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Non-Linear First-Order Differential Boundary Problems with Multipoint and Integral Conditions

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Abstract: This paper considers boundary value problem (BVP) for nonlinear first-order differential problems with multipoint and integral boundary conditions. A suitable Green function was constructed for the first time in order to reduce this problem into a corresponding integral equation. So that by using the Banach contraction mapping principle (BCMP) and Schaefer's fixed point theorem (SFPT) on the integral equation, we can show that the solution of the multipoint problem exists and it is unique.

Keywords: existence and uniqueness; ODE; multipoint and integral boundary conditions; Banach contraction map; Green function; Schaefer fixed point

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1. Introduction

Boundary value problems for ODEs, with special initial-boundary conditions, are intensively investigated for their many applications in physics and mathematics [1,2] in a wide range of problems from vibrations to the theory of elasticity [3]. In mathematical terms, these problems are often described by the multipoint boundary value problems [1]. This theory was mainly described in the original papers of Il'in and Moiseev [4], with further developments by several authors who contributed with fundamental results based on the Leray–Schauder Continuation Theorem and corresponding nonlinear generalizations, the degree theory, and fixed point theorem (FPT).

Many authors have studied various aspects of boundary value problems with multipoint boundary conditions for the differential equations having broad applications in several branches of physics and applied mathematics [5–11].

Differential equations with integral boundary conditions also have many applications in modeling and analyzing of many physical systems as blood flow problems, chemical engineering, thermoelasticity, underground water flow, population dynamics, etc. [12–33].

In this paper, an original approach based on the construction of a suitable Green function is proposed for the analysis of the multipoint BPV, so that a problem on a differential system is converted into an equivalent integral equation. Comparing with the results obtained by Multy and Sivasundaram [34], we do not use the fundamental matrix of the equation. The main advantage of our choice is that we don't require the existence of the derivative of the equation with respect to the phase coordinates. Then the uniqueness of the solution is studied for the integral equation by means of the Banach contraction mapping principle (BCMP), while the existence is also shown by using Schaefer's fixed point theorem (SFPT).

The organization of the paper is as follows: in Section 2 are given some preliminary remarks about this problem, together with some related definitions and known methods.

Section 3 deals with the proof of uniqueness, while in Section 4 is given a proof of the existence by means of the fixed point theorem. Some applications are given in Section 5. Conclusion and future perspectives are discussed in Section 6.

2. Preliminary Remarks

Let us start by considering the following nonlinear differential system

$$\dot{x}(t) = f(t, x(t)),\ t \in [0, T], \tag{1}$$

with multipoint boundary conditions

$$\sum_{i=0}^{m} l_i x(t_i) + \int_0^T n(t)x(t)dt = \alpha\ , \tag{2}$$

where $l_i,\ i = 1, 2, \ldots, m$ are n-order constant matrices with $\det N \neq 0,\ N = \sum\limits_{i=0}^{m} l_i + \int\limits_0^T n(t)dt$; $f : [0, T] \times R^n \to R^n\ n : [0.T] \to R^{n\times n}$ are some given continuous functions; the points $t_0, t_1, \ldots, t_m$ are arbitrarely chosen in the finite interval $0 = t_0 < t_1 < \ldots < t_{m-1} < t_m = T$. Let $C([0, T]; R^n)$ be the Banach space of all continuous functions from $[0, T]$ into R^n with the norm $\|x\| = \max\{|x(t)| :\ t \in [0, T]\}$.

In general the solution of (1)–(2) is characterized by the following:

Definition 1. *A function $x \in C^1([0, T]; R^n)$ is a solution of (1) and (2) if $\dot{x}(t) = f(t, x(t)),$ and for each $t \in [0, T]$ boundary conditions (2) are fulfilled.*

Let us now study the following problem:

$$\dot{x}(t) = y(t),\ t \in [0, T], \tag{3}$$

$$\sum_{i=1}^{m} l_i x(t_i) + \int_0^T n(t)x(t)dt = \alpha. \tag{4}$$

We have that

Lemma 1. *For $y \in C([0, T]; R^n)$ the solution of the BVP (3) and (4) is unique and it is given by*

$$x(t) = N^{-1}\alpha + \int_0^T G(t, \tau)y(\tau)d\tau,$$

where

$$G(t, \tau) = \begin{cases} G_1(t, \tau),\ \ t \in [0, t_1], \\ G_2(t, \tau),\ \ t \in [t_1, t_2), \\ \ldots\ldots\ldots\ldots \\ G_m(t, \tau),\ \ t \in [t_{m-1}, T]. \end{cases}$$

with

$$G_i(t,\tau) = \begin{cases} N^{-1}\left(l_0 + \int\limits_0^{\tau} n(s)ds\right), \; t_0 \le \tau \le t_1, \\ N^{-1}\left(\sum\limits_{k=0}^{1} l_k + \int\limits_0^{\tau} n(s)ds\right), \; t_1 < \tau \le t_2, \\ \dots\dots\dots\dots \\ N^{-1}\left(\sum\limits_{k=0}^{i-1} l_k + \int\limits_0^{\tau} n(s)ds\right), \; t_{i-1} < \tau \le t_i, \\ N^{-1}\left(\sum\limits_{k=0}^{i} l_k + \int\limits_0^{\tau} n(s)ds\right), \; t_i < \tau \le t, \\ -N^{-1}\left(\sum\limits_{k=i+1}^{m} l_k + \int\limits_{\tau}^{T} n(s)ds\right), \; t < \tau \le t_{i+1}, \\ -N^{-1}\left(\sum\limits_{k=i+2}^{m} l_k + \int\limits_{\tau}^{T} n(s)ds\right), \; t_{i+1} < \tau \le t_{i+2}, \\ \dots\dots\dots\dots\dots\dots \\ -N^{-1}\left(l_m + \int\limits_{\tau}^{T} n(s)ds\right), \; t_{m-1} < \tau \le T. \end{cases}$$

Proof. For any $t \in (0,T)$ the solution $x = x(\cdot)$ fulfills

$$x(t) = x_0 + \int\limits_0^t y(\tau)d\tau, \tag{5}$$

where x_0 is an arbitrary constant vector. Let us chose x_0 in such a way that $x(t)$ fulfills Equation (4). There follows

$$\sum_{i=0}^{m} l_i[x_0 + \int\limits_0^{t_i} y(s)ds] + \int\limits_0^T n(t)\left(x_0 + \int\limits_0^t y(s)ds\right)dt = \alpha \, ,$$

which implies

$$x_0 = N^{-1}\alpha - N^{-1}\left[\sum_{i=1}^{m} l_i \int\limits_0^{t_i} y(s)ds + \int\limits_0^T n(t)\int\limits_0^t y(s)dsdt\right]. \tag{6}$$

If we put this value into Equation (5), we get

$$x(t) = N^{-1}\alpha - N^{-1}\left[\sum_{i=1}^{m} l_i \int\limits_0^{t_i} y(s)ds + \int\limits_0^T n(t)\int\limits_0^t y(s)dsdt\right] + \int\limits_0^t y(s)ds. \tag{7}$$

Since the equality

$$\int\limits_0^T n(t)\int\limits_0^t y(s)dsdt = \int\limits_0^T \int\limits_t^T n(s)dsy(t)dt$$

holds, from Equation (7) we get

$$x(t) = N^{-1}\alpha - N^{-1}\left[\sum_{i=1}^{m} l_i \int_0^{t_i} y(s)ds + \int_0^T \int_t^T n(s)dsy(t)dt\right] + \int_0^t y(s)ds. \tag{8}$$

From where, by taking $t \in [0, t_1]$, there follows

$$\begin{aligned}
x(t) &= N^{-1}\alpha - N^{-1}\left(l_1\int_0^t y(\tau)d\tau + l_1\int_t^{t_1} y(\tau)d\tau\right) - N^{-1}\left(l_2\int_0^t \mu(\tau)d\tau + l_2\int_t^{t_1} \mu(\tau)d\tau\right) \\
&-N^{-1}l_2\int_{t_1}^{t_2} y(\tau)d\tau - N^{-1}\left(l_3\int_0^t y(\tau)d\tau + l_3\int_t^{t_1} y(\tau)d\tau\right) - N^{-1}l_3\left(\sum_{i=1}^{2}\int_{t_i}^{t_{i+1}} y(\tau)d\tau\right) - \dots \\
&-N^{-1}\left(\int_0^t\int_\tau^T n(s)dsy(\tau)d\tau + \int_t^{t_1}\int_\tau^T n(s)dsy(\tau)d\tau\right) - N^{-1}\left(\sum_{i=1}^{m}\int_{t_i}^{t_{i+1}}\int_\tau^T n(s)dsy(\tau)d\tau\right) \\
&+\int_0^t y(\tau)d\tau.
\end{aligned}$$

Let us write this equation in the equivalent form

$$\begin{aligned}
x(t) &= N^{-1}\alpha + \int_0^t \left(E - N^{-1}\sum_{i=1}^{m} l_i - N^{-1}\int_\tau^T n(s)ds\right)y(\tau)d\tau \\
&-N^{-1}\int_t^{t_1}\left(\sum_{i=1}^{m} l_i + \int_\tau^T n(s)ds\right)y(\tau)d\tau - N^{-1} - \int_{t_1}^{t_2} N^{-1}\left(\sum_{i=2}^{m} l_i + \int_\tau^T n(s)ds\right)y(\tau)d\tau \\
&-\int_{t_2}^{t_3} N^{-1}\left(\sum_{i=3}^{m} l_i + \int_\tau^T n(s)ds\right)y(\tau)d\tau - \dots - \int_{t_{m-1}}^{T} N^{-1}\left(l_m + \int_\tau^T n(s)ds\right)y(\tau)d\tau.
\end{aligned} \tag{9}$$

where E is the identity matrix.

Since

$$E - N^{-1}\sum_{i=1}^{m} l_i - N^{-1}\int_\tau^T n(s)ds = N^{-1}\left(l_0 + \int_0^\tau n(s)ds\right)$$

takes place, there follows

$$\begin{aligned}
x(t) &= N^{-1}\alpha + \int_0^t N^{-1}\left(l_0 + \int_0^\tau n(s)ds\right)y(\tau)d\tau - \\
&N^{-1}\int_t^{t_1}\left(\sum_{i=1}^{m} l_i + \int_\tau^T n(s)ds\right)y(\tau)d\tau - \int_{t_1}^{t_2} N^{-1}\left(\sum_{i=2}^{m} l_i + \int_\tau^T n(s)ds\right)y(\tau)d\tau \\
&-\int_{t_2}^{t_3} N^{-1}\left(\sum_{i=3}^{m} l_i + \int_\tau^T n(s)ds\right)y(\tau)d\tau - \dots \\
&-\int_{t_{m-1}}^{T} N^{-1}\left(l_m + \int_\tau^T n(s)ds\right)y(\tau)d\tau.
\end{aligned}$$

If we define:

$$G_1(t,\tau) = \begin{cases} N^{-1}\left(l_0 + \int\limits_0^{\tau} n(s)ds\right),\ t_0 \le \tau \le t, \\ -N^{-1}\left(\sum\limits_{i=1}^{m} l_i + \int\limits_{\tau}^{T} n(s)ds\right),\ t < \tau \le t_1, \\ -N^{-1}\left(\sum\limits_{i=2}^{m} l_i + \int\limits_{\tau}^{T} n(s)ds\right),\ t_1 < \tau \le t_2, \\ -N^{-1}\left(\sum\limits_{i=3}^{m} l_i + \int\limits_{\tau}^{T} n(s)ds\right),\ t_2 < \tau \le t_3, \\ \dots\dots\dots\dots\dots\dots \\ -N^{-1}\left(l_m + \int\limits_{\tau}^{T} n(s)ds\right),\ t_{m-1} < \tau \le T. \end{cases}$$

Equation (9) becomes,

$$x(t) = N^{-1}\alpha + \int\limits_0^T G_1(t,\tau)y(\tau)d\tau \tag{10}$$

For $t \in (t_1, t_2]$, Equation (8) gives

$$\begin{aligned} &x(t) = N^{-1}\alpha - N^{-1}\sum_{i=1}^{m} l_i \int\limits_0^{t_1} y(\tau)d\tau - N^{-1}\sum_{i=2}^{m} l_i\left(\int\limits_{t_1}^{t} y(\tau)d\tau + \int\limits_{t}^{t_2} y(\tau)d\tau\right) \\ &-N^{-1}\sum_{i=3}^{m} l_i\int\limits_{t_2}^{t_3} y(\tau)d\tau - \ldots - N^{-1}l_m\int\limits_{t_{m-1}}^{T} y(\tau)d\tau + \int\limits_0^{t_1} y(\tau)d\tau + \int\limits_{t_1}^{t} y(\tau)d\tau \\ &-\int\limits_0^{t_1} N^{-1}\left(\int\limits_{\tau}^{T} n(s)dsy(\tau)\right)d\tau - \left(\int\limits_{t_1}^{t} N^{-1}\left(\int\limits_{\tau}^{T} n(s)dsy(\tau)\right)d\tau + \int\limits_{t}^{t_2} N^{-1}\left(\int\limits_{\tau}^{T} n(s)dsy(\tau)\right)d\tau\right) \\ &-\sum_{i=2}^{m-1}\int\limits_{t_i}^{t_{i+1}} N^{-1}\int\limits_{\tau}^{T} n(s)dsy(\tau)d\tau = N^{-1}\alpha + \int\limits_0^{t_1}\left(E - N^{-1}\sum_{i=1}^{m} l_i - N^{-1}\int\limits_{\tau}^{T} n(s)ds\right)y(\tau)d\tau \\ &+\int\limits_{t_1}^{t}\left(E - N^{-1}\sum_{i=2}^{m} l_i - N^{-1}\int\limits_{\tau}^{T} n(s)ds\right)y(\tau)d\tau + \int\limits_{t}^{t_2}\left(-\sum_{i=2}^{m} N^{-1}l_i - N^{-1}\int\limits_{\tau}^{T} n(s)ds\right)y(\tau)d\tau \\ &+\int\limits_{t_2}^{t_3}\left(-\sum_{i=3}^{m} N^{-1}l_i - N^{-1}\int\limits_{\tau}^{T} n(s)ds\right)y(\tau)d\tau - \ldots + \int\limits_{t_{m-1}}^{T}\left(-N^{-1}l_m - N^{-1}\int\limits_{\tau}^{T} n(s)ds\right)y(\tau)d\tau \\ &= N^{-1}\alpha + \int\limits_0^{t_1} N^{-1}\left(l_0 + \int\limits_0^{\tau} n(s)ds\right)y(\tau)d\tau + \int\limits_{t_1}^{t} N^{-1}\left(l_0 + l_1 + \int\limits_0^{\tau} n(s)ds\right)y(\tau)d\tau \\ &+\int\limits_{t}^{t_2}\left(-\sum_{i=2}^{m} N^{-1}l_i - N^{-1}\int\limits_{\tau}^{T} n(s)ds\right)y(\tau)d\tau + \int\limits_{t_2}^{t_3}\left(-\sum_{i=3}^{m} N^{-1}l_i - N^{-1}\int\limits_{\tau}^{T} n(s)ds\right)y(\tau)d\tau - \ldots \\ &+\int\limits_{t_{m-1}}^{T}\left(-N^{-1}l_m - N^{-1}\int\limits_{\tau}^{T} n(s)ds\right)y(\tau)d\tau. \end{aligned}$$

So that by defining

$$G_2(t,\tau) = \begin{cases} N^{-1}\left(l_0 + \int\limits_0^{\tau} n(s)ds\right),\ t_0 \le \tau \le t_1, \\ N^{-1}\left(l_0 + l_1 + \int\limits_0^{\tau} n(s)ds\right),\ t_1 < \tau \le t, \\ -N^{-1}\left(\sum\limits_{i=2}^{m} l_i + \int\limits_{\tau}^{T} n(s)ds\right),\ t < \tau \le t_2, \\ -N^{-1}\left(\sum\limits_{i=3}^{m} l_i + \int\limits_{\tau}^{T} n(s)ds\right),\ t_2 < \tau \le t_3, \\ \dots\dots\dots\dots\dots\dots \\ -N^{-1}\left(l_m + \int\limits_{\tau}^{T} n(s)ds\right),\ t_{m-1} < \tau \le T. \end{cases}$$

it is obtained

$$x(t) = N^{-1}\alpha + \int\limits_0^T G_2(t,\tau)\mu(\tau)d\tau.$$

Continuing this process in a similar way, for the next segment $t \in (t_i, t_{i+1}]$, we get

$$x(t) = N^{-1}\alpha + \int\limits_0^T G_i(t,\tau)\mu(\tau)d\tau,$$

where

$$G_i(t,\tau) = \begin{cases} N^{-1}\left(l_0 + \int\limits_0^{\tau} n(s)ds\right),\ t_0 \le \tau \le t_1, \\ N^{-1}\left(\sum\limits_{i=0}^{1} l_i + \int\limits_0^{\tau} n(s)ds\right),\ t_1 < \tau \le t_2, \\ \dots\dots\dots\dots\dots \\ N^{-1}\left(\sum\limits_{k=0}^{i-1} l_k + \int\limits_0^{\tau} n(s)ds\right),\ t_{i-1} < \tau \le t_i, \\ N^{-1}\left(\sum\limits_{k=0}^{i} l_k + \int\limits_0^{\tau} n(s)ds\right),\ t_i < \tau \le t, \\ -N^{-1}\left(\sum\limits_{k=i+1}^{m} l_i + \int\limits_{\tau}^{T} n(s)ds\right),\ t < \tau \le t_{i+1}, \\ -N^{-1}\left(\sum\limits_{k=i+2}^{m} l_i + \int\limits_{\tau}^{T} n(s)ds\right),\ t_{i+1} < \tau \le t_{i+2}, \\ \dots\dots\dots\dots\dots\dots \\ -N^{-1}\left(l_m + \int\limits_{\tau}^{T} n(s)ds\right),\ t_{m-1} < \tau \le T. \end{cases}$$

and so on. There follows that Equations (3)–(4) can be expressed by

$$x(t) = N^{-1}\alpha + \int\limits_0^T G(t,\tau)\mu(\tau)d\tau.$$

So that the proof is given. □

3. Uniqueness of the Solution

The uniqueness of the solution of problem (1) and (2) is proven here by taking into account the following:

Lemma 2. *Let $f \in C([0,T] \times R^n; R^n)$, then $x(t)$ is solution of the BVP (1)–(2) iff $x(t)$ is a solution of the following integral equation*

$$x(t) = N^{-1}\alpha + \int_0^T G(t,\tau) f(\tau, x(\tau)) d\tau. \tag{11}$$

Proof. The proof is obtained similarly to Lemma 1 so that by a direct computation, we can see that the solution of Equation (11) fulfills BVP (1) and (2). □

Let us now assume that:

Hypothesis 1 (H1). $f : [0,T] \times R^n \to R^n$ *is a continuous function;*

Hypothesis 2 (H2). *There exists a constant $M \geq 0$ such that the inequality*

$$|f(t,x) - f(t,y)| \leq M|x - y|$$

holds for each $t \in [0,T]$ and all $x, y \in R^n$;

Hypothesis 3 (H3). *There exists a constant $K \geq 0$ such that $|f(t,x)| \leq K$ for each $t \in [0,T]$ and all $x \in R^n$.*

We can show that:

Theorem 1. *[Uniqueness]. By assuming that, (H1) and (H2) holds and*

$$L = TSM < 1, \tag{12}$$

where

$$S = \max_{[0,T]\times[0,T]} \|G(t,\tau)\|.$$

Then BVP (1), (2) admits a unique solution on the interval $[0,T]$.

Proof. To show this, let us transform (1) and (2) into a fixed point problem. Let

$$F : C([0,T]; R^n) \to C([0,T]; R^n)$$

be an operator, defined as

$$(Fx)(t) = N^{-1}\alpha + \int_0^T G(t,\tau) f(\tau, x(\tau)) d\tau \tag{13}$$

whose fixed points are solutions of Equations (1) and (2).

Setting $\max_{[0,T]} |f(t,0)| = M_f$ and taking $r \geq \frac{\|N^{-1}d\| + M_f TS}{1-L}$ we show that $FB_r \subset B_r$, where

$$B_r = \{x \in C([0,T]R^n) : \|x\| \leq r\}$$

For $x \in B_r$, by using (H2), we get

$$\|Fx(t)\| \leq \left\|N^{-1}\alpha\right\| + \int_0^T |G(t,\tau)|(|f(\tau,x(\tau)) - f(\tau,0)| + |f(\tau,0)|)d\tau$$
$$\leq \left\|N^{-1}d\right\| + S\int_0^T \left(M|x| + M_f\right)dt \leq \left\|N^{-1}d\right\| + SMrT + M_fTS \leq \frac{\left\|N^{-1}\alpha\right\| + M_fTS}{1-L} \leq r$$

Let us show now that F is a contraction map for any $x, y \in B_r$. Thus we write

$$|Fx - Fy| \leq \int_0^T |G(t,\tau)(f(\tau,x(\tau)) - f(\tau,y(\tau))|d\tau \leq \int_0^T |G(t,\tau)||f(\tau,x(\tau)) - f(\tau,y(\tau))|d\tau$$
$$\leq MS\int_0^T |x(t) - y(t)|dt \leq MTS\max_{[0,T]} |x(t) - y(t)| \leq MTS\|x - y\|$$

or

$$\|Fx - Fy\| \leq L\|x - y\|$$

It shows that according to (12), F is a contraction map and therefore (1) and (2) admits a unique solution. □

4. Existence of the Solution

Theorem 2. *[Existence] Let us assume that (H1)–(H3) hold. Then there exists at least one solution of (1), (2) on [0, T].*

Proof. By taking into account its definition (13), we can use the SFPT to show that there exists a fixed point for F. The multistep proof is as follows:

Step 1: F is a continuous operator. In order to show this, let $\{x_n\}$ be a sequence such that $x_n \to x$ in $C([0,T];R^n)$. There follows that, for $t \in [0,T]$

$$|(Fx)(t) - (Fx_n)(t)| = \left|\int_0^T G(t,\tau)(f(\tau,x(\tau)) - f(\tau,x_n(\tau)))d\tau\right|$$
$$\leq TSM|x(t) - x_n(t)| \leq L\|x - x_n\|.$$

From this we get $\|(Fx)(t) - (Fx_n)(t)\| \to 0$ as $n \to \infty$, which implies that F is a continuous operator.

Step 2: The operator F maps bounded sets into bounded sets in $C([0,T];R^n)$. In order to show this, it is enough to prove that for any $\eta > 0$ there exists a positive constant ω such that for each

$$x \in B_\eta = \{x \in C([0,T];R^n) : \|x\| \leq \eta\}$$

it is $\|F(x)\| \leq \omega$. So that for each $t \in [0,T]$ we get

$$|(Fx)(t)| \leq \left\|N^{-1}\alpha\right\| + TSK$$

From where there follows

$$\|F(x)\| \leq \left\|N^{-1}\alpha\right\| + TSK = \omega,$$

Step 3: Let us show now that F maps bounded sets into equicontinuous sets in $C([0,T];R^n)$. Let $\xi_1, \xi_2 \in [0,T],\ \xi_1 < \xi_2,\ B_\eta$ be a bounded set in $C([0,T];R^n)$ as shown in Step 2, and let $x \in B_\eta$.

Case 1. $\xi_1, \xi_2 \in [t_i, t_{i+1}]$. Then,

$$F(x(\xi_2)) - F(x(\xi_1)) = \int_{t_i}^{\xi_2} N^{-1}\left(\sum_{k=0}^{i} l_k + \int_0^{\tau} n(s)ds\right) f(\tau, x(\tau))d\tau$$
$$- \int_{\xi_2}^{t_{i+1}} N^{-1}\left(\sum_{k=i+1}^{m} l_k + \int_{\tau}^{T} n(s)ds\right) f(\tau, x(\tau))d\tau$$
$$- \int_{t_i}^{\xi_{\backslash 1}} N^{-1}\left(\sum_{k=0}^{i} l_k + \int_0^{\tau} n(s)ds\right) f(\tau, x(\tau))d\tau$$
$$+ \int_{\xi_1}^{t_{i+1}} N^{-1}\left(\sum_{k=i+1}^{m} l_k + \int_{\tau}^{T} n(s)ds\right) f(\tau, x(\tau))d\tau$$
$$= \int_{\xi_1}^{\xi_2} N^{-1}\left(\sum_{k=0}^{i} l_k + \int_0^{\tau} n(s)ds\right) f(\tau, x(\tau))d\tau + \int_{\xi_1}^{\xi_2} N^{-1}\left(\sum_{k=i+1}^{m} l_k + \int_{\tau}^{T} n(s)ds\right) f(\tau, x(\tau))d\tau$$
$$= \int_{\xi_1}^{\xi_2} f(\tau, x(\tau))d\tau.$$

Case 2. $\xi_1 \in [t_{i-1}, t_i),\ \xi_2 \in [t_i, t_{i+1}]$. Then

$$F(x(\xi_2)) - F(x(\xi_1)) =$$
$$= \int_{t_{i-1}}^{t_i} N^{-1}\left(\sum_{k=0}^{i-1} l_k + \int_0^{\tau} n(s)ds\right) f(\tau, x(\tau))d\tau + \int_{t_i}^{\xi_2} N^{-1}\left(\sum_{k=0}^{i} l_k + \int_0^{\tau} n(s)ds\right) f(\tau, x(\tau))d\tau$$
$$- \int_{\xi_2}^{t_{i+1}} N^{-1}\left(\sum_{k=i+1}^{m} l_k + \int_{\tau}^{T} n(s)ds\right) f(\tau, x(\tau))d\tau$$
$$- \int_{t_{i-1}}^{\xi_{\backslash 1}} N^{-1}\left(\sum_{k=0}^{i-1} l_k + \int_{\tau}^{T} n(s)ds\right) f(\tau, x(\tau))d\tau + \int_{\xi_1}^{t_i} N^{-1}\left(\sum_{k=i}^{m} l_k + \int_{\tau}^{T} n(s)ds\right) f(\tau, x(\tau))d\tau$$
$$+ \int_{t_i}^{t_{i+1}} N^{-1}\left(\sum_{k=i+1}^{m} l_k + \int_{\tau}^{T} n(s)ds\right) f(\tau, x(\tau))d\tau$$
$$= \int_{\xi_1}^{t_i} f(\tau, x(\tau))d\tau + \int_{t_i}^{\xi_2} f(\tau, x(\tau))d\tau = \int_{\xi_1}^{\xi_2} f(\tau, x(\tau))d\tau.$$

The r.h.s. of this equation tends to zero for $t_2 \to t_1$. As a consequence of Steps 1–3 and by taking into account the Ascoli–Arzela theorem, there follows that $F : C([0, T]; R^n) \to C([0, T]; R^n)$ is continuous.

Step 4: Let us show now that the set $\Delta = \{x \in C([0, T]; R^n) : x = \lambda F(x)\ for\ some\ 0 < \lambda < 1\}$ is bounded. Let $x \in \Delta$, then, $x = \lambda F(x)$ for some $0 < \lambda < 1$ so that, for each $t \in [0, T]$ we have

$$x(t) = \lambda N^{-1}\alpha + \lambda \int_0^T G(t, \tau) f(\tau, x(\tau))d\tau.$$

So that

$$\|x\| \le \left\|N^{-1}\alpha\right\| + SMT.$$

Therefore, Δ is bounded and we can conclude that the operator F admits at least one fixed point. As a consequence, there exists at least one solution for the problem (1) and (2) on the interval $[0, T]$. □

Some more problems for the two-point and the three-point boundary value conditions are studied in [5,30–34].

5. Example: Analysis of the Vibrations of a Non-Homogeneous String

The existence and uniqueness of the solution for a nonlinear first-order equation with multipoint boundary conditions are given for a concrete example.

Example. Let

$$\begin{cases} \dot{x}_1 = \sin\alpha\, x_2, \\ \dot{x}_2 = \cos\beta x_1, \end{cases} \quad t \in [0,2] \tag{14}$$

be a given differential system with the following three-point and integral boundary conditions:

$$\begin{aligned} \tfrac{1}{2}x_1(0) - \tfrac{1}{2}x_2(0) + x_2(1) - \tfrac{1}{2}x_2(2) + \int_0^2 \tfrac{1}{4}x_1(t)dt = 1, \\ -\tfrac{1}{2}x_1(0) + x_1(1) - \tfrac{1}{2}x_1(2) + \tfrac{1}{2}x_2(2) + \int_0^2 \tfrac{1}{4}x_2(t)dt = 0. \end{aligned} \tag{15}$$

Evidently,

$$\int_0^2 n(t)dt = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

Obviously for this case

$$l_0 = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix},\ l_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, l_2 = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$N = l_0 + l_1 + l_2 + \int_0^2 n(t)dt = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

Then for $t \in [0,1]$ we obtain

$$G_1(t,\tau) = \begin{cases} \begin{pmatrix} 0.5 + 0.25\tau & -0.5 \\ -0.5 & 0.25\tau \end{pmatrix}, & 0 \le \tau \le t \\ \begin{pmatrix} 0.25(\tau-2) & -0.5 \\ -0.5 & 0.25(\tau-2)+0.5 \end{pmatrix}, & t < \tau \le 1, \\ \begin{pmatrix} 0.25(\tau-2) & 0.5 \\ 0.5 & 0.25(\tau-2)-0.5 \end{pmatrix}, & 1 < \tau \le 2, \end{cases}$$

and for $t \in (1,2]$

$$G_2(t,\tau) = \begin{cases} \begin{pmatrix} 0.5 + 0.25\tau & -0.5 \\ -0.5 & 0.25\tau \end{pmatrix}, & 0 \le \tau \le 1 \\ \begin{pmatrix} 0.25\tau + 0.5 & 0.5 \\ 0.5 & 0.25\tau \end{pmatrix}, & 1 < \tau \le t, \\ \begin{pmatrix} 0.25(\tau-2) & 0.5 \\ 0.5 & 0.25(\tau-2)-0.5 \end{pmatrix}, & t < \tau \le 2, \end{cases}$$

Here

$$\|S\| \le 2,\ T = 2 \text{ and } K = \max\{|\alpha|, |\beta|\}$$

So $L = TSK = 2 \cdot 2 \cdot \max\{|\alpha|, |\beta|\} < 1$.

Thus $\max\{|\alpha|, |\beta|\} < \frac{1}{4}$. From here, we can easily see that the given system (14)–(15) has a unique solution.

The solution of system (14)–(15) involves elliptic functions; therefore, the exact solution is a quite impossible problem. However, in some special cases, it is possible to obtain the exact form of the two functions $x_1(t),\ x_2(t),$ which also fulfill the boundary-integral conditions (15). In fact, let us compute the solution of system (14)–(15) after linearization of the trigonometric functions of (14), that is

$$\left\{\begin{array}{rcl} \dot{x_1} &=& \alpha\, x_2 \\ \dot{x_2} &=& 1\,, \end{array}\right. \tag{16}$$

$$\left\{\begin{array}{rcl} x_1(t) &=& x_1(0) + \alpha\, x_2(0)t + \frac{1}{2}\alpha t^2 \\ x_2(t) &=& x_2(0) + t\,. \end{array}\right.$$

Moreover, if we also assume that the initial conditions are as follows

$$\left\{\begin{array}{rcl} x_1(0) &=& 1 + \frac{5}{12}\alpha - \frac{1}{4}\alpha^2 \\ x_2(0) &=& \frac{\alpha}{2} - \frac{3}{2} \end{array}\right.$$

the two functions $x_1(t),\ x_2(t),$ also fulfill the three-point boundary-integral conditions (15). So that, at least in the linearized case, we have explicitly computed the solution of the three-point boundary-integral problem.

At the second-order approximation

$$\left\{\begin{array}{rcl} \dot{x_1} &=& \alpha\, x_2 \\ \dot{x_2} &=& 1 - \frac{\beta^2}{2}x_1^2 \end{array}\right. \tag{17}$$

the solution can be obtained by solving the equation

$$\ddot{x_1} + \frac{\beta^2}{2}x_1^2 = \alpha$$

which, however, is expressed in terms of Weierstrass elliptic functions.

6. Conclusions

In this paper, a proof of the existence and uniqueness is proved for the solution for a class of nonlinear differential equations with some special boundary conditions. These theorems might be useful in the analysis of several physical problems arising in applied fields, such as problems with impulsive conditions, or wave propagations in non-homogeneous media. So, when the hypotheses of the theorems are fulfilled, then the solution exists and is unique.

The approach given here may be applied to the special cases, for instance, if a physical process is described in terms of a multipoint boundary and is subjected to an impulsive effect at certain points, then it can be studied by the following problem:

$$\dot{x} = f(t,x),\ t \in\ [0,T],$$

under multipoint and integral boundary conditions

$$\sum_{i=0}^{m} l_i x(t_i) + \int_0^T n(t)x(t)dt = \alpha,$$

with impulsive conditions

$$x\left(\tau_j^+\right) - x(\tau_j) = I_j\big(x(\tau_j)\big),\ j = 1,2,\ldots,k.$$

Here $0 = t_0 < t_1 < \ldots < t_{m-1} < t_m = T;\ 0 < \tau_1 < \tau_2 < \ldots < \tau_k < T; n(t) \in R^{n\times n}$ is a given function; $l_i \in R^{n\times n},\ i = 1, 2, \ldots, m$ are given matrices; $\alpha \in R^n$ is a given vector, and

$$\det N \neq 0,\ N = \sum_{i=0}^{m} l_i + \int_0^T n(t)dt.$$

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