpha)$ and so $\frac{1}{\exp(-i\pi n\alpha)-\exp(i\pi n\alpha)} \sim \exp(-i\pi n\alpha)$ has exponential decay.

Either way, for $\mathrm{Re}(\alpha) > 0$ and $\alpha \notin \mathbb{R}$, the series converges absolutely and locally uniformly, just like the original series (1). □

Proof using ratio test and Stirling's formula. This is a more elementary way to prove convergence of a power series, going back to basics with the ratio test instead of relying on knowledge of the series for the original Mittag-Leffler function. We use Stirling's formula for the asymptotics of the gamma functions for large n; the ratio between consecutive terms is

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{\Gamma(-n\alpha-\alpha)}{\Gamma(-n\alpha)}z \\ &\sim \frac{\sqrt{\frac{2\pi}{-n\alpha-\alpha}}\left(\frac{-n\alpha-\alpha}{e}\right)^{-n\alpha-\alpha}}{\sqrt{\frac{2\pi}{-n\alpha}}\left(\frac{-n\alpha}{e}\right)^{-n\alpha}}z \\ &\sim \left(\frac{n+1}{n}\right)^{-n\alpha}\left(\frac{-\alpha}{e}\right)^{-\alpha}(n+1)^{-\alpha}z \sim \Big(-\alpha(n+1)\Big)^{-\alpha}z\end{aligned}$$

as $n \to \infty$. The limit is zero if $\mathrm{Re}(\alpha) > 0$, so in this case the series converges absolutely and locally uniformly as required. We still require the assumption $\alpha \notin \mathbb{R}$ to avoid having any zero terms. □

The following result was stated in [35], but the proof was only outlined. We present here the complete proof.

Proposition 2 ([35])**.** *The modified Mittag-Leffler function $E^{\alpha}(z)$, defined for $\mathrm{Re}(\alpha) > 0$ by Definition 1, has an analytic continuation to all $\alpha \in \mathbb{C}\backslash\mathbb{R}$ given by the following complex integral:*

$$E^{\alpha}(z) = \frac{1}{-2i}\int_H e^t t^{-1}\mathfrak{S}_{\alpha}\left(zt^{-\alpha}\right)\mathrm{d}t,$$

where H is the standard Hankel contour (starting and ending at negative real infinity and wrapping counterclockwise around the origin) and $\mathfrak{S}_{\alpha}$ is the function defined by

$$\mathfrak{S}_{\alpha}(x) = \sum_{n=1}^{\infty}\frac{x^n}{\sin(\pi n\alpha)}, \qquad x \in \mathbb{R}, \alpha \in \mathbb{C}\backslash\mathbb{R}.$$

Proof. We follow the method of [36], and proceed as follows using the standard contour integral representation of the inverse gamma function:

$$\begin{aligned}E^{\alpha}(z) &= \sum_{n=1}^{\infty}\frac{\pi}{\sin(-\pi n\alpha)}\cdot\frac{z^n}{\Gamma(n\alpha+1)} \\ &= \sum_{n=1}^{\infty}\frac{\pi z^n}{\sin(-\pi n\alpha)}\cdot\frac{1}{2\pi i}\int_H e^t t^{-n\alpha-1}\,\mathrm{d}t \\ &= \frac{1}{-2i}\sum_{n=1}^{\infty}\int_H e^t t^{-n\alpha-1}\frac{z^n}{\sin(\pi n\alpha)}\,\mathrm{d}t \\ &= \frac{1}{-2i}\int_H e^t t^{-1}\sum_{n=1}^{\infty}\frac{\left(zt^{-\alpha}\right)^n}{\sin(\pi n\alpha)}\,\mathrm{d}t,\end{aligned}$$

where the interchange of summation and integration is permitted by locally uniform convergence of the series. Note that locally uniform convergence of the series for $\mathfrak{S}_\alpha$ is guaranteed by the ratio test combined with an exponential-decay argument for dividing by a sine function similar to that in the first proof of Proposition 1 above. □

2.2. Extension to the 2-Parameter and 3-Parameter Cases

The original Mittag-Leffler function (1) has been modified, using the functional equation for the gamma function, as described in Definition 1 and the subsequent discussion. This modified Mittag-Leffler function $E^\alpha(z)$ depends on one variable z and one parameter α, just like the original Mittag-Leffler function $E_\alpha(z)$.

In a similar way, it is also possible to modify the 2-parameter Mittag-Leffler function (2) and the 3-parameter Mittag-Leffler function (3), thereby obtaining modified Mittag-Leffler functions with two and three parameters. We start with the 2-parameter version.

Definition 2. *The modified 2-parameter Mittag-Leffler function $E^{\alpha,\beta}(z)$ is defined by the following series for all $z \in \mathbb{C}$ and $\alpha, \beta \in \mathbb{C}$ satisfying $\mathrm{Re}(\alpha) > 0$ and α, β not both real and $n\alpha + \beta \notin \mathbb{N}$ for any $n \in \mathbb{N}$:*

$$E^{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \Gamma(1 - n\alpha - \beta) z^n \tag{6}$$

and by analytic continuation for all $\alpha, \beta \in \mathbb{C}$ satisfying α, β not both real and $n\alpha + \beta \notin \mathbb{N}$ for any $n \in \mathbb{N}$.

Theorem 1. *The infinite power series (6) is locally uniformly convergent for all $z \in \mathbb{C}$, for any fixed $\alpha, \beta \in \mathbb{C}$ satisfying $\mathrm{Re}(\alpha) > 0$ and α, β not both real and $n\alpha + \beta \notin \mathbb{N}$ for any $n \in \mathbb{N}$.*

Proof using reflection formula. This follows similar lines as the first proof of Proposition 1:

$$\begin{aligned} E^{\alpha,\beta}(z) &= \sum_{n=0}^{\infty} \frac{\pi}{\sin(\pi(n\alpha + \beta))} \cdot \frac{z^n}{\Gamma(n\alpha + \beta)} \\ &= 2\pi i \sum_{n=0}^{\infty} \frac{1}{\exp(i\pi(n\alpha + \beta)) - \exp(-i\pi(n\alpha + \beta))} \cdot \frac{z^n}{\Gamma(n\alpha + \beta)}, \end{aligned}$$

where this series is identical to the original Mittag-Leffler function (2) except for the extra factor involving two exponential functions in the denominator. We split into three cases to consider the behaviour of this factor:

- If $\mathrm{Im}(\alpha) > 0$, then $\exp(i\pi(n\alpha + \beta)) - \exp(-i\pi(n\alpha + \beta)) \sim \exp(-i\pi(n\alpha + \beta))$ for sufficiently large n, and so
$$\frac{1}{\exp(i\pi(n\alpha + \beta)) - \exp(-i\pi(n\alpha + \beta))} \sim \exp(i\pi(n\alpha + \beta))$$
has exponential decay as $n \to \infty$.
- If $\mathrm{Im}(\alpha) < 0$, then $\exp(i\pi(n\alpha + \beta)) - \exp(-i\pi(n\alpha + \beta)) \sim \exp(i\pi(n\alpha + \beta))$ for sufficiently large n, and so
$$\frac{1}{\exp(i\pi(n\alpha + \beta)) - \exp(-i\pi(n\alpha + \beta))} \sim \exp(-i\pi(n\alpha + \beta))$$
has exponential decay as $n \to \infty$.
- If $\mathrm{Im}(\alpha) = 0$, then $\mathrm{Im}(\beta) \neq 0$ by assumption. The extra term is bounded by a constant as $n \to \infty$, namely, either
$$\frac{1}{\exp(\pi \mathrm{Im}\beta) - \exp(-\pi \mathrm{Im}\beta)} \quad \text{or} \quad \frac{1}{\exp(-\pi \mathrm{Im}\beta) - \exp(\pi \mathrm{Im}\beta)}$$

according to the sign of $\mathrm{Im}(\beta)$.

In any case, provided $\mathrm{Re}(\alpha) > 0$ so that the original series (2) converges, the new series (6) also converges absolutely and locally uniformly. We assume throughout that the bottom never cancels out exactly to zero; i.e., that $n\alpha + \beta$ is never an integer for any n. □

Proof using ratio test and Stirling's formula. Again the calculations here are similar to those in the second proof of Proposition 1:

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{\Gamma(1-(n+1)\alpha-\beta)}{\Gamma(1-n\alpha-\beta)} \\
&\sim \frac{\left(\frac{-(n+1)\alpha-\beta}{e}\right)^{-(n+1)\alpha-\beta}\sqrt{2\pi(-(n+1)\alpha-\beta)}}{\left(\frac{-n\alpha-\beta}{e}\right)^{-n\alpha-\beta}\sqrt{2\pi(-n\alpha-\beta)}} \\
&\sim \left(\frac{-(n+1)\alpha-\beta}{e}\right)^{-\alpha}\left(\frac{(n+1)\alpha+\beta}{n\alpha+\beta}\right)^{-\beta}\left(\frac{(n+1)\alpha+\beta}{n\alpha+\beta}\right)^{-n\alpha} \\
&\sim \frac{e^{\alpha}}{(-(n+1)\alpha-\beta)^{\alpha}}\left(\frac{n\alpha+\beta}{(n+1)\alpha+\beta}\right)^{n\alpha} \\
&\sim \frac{1}{\big(-(n+1)\alpha-\beta\big)^{\alpha}}
\end{aligned}$$

as $n \to \infty$. The limit is zero if $\mathrm{Re}(\alpha) > 0$, so in this case the series converges absolutely and locally uniformly as required. We still require the assumption $n\alpha + \beta \notin \mathbb{N}$ to avoid having any zero terms. □

Theorem 2 (Complex integral representation of modified 2-parameter Mittag-Leffler function)**.** *The modified 2-parameter Mittag-Leffler function, defined above under the assumption* $\mathrm{Re}(\alpha) > 0$*, has an analytic continuation to* $\alpha, \beta \in \mathbb{C}$ *satisfying* α, β *not both real and* $n\alpha + \beta \notin \mathbb{N}$ *for any* $n \in \mathbb{N}$*, given by the following complex integral:*

$$E^{\alpha,\beta}(z) = \frac{1}{2i}\int_H e^t t^{-\beta}\mathfrak{S}_{\alpha,\beta}\left(zt^{-\alpha}\right)\mathrm{d}t,$$

where H *is the standard Hankel contour (starting and ending at negative real infinity and wrapping counterclockwise around the origin) and* $\mathfrak{S}_{\alpha,\beta}$ *is the function defined by*

$$\mathfrak{S}_{\alpha,\beta}(x) = \sum_{n=0}^{\infty}\frac{x^n}{\sin\left(\pi(n\alpha+\beta)\right)}, \qquad \alpha, \beta \text{ not both real, } n\alpha+\beta \notin \mathbb{N}\ \forall n.$$

Proof. Similarly to Proposition 2, we use the contour integral representation of the inverse gamma function:

$$\begin{aligned}
E^{\alpha,\beta}(z) &= \sum_{n=0}^{\infty}\Gamma(1-n\alpha-\beta)z^n \\
&= \sum_{n=0}^{\infty}\frac{\pi}{\sin(\pi(n\alpha+\beta))}\frac{z^n}{\Gamma(n\alpha+\beta)} \\
&= \sum_{n=0}^{\infty}\frac{\pi z^n}{\sin(\pi(n\alpha+\beta))}\cdot\frac{1}{2\pi i}\int_H e^t t^{-n\alpha-\beta}\,\mathrm{d}t \\
&= \frac{1}{2i}\int_H e^t t^{-\beta}\left[\sum_{n=0}^{\infty}\frac{(zt^{-\alpha})^n}{\sin(\pi(n\alpha+\beta))}\right]\mathrm{d}t \\
&= \frac{1}{2i}\int_H e^t t^{-\beta}\mathfrak{S}_{\alpha,\beta}\left(zt^{-\alpha}\right)\mathrm{d}t,
\end{aligned}$$

as required, where the interchange of summation and integration is permitted by locally uniform convergence of the series. Note that, as before, locally uniform convergence of the series for $\mathfrak{S}_{\alpha,\beta}$ is guaranteed by the ratio test combined with an exponential-decay argument for dividing by a sine function similar to that in the first proof of Theorem 1 above. □

Definition 3. *The modified 3-parameter Mittag-Leffler function $E_{\gamma}^{\alpha,\beta}(z)$ is defined by the following series for all $z \in \mathbb{C}$ and $\alpha, \beta, \gamma \in \mathbb{C}$ satisfying $\mathrm{Re}(\alpha) > 0$ and α, β not both real and $n\alpha + \beta \notin \mathbb{N}$ for any $n \in \mathbb{N}$:*

$$E_{\gamma}^{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{n!} \Gamma(1 - n\alpha - \beta) z^n = \sum_{n=0}^{\infty} \frac{\Gamma(\gamma + n)}{\Gamma(\gamma) n!} \Gamma(1 - n\alpha - \beta) z^n \tag{7}$$

and by analytic continuation for all $\alpha, \beta, \gamma \in \mathbb{C}$ satisfying α, β not both real and $n\alpha + \beta \notin \mathbb{N}$ for any $n \in \mathbb{N}$.

Theorem 3. *The infinite power series (7) is locally uniformly convergent for all $z \in \mathbb{C}$, for any fixed $\alpha, \beta, \gamma \in \mathbb{C}$ satisfying $\mathrm{Re}(\alpha) > 0$ and α, β not both real and $n\alpha + \beta \notin \mathbb{N}$ for any $n \in \mathbb{N}$.*

Proof. This follows almost directly from the result of Theorem 1, either by the first method (reflection formula) or by the second method (ratio test).

Using the reflection formula, we find

$$E_{\gamma}^{\alpha,\beta}(z) = 2\pi i \sum_{n=0}^{\infty} \frac{1}{\exp(i\pi(n\alpha + \beta)) - \exp(-i\pi(n\alpha + \beta))} \cdot \frac{\Gamma(\gamma + n) z^n}{\Gamma(\gamma)\Gamma(n\alpha + \beta) n!}$$

which is identical to the original 3-parameter Mittag-Leffler series (3) except for the extra factor which is exactly the same as in Theorem 1 and therefore gives the same convergence properties under the same conditions.

Using the ratio test, we find

$$\frac{a_{n+1}}{a_n} = \frac{(\gamma + n)\Gamma(1 - (n+1)\alpha - \beta)}{(n+1)\Gamma(1 - n\alpha - \beta)} \sim \frac{\Gamma(1 - (n+1)\alpha - \beta)}{\Gamma(1 - n\alpha - \beta)},$$

which is exactly the same as in Theorem 1 and therefore gives the same convergence properties under the same conditions. □

Theorem 4 (Complex integral representation of modified 3-parameter Mittag-Leffler function)**.** *The modified 3-parameter Mittag-Leffler function, defined above under the assumption $\mathrm{Re}(\alpha) > 0$, has an analytic continuation to $\alpha, \beta, \gamma \in \mathbb{C}$ satisfying α, β not both real and $n\alpha + \beta \notin \mathbb{N}$ for any $n \in \mathbb{N}$, given by the following complex integral:*

$$E_{\gamma}^{\alpha,\beta}(z) = \frac{1}{2i} \int_H e^t t^{-\beta} \mathfrak{S}_{\alpha,\beta}^{\gamma}(z t^{-\alpha}) \, \mathrm{d}t,$$

where H is the standard Hankel contour (starting and ending at negative real infinity and wrapping counterclockwise around the origin) and $\mathfrak{S}_{\alpha,\beta}^{\gamma}$ is the function defined by

$$\mathfrak{S}_{\alpha,\beta}^{\gamma}(x) = \sum_{n=0}^{\infty} \frac{\Gamma(\gamma + n) x^n}{\Gamma(\gamma) n! \sin\left(\pi(n\alpha + \beta)\right)}, \qquad \alpha, \beta \text{ not both real, } n\alpha + \beta \notin \mathbb{N}\ \forall n.$$

Proof. Similarly to Proposition 2 and Theorem 2, we use the contour integral representation of the inverse gamma function:

$$\begin{aligned}
E_{\gamma}^{\alpha,\beta}(z) &= \sum_{n=0}^{\infty} \frac{\Gamma(\gamma+n)}{\Gamma(\gamma)n!}\Gamma(1-n\alpha-\beta)z^n \\
&= \sum_{n=0}^{\infty} \frac{\pi}{\sin(\pi(n\alpha+\beta))}\frac{\Gamma(\gamma+n)z^n}{\Gamma(\gamma)\Gamma(n\alpha+\beta)n!} \\
&= \sum_{n=0}^{\infty} \frac{\pi\Gamma(\gamma+n)z^n}{\Gamma(\gamma)n!\sin(\pi(n\alpha+\beta))}\frac{1}{2\pi i}\int_H e^t t^{-n\alpha-\beta}\,\mathrm{d}t \\
&= \frac{1}{2i}\int_H e^t t^{-\beta}\left[\sum_{n=0}^{\infty}\frac{\Gamma(\gamma+n)(zt^{-\alpha})^n}{\Gamma(\gamma)n!\sin(\pi(n\alpha+\beta))}\right]\mathrm{d}t \\
&= \frac{1}{2i}\int_H e^t t^{-\beta}\mathfrak{S}_{\alpha,\beta}^{\gamma}(zt^{-\alpha})\,\mathrm{d}t,
\end{aligned}$$

where the interchange of summation and integration is permitted by locally uniform convergence of the series. Note that locally uniform convergence of the series for $\mathfrak{S}_{\alpha,\beta}^{\gamma}$ is guaranteed by the same property of $\mathfrak{S}_{\alpha,\beta}$, since the ratio test gives almost exactly the same expression for both. □

Remark 1. *The condition required in the above definitions and theorems, that $n\alpha+\beta \notin \mathbb{N}$ for any $n \in \mathbb{N}$, may at first seem to be very restrictive. However, this is simply the requirement that all the terms of the series itself are well-defined. If we ever have $n\alpha+\beta \in \mathbb{N}$ for some n, then $\Gamma(1-n\alpha-\beta)$ is not defined for this value of n, and so the series itself makes no sense. This condition is added simply to ensure that our definitions can actually make sense, even before convergence considerations.*

3. Extensions of Fractional Operators

3.1. Contour Integral Formulae for Prabhakar Fractional Operators

Definition 4 ([14,25,37])**.** *The Prabhakar fractional integral of a function $f \in L^1[a,b]$, with parameters $\alpha,\beta,\gamma,\delta \in \mathbb{C}$ satisfying $\mathrm{Re}(\alpha)>0$ and $\mathrm{Re}(\beta)>0$, is defined as*

$${}_{a}^{P}I_{x}^{\alpha,\beta,\gamma,\delta}f(x) = \int_a^x (x-\xi)^{\beta-1}E_{\alpha,\beta}^{\gamma}\big(\delta(x-\xi)^{\alpha}\big)f(\xi)\,\mathrm{d}\xi, \tag{8}$$

using the 3-parameter Mittag-Leffler function (3) as a kernel function. This operator can also be written as an infinite series of Riemann–Liouville fractional integrals, as follows:

$${}_{a}^{P}I_{x}^{\alpha,\beta,\gamma,\delta}f(x) = \sum_{n=0}^{\infty}\frac{(\gamma)_n\delta^n}{n!}\,{}_{a}^{RL}I_{x}^{n\alpha+\beta}f(x). \tag{9}$$

The Prabhakar fractional derivative of a smooth function $f(x)$, with parameters $\alpha,\beta,\gamma,\delta \in \mathbb{C}$ satisfying $\mathrm{Re}(\alpha)>0$ and $\mathrm{Re}(\beta)\geq 0$, is defined as

$${}_{a}^{P}D_{x}^{\alpha,\beta,\gamma,\delta}f(x) = \frac{\mathrm{d}^m}{\mathrm{d}x^m}\,{}_{a}^{P}I_{x}^{\alpha,m-\beta,-\gamma,\delta}f(x), \qquad m := \lfloor \mathrm{Re}(\beta)\rfloor + 1. \tag{10}$$

Using composition properties of Riemann–Liouville derivatives and integrals, this operator can be written as an infinite series similar to (9):

$${}_{a}^{P}D_{x}^{\alpha,\beta,\gamma,\delta}f(x) = \sum_{n=0}^{\infty}\frac{(-\gamma)_n\delta^n}{n!}\,{}_{a}^{RL}I_{x}^{n\alpha-\beta}f(x), \tag{11}$$

where the operator denoted by ${}^{RL}_{a}I^{n\alpha-\beta}_{x}$ is either a Riemann–Liouville integral or a Riemann–Liouville derivative depending on the sign of $\mathrm{Re}(n\alpha-\beta)$.

Note that the variable x in the above definition is assumed to be real, in the fixed interval $[a,b]$. In the previous paper [35], the Atangana–Baleanu fractional operators were extended from the real line to the complex plane, using a complex contour integral approach and the modified 1-parameter Mittag-Leffler function (5). Now we seek to do the same for the Prabhakar fractional operators, using the modified 3-parameter Mittag-Leffler function which we have defined in this paper.

Theorem 5. *The analytic continuation of the Prabhakar fractional integral is given by*

$$ {}^{P}_{a}I^{\alpha,\beta,\gamma,\delta}_{z}f(z)=\frac{1}{2\pi i}\int_{H_a^z}(\zeta-z)^{\beta-1}E^{\alpha,\beta}_{\gamma}\big(\delta(\zeta-z)^{\alpha}\big)f(\zeta)\,\mathrm{d}\zeta, \tag{12}$$

where $E^{\alpha,\beta}_{\gamma}(x)$ is the modified 3-parameter Mittag-Leffler function defined by (7) above, and the complex contour of integration H_a^z is the Hankel-type contour which starts above a on the branch cut from z, wraps around z in a counterclockwise sense, and returns to a.

This formula (12) also covers Prabhakar fractional differentiation, under the convention (following from the semigroup property and series formula) that ${}^{P}_{a}D^{\alpha,\beta,\gamma,\delta}_{x}f(x)={}^{P}_{a}I^{\alpha,-\beta,-\gamma,\delta}_{x}f(x)$. In other words, we have

$$ {}^{P}_{a}D^{\alpha,\beta,\gamma,\delta}_{z}f(z)=\frac{1}{2\pi i}\int_{H_a^z}(\zeta-z)^{\beta-1}E^{\alpha,-\beta}_{-\gamma}\big(\delta(\zeta-z)^{\alpha}\big)f(\zeta)\,\mathrm{d}\zeta.$$

The assumption on the parameters $\alpha,\beta,\gamma,\delta$ for this Theorem is that α,β not both real and $n\alpha+\beta\notin\mathbb{N}$ for any $n\in\mathbb{N}$.

Proof. We use the series formula for Prabhakar fractional calculus, noting that both (9) for integrals and (11) for derivatives become the same formula under the convention ${}^{P}_{a}D^{\alpha,\beta,\gamma,\delta}_{x}f(x)={}^{P}_{a}I^{\alpha,-\beta,-\gamma,\delta}_{x}f(x)$. We have

$$\begin{aligned}
{}^{P}_{a}I^{\alpha,\beta,\gamma,\delta}_{x}f(x)&=\sum_{n=0}^{\infty}\frac{(\gamma)_n\delta^n}{n!}\,{}^{RL}_{a}I^{n\alpha+\beta}_{x}f(x)\\
&=\sum_{n=0}^{\infty}\frac{(\gamma)_n\delta^n}{n!}\,{}^{\mathbb{C}}_{a}I^{n\alpha+\beta}_{x}f(x)\\
&=\sum_{n=0}^{\infty}\frac{(\gamma)_n\delta^n}{n!}\frac{\Gamma(1-n\alpha-\beta)}{2\pi i}\int_{H_a^z}(\zeta-z)^{n\alpha+\beta-1}f(\zeta)\,\mathrm{d}\zeta\\
&=\frac{1}{2\pi i}\int_{H_a^z}(\zeta-z)^{\beta-1}\left[\sum_{n=0}^{\infty}\frac{(\gamma)_n}{n!}\Gamma(1-n\alpha-\beta)\big(\delta(\zeta-z)^{\alpha}\big)^{n}\right]f(\zeta)\,\mathrm{d}\zeta\\
&=\frac{1}{2\pi i}\int_{H_a^z}(\zeta-z)^{\beta-1}E^{\alpha,\beta}_{\gamma}\big(\delta(\zeta-z)^{\alpha}\big)f(\zeta)\,\mathrm{d}\zeta.
\end{aligned}$$

The above manipulation is valid provided that $\mathrm{Re}(\alpha)>0$. Note that we do not need any assumption on $\mathrm{Re}(\beta)$ since the case $\mathrm{Re}(\beta)>0$ is covered by the Prabhakar fractional integral and the case $\mathrm{Re}(\beta)\le 0$ by the Prabhakar fractional derivative.

The final formula, however—the right-hand side of Equation (12)—is well-defined and analytic for any $\alpha,\beta,\gamma,\delta$ satisfying α,β not both real and $n\alpha+\beta\notin\mathbb{N}$ for any $n\in\mathbb{N}$, by the analytic continuation of $E^{\alpha,\beta}_{\gamma}$ given in Theorem 4. Therefore, (12) provides the analytic continuation of the Prabhakar fractional integral and derivative, even to the cases where $\mathrm{Re}(\alpha)>0$ no longer applies. □

3.2. Contour Integral Formulae for Atangana–Baleanu Fractional Operators

Definition 5 ([27,38])**.** *The Atangana–Baleanu fractional integral of a function $f \in L^1[a,b]$, with parameter $\alpha \in (0,1)$, is defined as*

$$ {}^{AB}_{a}I^{\alpha}_{x}f(x) = \frac{1-\alpha}{B(\alpha)}f(x) + \frac{\alpha}{B(\alpha)}\,{}^{RL}_{a}I^{\alpha}_{x}f(x), \tag{13}$$

The Atangana–Baleanu fractional derivatives of a function $f \in C^1[a,b]$, with parameter $\alpha \in (0,1)$, of Riemann–Liouville and Caputo types respectively, are defined as:

$$ {}^{ABR}_{a}D^{\alpha}_{x}f(x) = \frac{B(\alpha)}{1-\alpha}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x E_{\alpha}\left(\frac{-\alpha}{1-\alpha}(x-\xi)^{\alpha}\right)f(\xi)\,\mathrm{d}\xi, \tag{14}$$

$$ {}^{ABC}_{a}D^{\alpha}_{x}f(x) = \frac{B(\alpha)}{1-\alpha}\int_a^x E_{\alpha}\left(\frac{-\alpha}{1-\alpha}(x-\xi)^{\alpha}\right)f'(\xi)\,\mathrm{d}\xi, \tag{15}$$

using the 1-parameter Mittag-Leffler function (1) *as a kernel function. These operators can also be written as infinite series of Riemann–Liouville fractional integrals, as follows:*

$$ {}^{ABR}_{a}D^{\alpha}_{x}f(x) = \frac{B(\alpha)}{1-\alpha}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^n {}^{RL}_{a}I^{n\alpha+1}_{x}f(x) \tag{16}$$

$$ = \frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^n {}^{RL}_{a}I^{n\alpha}_{x}f(x), \tag{17}$$

$$ {}^{ABC}_{a}D^{\alpha}_{x}f(x) = \frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^n {}^{RL}_{a}I^{n\alpha+1}_{x}f'(x). \tag{18}$$

In the paper [35], these definitions were extended beyond $\alpha \in (0,1)$ to complex values of α. The Atangana–Baleanu (AB) integral is easy to extend to complex α and complex x, just using the well-known extension of the Riemann–Liouville integral:

$$ {}^{AB}_{a}I^{\alpha}_{z}f(z) = \frac{1}{2\pi i B(\alpha)}\int_{H_a^z}\left(\frac{1-\alpha}{\zeta-z} + \frac{\alpha\Gamma(1-\alpha)}{(\zeta-z)^{1-\alpha}}\right)f(\zeta)\,\mathrm{d}\zeta.$$

For the AB derivatives (of both types), the complex contour formulae written in [35] were as follows:

$$ {}^{ABR}_{a}D^{\alpha}_{z}f(z) = \frac{B(\alpha)}{2\pi i(1-\alpha)}\cdot\frac{\mathrm{d}}{\mathrm{d}z}\int_{H_a^z}E^{\alpha}\left(\frac{-\alpha}{1-\alpha}\zeta-z)^{\alpha}\right)f(\zeta)\,\mathrm{d}\zeta,$$

$$ {}^{ABC}_{a}D^{\alpha}_{z}f(z) = \frac{B(\alpha)}{2\pi i(1-\alpha)}\int_{H_a^z}E^{\alpha}\left(\frac{-\alpha}{1-\alpha}\zeta-z)^{\alpha}\right)f'(\zeta)\,\mathrm{d}\zeta,$$

where here the notation E^{α} refers to the incorrectly defined function from [35],

$$ E^{\alpha}(z) = \sum_{n=0}^{\infty}\Gamma(-n\alpha)z^n,$$

this being incorrect because of the $n=0$ term.

The correct version of these formulae is given by slightly modifying them as follows:

$$ {}^{ABR}_{a}D^{\alpha}_{z}f(z) = \frac{B(\alpha)}{2\pi i(1-\alpha)}\cdot\frac{\mathrm{d}}{\mathrm{d}z}\int_{H_a^z}\left[\frac{1}{\zeta-z} + E^{\alpha}\left(\frac{-\alpha}{1-\alpha}\zeta-z)^{\alpha}\right)\right]f(\zeta)\,\mathrm{d}\zeta,$$

$$ {}^{ABC}_{a}D^{\alpha}_{z}f(z) = \frac{B(\alpha)}{2\pi i(1-\alpha)}\int_{H_a^z}\left[\frac{1}{\zeta-z} + E^{\alpha}\left(\frac{-\alpha}{1-\alpha}\zeta-z)^{\alpha}\right)\right]f'(\zeta)\,\mathrm{d}\zeta - f(a),$$

where this time the notation E^{α} refers to the well-defined function from (5) above. These formulae are obtained by treating the $n=0$ term separately, starting from the series formulae (16) and (18), and in the Caputo case using the fact that ${}^{RL}_{a}I^{1}_{x}f'(x) = f(x) - f(a)$. However, we can obtain more elegant formulae by considering instead the series formula (17) and using the 2-parameter modified Mittag-Leffler function defined in Section 2.2 above.

Theorem 6. *The analytic continuation of the AB fractional derivative of Riemann–Liouville type is given by*

$$ {}^{ABR}_{\ a}D^{\alpha}_{z}f(z) = \frac{B(\alpha)}{2\pi i(1-\alpha)} \int_{H_a^z} E^{\alpha,0}\left(\frac{-\alpha}{1-\alpha}(\zeta - z)^{\alpha}\right) \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta, \tag{19} $$

where $E^{\alpha,\beta}(x)$ is the modified 2-parameter Mittag-Leffler function defined by (6) above, and the complex contour of integration H_a^z is the Hankel-type contour which starts above a on the branch cut from z, wraps around z in a counterclockwise sense, and returns to a.

The analytic continuation of the AB fractional derivative of Caputo type can then be deduced using the relationship between ABR and ABC derivatives given by the fundamental theorem of calculus:

$$ {}^{ABC}_{\ a}D^{\alpha}_{z}f(z) = \frac{B(\alpha)}{2\pi i(1-\alpha)} \int_{H_a^z} E^{\alpha,0}\left(\frac{-\alpha}{1-\alpha}(\zeta - z)^{\alpha}\right) \frac{f(\zeta)}{\zeta - z} \,\mathrm{d}\zeta - \frac{B(\alpha)}{1-\alpha} E_{\alpha}\left(\frac{-\alpha}{1-\alpha} z^{\alpha}\right) f(a). $$

The assumption on the parameter α for this Theorem is simply $\alpha \in \mathbb{C}\backslash\mathbb{R}$.

Proof. We start from the series formula (17) for the ABR fractional derivative, and use the complex integral representation for the Riemann–Liouville fractional integral:

$$ \begin{aligned} {}^{ABR}_{\ a}D^{\alpha}_{x}f(x) &= \frac{B(\alpha)}{1-\alpha} \sum_{n=0}^{\infty} \left(\frac{-\alpha}{1-\alpha}\right)^{n} {}^{RL}_{a}I^{n\alpha}_{x} f(x) \\ &= \frac{B(\alpha)}{1-\alpha} \sum_{n=0}^{\infty} \left(\frac{-\alpha}{1-\alpha}\right)^{n} \frac{\Gamma(-n\alpha+1)}{2i\pi} \int_{H_a^z} (\zeta - z)^{n\alpha-1} f(\zeta) \,\mathrm{d}\zeta. \end{aligned} $$

Note that this substitution is valid for all values of n, since the complex contour formula (4) is valid for any order of differentiation not in $\mathbb{Z}^{-}$. Here the orders of differentiation are $-n\alpha$ for $n \geq 0$, which is either zero or nonreal. (This is why, when using (16) instead of (17), we needed to treat $n = 0$ separately: because in the case of (16), the $n = 0$ term gives order of differentation -1 which is in $\mathbb{Z}^{-}$).

Continuing, and using the fact that the series formulae for AB derivatives are locally uniformly convergent [38]:

$$ \begin{aligned} {}^{ABR}_{\ a}D^{\alpha}_{x}f(x) &= \frac{B(\alpha)}{1-\alpha} \frac{1}{2i\pi} \int_{H_a^z} f(\zeta) \sum_{n=0}^{\infty} \Gamma(-n\alpha+1) \left(\frac{-\alpha}{1-\alpha}\right)^{n} (\zeta - z)^{n\alpha-1} \,\mathrm{d}\zeta \\ &= \frac{B(\alpha)}{1-\alpha} \frac{1}{2i\pi} \int_{H_a^z} \frac{f(\zeta)}{\zeta - z} E^{\alpha,0}\left(\frac{-\alpha}{1-\alpha}(\zeta - z)^{\alpha}\right) \,\mathrm{d}\zeta, \end{aligned} $$

which is the desired result for ABR derivatives.

For the case of ABC derivatives, we simply use the following relationship between ABR and ABC following from the series formulae (16)–(18):

$$\begin{aligned}
{}^{ABC}_{\ a}D^{\alpha}_{x}f(x) &= \frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{n}\,{}^{RL}_{\ a}I^{n\alpha}_{x}\left({}^{RL}_{\ a}I^{1}_{x}f'(x)\right)\\
&= \frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{n}\,{}^{RL}_{\ a}I^{n\alpha}_{x}\Big(f(x)-f(a)\Big)\\
&= {}^{ABR}_{\ \ a}D^{\alpha}_{x}f(x)-\frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{n}\,{}^{RL}_{\ a}I^{n\alpha}_{x}(1)f(a)\\
&= {}^{ABR}_{\ \ a}D^{\alpha}_{x}f(x)-\frac{B(\alpha)}{1-\alpha}\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^{n}\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}f(a)\\
&= {}^{ABR}_{\ \ a}D^{\alpha}_{x}f(x)-\frac{B(\alpha)}{1-\alpha}E_{\alpha}\left(\frac{-\alpha}{1-\alpha}x^{\alpha}\right)f(a).
\end{aligned}$$

The last term here is the initial value term which gives the desired result for ABC derivatives. □

Remark 2. *Note that using $\beta = 0$ in the usual 2-parameter or 3-parameter Mittag-Leffler functions would not be possible when using them as kernel functions, because it would lead to a non-integrable singularity. However, in the complex setting, this is fine since $\frac{1}{\zeta - z}$ can be integrated using Cauchy's integral formula.*

Remark 3. *It is known that the Atangana–Baleanu fractional operators are special cases of the Prabhakar fractional calculus. Indeed, this is an obvious fact for the AB derivative, since it (like the Prabhakar operators) is defined using an integral transform with Mittag-Leffler kernel. The AB integral, on the other hand, is simply the linear combination of a function with its Riemann–Liouville integral, with no Mittag-Leffler functions involved in the definition; it was only noticed recently in [39] that it too is a special case of Prabhakar. The relationships are given by*

$$\begin{aligned}
{}^{AB}_{\ a}I^{\alpha}_{x}f(x) &= \frac{1-\alpha}{B(\alpha)}\,{}^{P}_{a}I^{\alpha,0,-1,\frac{-\alpha}{1-\alpha}}_{x}f(x),\\
{}^{ABR}_{\ \ a}D^{\alpha}_{x}f(x) &= \frac{1-\alpha}{B(\alpha)}\,{}^{P}_{a}D^{\alpha,0,-1,\frac{-\alpha}{1-\alpha}}_{x}f(x).
\end{aligned}$$

Using this, it is possible to deduce the result of Theorem 6 directly from that of Theorem 5. Note that the multiplier $(\zeta - z)^{\beta-1}$ appearing in (12), which is a typical power function multiplier found when dealing with Mittag-Leffler function kernels, in the AB case becomes simply $\frac{1}{\zeta - z}$, which is a typical multiplier found in complex analysis according to Cauchy's integral formula.

3.3. Series for Negative α

In the paper [36], a series formula is given for the Mittag-Leffler function $E_{\alpha}(z)$ which is valid for negative real numbers α, by using a functional equation that emerges from the complex integral representation. The same functional equation approach works to prove similar series formulae for the two-parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$ and for complex α with $\mathrm{Re}(\alpha) < 0$. We state the general result as follows.

Proposition 3 ([36])**.** *The analytic continuation of the two-parameter Mittag-Leffler function $E_{\alpha,\beta}(z)$, originally defined by (2), to the domain $\alpha, \beta \in \mathbb{C}, \mathrm{Re}(\alpha) < 0$ is given by the following locally uniformly convergent series:*

$$E_{\alpha,\beta}(z) = \sum_{n=1}^{\infty}\frac{-z^{-n}}{\Gamma(-n\alpha+\beta)}. \tag{20}$$

Proof. From the complex integral representation of the two-parameter Mittag-Leffler function,

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i}\int_H \frac{t^{\alpha-\beta}e^t}{t^\alpha - z}\,\mathrm{d}t = \frac{1}{2\pi i}\int_H \frac{e^t}{t^\beta - zt^{\beta-\alpha}}\,\mathrm{d}t,$$

valid for all $\alpha, \beta \in \mathbb{C}$, we use the algebraic identity

$$\frac{1}{t^\beta - zt^{\beta-\alpha}} = \frac{1}{t^\beta} - \frac{1}{t^\beta - z^{-1}t^{\alpha+\beta}} \tag{21}$$

to obtain

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i}\int_H \frac{e^t}{t^\beta}\,\mathrm{d}t - \frac{1}{2\pi i}\int_H \frac{e^t}{t^\beta - z^{-1}t^{\alpha+\beta}}\,\mathrm{d}t = \frac{1}{\Gamma(\beta)} - E_{-\alpha,\beta}(z^{-1}),$$

valid for all $\alpha, \beta \in \mathbb{C}$. Then, if $\mathrm{Re}(\alpha) < 0$, we have $\mathrm{Re}(-\alpha) > 0$ and therefore we can use the original series formula (2) for $E_{-\alpha,\beta}(z^{-1})$. Cancelling the $n = 0$ term, this gives the desired series for $E_{\alpha,\beta}(z)$ in the case of $\mathrm{Re}(\alpha) < 0$. □

Remark 4. *The same technique cannot be used to give an elegant series representation of the 3-parameter Mittag-Leffler function $E^{\gamma}_{\alpha,\beta}(z)$ for negative values of α. This is because the complex integral representation of this function involves a γth power:*

$$E^{\gamma}_{\alpha,\beta}(z) = \frac{1}{2\pi i}\int_H \frac{t^{-\beta}e^t}{(1 - zt^{-\alpha})^\gamma}\,\mathrm{d}t,$$

and there is no analogue of the identity (21) *for reciprocals of γth powers.*

Remark 5. *Furthermore, the technique of Proposition 3 cannot be applied to our modified Mittag-Leffler functions either, even in the 1-parameter and 2-parameter cases. The complex integral representations of Proposition 2 and Theorem 2 have integrands involving the functions $\mathfrak{S}_\alpha$ and $\mathfrak{S}_{\alpha,\beta}$ which do not have simple identities like* (21) *between them.*

4. Conclusions and Further Work

This paper serves as a continuation of the work of [35], in which the first modified Mittag-Leffler function was defined and used to extend Atangana–Baleanu fractional operators into the complex context. Here we have corrected an omission in [35], in which the issues surrounding the $n = 0$ term of the Mittag-Leffler series were overlooked. We have defined modified Mittag-Leffler functions of one, two, and three parameters, and rigorously checked the convergence issues for the series in each case.

The power of Mittag-Leffler functions and their series in fractional calculus cannot be understated. Several important operators of fractional calculus are defined using Mittag-Leffler functions, and the series formulae for these operators have been useful in proving a number of useful properties. Our modified Mittag-Leffler functions and their series can be used to provide new formulae for the same operators, which are valid in larger domains than the original ones. We showed how both the Prabhakar and the Atangana–Baleanu operators can be applied to find fractional derivatives and integrals of functions of a complex variable as well as real functions.

The work contained in this paper will be useful for ongoing research into these fractional-calculus operators and their applications. It was already seen, for example, in [12,40,41], that complex integral representations of Mittag-Leffler functions are useful in finding asymptotic expansions, and therefore in bounding and approximating the functions. In some cases, complex orders of fractional derivatives can be vital for modelling [42–44]. The analysis of fractional evolution processes in [12] even used Mittag-Leffler-type infinite series involving the gamma function at negative parameters, such as $\Gamma(1 - n\alpha)$, similarly to the functions we have introduced in this paper. Therefore, we expect our formulae to find applications in the future.

Furthermore, we investigated another way of extending Mittag-Leffler functions by analytic continuation, namely the series for negative α. Although this approach does not work directly on the modified Mittag-Leffler functions defined here, we believe it will be useful in the analysis of Atangana–Baleanu and Prabhakar fractional operators. Further research in this direction is currently ongoing.

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