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Integral Representation of Fractional Derivative of Delta Function

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Abstract: Delta function is a widely used generalized function in various fields, ranging from physics to mathematics. How to express its fractional derivative with integral representation is a tough problem. In this paper, we present an integral representation of the fractional derivative of the delta function. Moreover, we provide its application in representing the fractional Gaussian noise.

1. Introduction

The delta function $\delta(t)$ is a generalized function. It has wide applications in various fields of sciences and engineering, e.g., Harris [1], Korn and Korn [2], Mitra and Kaiser [3], Palley [4], Li [5,6], Dirac [7], Robinson [8], Walton [9], Novomestky [10], Hoskins [11], simply citing a few. It is defined by

$$\langle \delta(t), x(t) \rangle = \int_{-\infty}^{\infty} \delta(t)x(t)dt = x(0), \tag{1}$$

where x is a test function (Gelfand and Vilenkin [12]). Its derivative of integer order $\delta^{(n)}(t)$ ($n = 1, 2, \ldots$) is well described in the domain of generalized functions, e.g., Gelfand and Vilenkin [12]. This is obtained by

$$\left\langle \delta^{(n)}(t), x(t) \right\rangle = \int_{-\infty}^{\infty} \delta^{(n)}(t)x(t)dt = (-1)^n x^{(n)}(0). \tag{2}$$

However, the reports for its fractional derivative $\delta^{(v)}(t)$ ($v > 0$) are rarely seen. In this regard, Li and Li [13] studied the fractional derivatives of generalized functions but they did not provide the concrete representation of $\delta^{(v)}(t)$. In the preprint [14], in 2020, Makris utilizes the Riemann–Liouville definition of the fractional derivative to describe $\delta^{(v)}(t)$ in the form

$$\delta^{(v)}(t - \xi) = \frac{1}{\Gamma(-v)} \int_{0^-}^{t} \frac{\delta(\tau - \xi)}{(t - \tau)^{1+v}} d\tau. \tag{3}$$

Nevertheless, the above is not enough, obviously, because it is just an abstract form for any $x^{(v)}(t)$. Recently, in the forum of Mathematics [15], a scholar said that "He does not know whether the current integral representation of the fractional derivative works on delta function or not." As a matter of fact, how to express the integral representation of $\delta^{(v)}(t)$ is a tough and or open problem.

There are several types of definitions of fractional derivatives, such as the Riemann–Liouville's (Liouville [16], Ross [17]), the Weyl's (Weyl [18], Raina and Koul [19]). We refer to [20–28] for other

types of definitions of fractional derivatives. This research uses the Weyl definition of a fractional derivative. We aim at presenting an integral representation of $\delta^{(v)}(t)$ and its application to representing the fractional Gaussian noise.

The rest of the paper is organized as follows. Preliminaries are provided in Section 2. The results are presented in Section 3. The application case is in Section 4, which is followed by conclusions.

2. Preliminaries

Lemma 1. *Any generalized function has derivatives of all orders* [12].

Lemma 2. *There exists the Fourier transform for any generalized function* [12].

Lemma 3. *The operations of derivative and integral are interchangeable in the domain of generalized functions* [12].

Denote by F and F^{-1} the operators of Fourier transform and its inverse, respectively. Then,

$$F[x(t)] = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}dt \triangleq X(\omega),$$

$$F^{-1}[X(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega = x(t).$$

(4)

The Fourier transform of $\delta(t)$ is acquired by

$$F[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)e^{-i\omega t}dt = 1.$$

(5)

The inverse Fourier transform of 1 is expressed by

$$F^{-1}(1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t}d\omega = \delta(t).$$

(6)

Taking into account of the Euler formula in the above, one has

$$\int_{-\infty}^{\infty} \cos \omega t d\omega = 2\pi\delta(t)$$

(7)

and

$$\int_{-\infty}^{\infty} \sin \omega t d\omega = 0.$$

(8)

Equation (7) can be written by

$$\delta(t) = \frac{1}{\pi} \int_{0}^{\infty} \cos \omega t d\omega.$$

(9)

The fractional derivative of order $v \geq 0$ of t^λ for $\lambda > 0$ is in the form (Miller and Ross [29] (p. 249))

$$\frac{d^v t^\lambda}{dt^v} = \frac{\Gamma(v - \lambda)}{\Gamma(-\lambda)} t^{-v+\lambda}.$$

(10)

Lemma 4. *Let X(ω) be the Fourier transform of x(t). Then, for v ≥ 0,*

$$F\left[x^{(v)}(t)\right] = (i\omega)^v X(\omega).$$ (11)

3. Integral Representation of $\delta^{(v)}(t)$

We now present an integral representation of the fractional derivative of $\delta(t)$ of order $v \geq 0$.

Theorem 1. *An integral representation of the fractional derivative of δ (t) of order v ≥ 0 is given by*

$$\delta^{(v)}(t) = \frac{1}{\pi}\int_0^\infty \frac{d^v \cos \omega t}{dt^v} d\omega.$$ (12)

Proof. Due to Equation (9), we have

$$\delta^{(v)}(t) = \frac{1}{\pi}\frac{d^v}{dt^v}\int_0^\infty \cos \omega t\, d\omega.$$ (13)

According to Lemma 3, we have

$$\frac{d^v}{dt^v}\int_0^\infty \cos \omega t\, d\omega = \int_0^\infty \frac{d^v \cos \omega t}{dt^v} d\omega.$$ (14)

Hence, Theorem 1 holds.　□

Theorem 2. *An integral representation of $\delta^{(v)}(t)$ for v ≥ 0 can be written by*

$$\delta^{(v)}(t) = \frac{1}{\pi}\int_0^\infty \omega^v \cos\left(\omega t - \frac{\pi v}{2}\right) d\omega.$$ (15)

Proof. Following Miller and Ross [29] (p. 248), one has

$$\frac{d^v \cos \omega t}{dt^v} = \omega^v \cos\left(\omega t - \frac{\pi v}{2}\right).$$ (16)

Replacing $\frac{d^v \cos \omega t}{dt^v}$ in Equation (12) with the above yields Equation (15).　□

Note that Equation (12) is a general representation of $\delta^{(v)}(t)$ because we have not designated a concrete form of $\cos^{(v)}\omega t$ in Theorem 1. Different types of fractional derivatives may yield different results for $\cos^{(v)}\omega t$ as can be seen from [29], Lavoie et al. [30], Li and Zeng [31], Li and Cai [32]. Equation (15) in Theorem 2 is the concrete representation of $\delta^{(v)}(t)$ with the Weyl fractional derivative.

At the end of Section 3, it is noted that $\delta(t)$ can be taken as a derivative of the unit step function $u(t)$, e.g., [1–3,5,12], and Yang et al. [33]. When using $u(t)$, we have $\delta^{(v)}(t) = u^{(v+1)}(t)$.

4. Discussions

We discuss an application of $\delta^{(v)}(t)$ in this section. Denote by $h(t)$ the impulse response of a linear system L (Figure 1). The system response $y(t)$ under the excitation $x(t)$ is acquired by

$$y(t) = x(t) * h(t),\tag{17}$$

where $*$ stands for the convolution. When $x(t) = \delta^{(v)}(t)$, we have
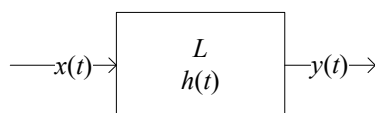
$$y(t) = \delta^{(v)}(t) * h(t) = h^{(v)}(t).\tag{18}$$

Figure 1. Linear system.

The above implies that the system with the impulse response $h(t)$ under the excitation $\delta^{(v)}(t)$ equals to the system with the impulse response $h^{(v)}(t)$ under the excitation $\delta(t)$ as shown in Figure 2.
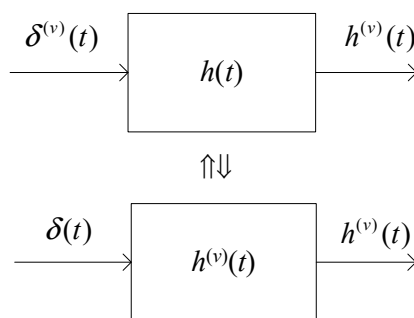

Figure 2. A fractional system resulting from $\delta^{(v)}(t)$.

Using Theorem 2 and Equation (18), we may write $h^{(v)}(t)$ by

$$h^{(v)}(t) = \delta^{(v)}(t) * h(t) = \frac{1}{\pi}\int\limits_{-\infty}^{\infty}\left[\int\limits_{0}^{\infty}\omega^v\cos\left(\omega\tau - \frac{\pi v}{2}\right)d\omega\right]h(t-\tau)d\tau.\tag{19}$$

Let $0 < H < 1$, where H is the Hurst parameter. Let

$$h(t) = \frac{\sqrt{V_H\sin(H\pi)\Gamma(2H+1)}}{-2\sin\left(\frac{(H-3/2)\pi}{2}\right)\Gamma\left(H-\frac{1}{2}\right)}\frac{\Gamma(-H)}{\Gamma\left(\frac{3}{2}-H\right)}t^H,\tag{20}$$

where

$$V_H = \Gamma(1-2H)\frac{\cos\pi H}{\pi H}.\tag{21}$$

The above $h(t)$ is a power function with dependence (Li [34]). According to Equation (10), we have

$$h^{\left(\frac{3}{2}\right)}(t) = \frac{\sqrt{V_H\sin(H\pi)\Gamma(2H+1)}}{-2\sin\left(\frac{(H-3/2)\pi}{2}\right)\Gamma\left(H-\frac{1}{2}\right)}t^{-\frac{3}{2}+H}.\tag{22}$$

Let $w(t)$ be the standard white noise. Denote by $G(t)$ the fractional Gaussian noise. Then, under the excitation $w(t)$, the response to a system with the impulse response function $h^{(1.5)}(t)$ is

$$G(t) = w(t) * h^{(3/2)}(t) = \frac{\sqrt{V_H \sin(H\pi)\Gamma(2H+1)}}{-2\sin\left(\frac{(H-3/2)\pi}{2}\right)\Gamma\left(H - \frac{1}{2}\right)} \int_{-\infty}^{\infty} \frac{w(\tau)}{(t-\tau)^{\frac{3}{2}-H}} d\tau. \tag{23}$$

The above is a new alternative representation of $G(t)$ that was previously reported by Li et al. [35]. To be precise, $G(t)$ may be taken as a response to a system with the impulse response function $h^{(1.5)}(t)$ under the excitation $w(t)$. An interesting alternative of Equation (23) is given by

$$G(t) = w^{(3/2)}(t) * h(t) = \frac{\sqrt{V_H \sin(H\pi)\Gamma(2H+1)}}{-2\sin\left(\frac{(H-3/2)\pi}{2}\right)\Gamma\left(H - \frac{1}{2}\right)} \frac{\Gamma(-H)}{\Gamma\left(\frac{3}{2} - H\right)} \int_{-\infty}^{\infty} w^{(3/2)}(\tau)(t-\tau)^H d\tau. \tag{24}$$

The above exhibits another new alternative representation of $G(t)$, which may be taken as the response to a system with the impulse response $h(t)$ under the excitation $w^{(1.5)}(t)$.

5. Conclusions

The results presented in Theorems 1 and 2 are the integral representations of the fractional derivative of the delta function, providing a solution to the tough problem. In addition, we discussed the application of the representation of the fractional derivative of the delta function in representing the fractional Gaussian noise.

Conflicts of Interest: The author declares no conflict of interest.

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